

# Pacific Journal of Mathematics

**ON THE UNIFORM CONVERGENCE OF A CERTAIN  
EIGENFUNCTION SERIES**

LUNA MISHOE AND G. C. FORD

# ON THE UNIFORM CONVERGENCE OF A CERTAIN EIGENFUNCTION SERIES

L. I. MISHOE AND G. C. FORD

**1. Introduction.** In the attempt to solve certain problems in mathematical physics, such as diffraction of an arbitrary pulse by a wedge as considered by Irvin Kay [1], one encounters a hyperbolic differential equation of the type

$$(a) \quad u_{xx} - q(x)u = u_{xt} - p(x)u_t$$

where  $u(x, t)$  must satisfy the boundary conditions  $u(1, t) = u(0, t) = 0$  and  $u(x, 0) = F(x)$ . In attempting to solve equation (a) by separation of variables, one is led to the consideration of expanding an arbitrary function  $F(x)$  in terms of the eigenfunctions  $u_n(x)$  of the equation

$$u'' + q(x)u + \lambda(p(x)u - u') = 0$$

satisfying the boundary conditions  $u(0) = u(1) = 0$ .

In the previous paper [2] by B. Friedman and L. I. Mishoe, it was proved that a function  $F(x)$  of bounded variation for  $0 \leq x \leq 1$  could be expanded in terms of the eigenfunctions  $u_n(x)$  of the system  $u'' + qu + \lambda(pu - u') = 0$ ,  $u(0) = u(1) = 0$ , provided  $F(0^+) + F(1^-) \exp\left(-\int_0^1 p dt\right) = 0$ . However, the question of uniform convergence of the series  $\sum_{n=1}^{\infty} a_n u_n(x)$  to  $F(x)$  was not considered. In this paper we establish sufficient conditions for the series  $\sum_{n=1}^{\infty} a_n u_n(x)$  to converge uniformly to  $F(x)$  for  $0 < x < 1$ .

The following theorem has already been proved [2]:

**THEOREM 1.** *Let  $F(x)$  be a function of bounded variation for  $0 \leq x \leq 1$ . Let  $u_n(x)$  be the eigenfunctions of the system*

$$(1) \quad (A + \lambda B)u = 0; \quad u(0) = u(1) = 0,$$

where  $A$  is the operator  $d^2/dx^2 + q(x)$ , and where  $B$  is the operator  $-d/dx + p(x)$ .

Let  $q(x)$  be continuous and  $p(x)$  have a continuous second derivative. Furthermore, let  $v_n(x)$  be the eigenfunctions of the system adjoint to (1). If

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$$(2) \quad F(0^+) + F(1^-) \exp\left(-\int_0^1 p(t)dt\right) = 0,$$

then the series

$$(3) \quad \sum_{-\infty}^{\infty} a_n u_n(x),$$

where

$$(4) \quad a_n = \int_0^1 F(\xi) \left[ \frac{p(\xi)v_n(\xi) + v_n'(\xi)}{C'(\lambda_n)} \right] d\xi,$$

and where the Wronskian  $\omega(x)$  of the two independent solutions  $u_1(x)$  and  $u_2(x)$  has the form

$$(5) \quad \omega(x) = u_1 u_2' - u_2 u_1' = C(\lambda) e^{\lambda x}$$

with

$$(6) \quad C(\lambda) = \lambda^{-1} \exp\left(-\int_0^1 p(t)dt\right) - \exp\left(-\lambda + \int_0^1 p(t)dt\right) + O(\lambda^{-2}),$$

converges to  $F(x)$  at every point where  $F(x)$  is continuous in  $0 < x < 1$ . At all other points, the series converges to  $\frac{1}{2}(F(x+0) + F(x-0))$ . If  $F(x)$  does not satisfy the boundary conditions (2), then the series (3) converges to

$$(7) \quad \frac{1}{2} \left[ F(x+0) + F(x-0) - \left\{ F(0^+) + F(1^-) \exp\left(-\int_0^1 p(t)dt\right) \right\} \exp\left(\int_0^x p(t)dt\right) \right].$$

In this paper, we prove:

**THEOREM 2.** *If  $F'(x)$  exists and is of bounded variation for  $0 \leq x \leq 1$ , then a sufficient condition for the series  $\sum_{-\infty}^{\infty} a_n u_n(x)$  to converge uniformly to  $F(x)$  for  $0 < x < 1$  is that  $F(0) = F(1) = 0$ .*

**2. An asymptotic form for  $C'(\lambda_n)$ .** Using (5) and the boundary conditions  $u(0) = u(1) = 0$  and  $u'(0) = u'(1) = 1$ , we have

$$(8) \quad C(\lambda) = e^{-\lambda} u_1(1, \lambda).$$

Then it follows that

$$(9) \quad C'(\lambda) = \frac{d}{d\lambda} C(\lambda) = -C(\lambda) + \frac{1}{2} e^{-\lambda} u_1(1, \lambda) + e^{-\lambda/2} \frac{d}{d\lambda} w_1(x, \lambda) \quad \text{at } x=1$$

where

$$(10) \quad u_1 = e^{\lambda x/2} w_1.$$

Now (10) transforms the equation  $(A + \lambda B)u = 0$  into  $w_1' + \left(q + \lambda p - \frac{\lambda^2}{4}\right)w_1 = 0$ .

It can be verified [2] that  $w_1$  satisfies the equation

$$(11) \quad w_1 = \frac{\sinh R(0, x)}{[r(x)r(0)]^{1/2}} - \int_0^x \frac{\sinh R(\xi, t)}{[r(x)r(\xi)]^{1/2}} g(\xi)w_1(\xi)d\xi$$

where

$$(12) \quad g(x) = p^2 + \frac{p''}{\lambda - 2p} + \frac{3p'^2}{(\lambda - 2p)^2} + q$$

and

$$(13) \quad r(x) = \frac{\lambda}{2} - p(x), \quad R(\xi, x) = \int_\xi^x r(t)dt.$$

We note that  $g(x)$  and  $g'(x) = \frac{d}{d\lambda}g(x)$  are bounded for  $|\lambda|$  sufficiently large.

Also, if in (11) we make the substitution

$$(14) \quad w_1 = \lambda^{-1} \exp\left(\frac{1}{2}|\sigma|x\right)Z_1(x)$$

where  $\sigma = \mathcal{R}\lambda$ , we note that  $Z_1(x)$  is bounded [2] for  $|\lambda|$  sufficiently large. Differentiating (11) with respect to  $\lambda$ , we obtain

$$(15) \quad w_1' = \frac{x \cosh R(0, x)}{2[r(x)r(0)]^{1/2}} - \frac{[r(x) + r(0)] \sinh R(0, x)}{4[r(x)r(0)]^{3/2}} \\ - \int_0^x \frac{\sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g(\xi)w_1'(\xi)d\xi - \int_0^x \frac{\sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g'(\xi)w_1(\xi)d\xi \\ - \int_0^x (x - \xi) \frac{\cosh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g(\xi)w_1(\xi)d\xi \\ + \frac{1}{4} \int_0^x \frac{[r(x) + r(\xi)] \sinh R(\xi, x)}{[r(x)r(\xi)]^{3/2}} g(\xi)w_1(\xi)d\xi.$$

If we substitute

$$(16) \quad w_1 = \lambda^{-1}\rho(x) \exp\left(\frac{1}{2}|\sigma|x\right),$$

we obtain that

$$(17) \quad \rho(x) = \frac{\lambda x \exp\left(-\frac{1}{2}|\sigma|x\right) \cosh R(0, x)}{[r(x)r(0)]^{1/2}} \\ - \frac{\lambda[r(x) + r(0)] \exp\left(-\frac{1}{2}|\sigma|x\right) \sinh R(0, x)}{4[r(x)r(0)]^{3/2}}$$

$$\begin{aligned}
 & - \int_0^x \frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)] \sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g(\xi)\rho(\xi)d\xi \\
 & - \int_0^x \frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)] \sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g'(\xi)Z_1(\xi)d\xi \\
 & - \int_0^x \frac{1}{2}(x-\xi) \frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)] \cosh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g(\xi)Z_1(\xi)d\xi \\
 & + \int_0^x \frac{[r(x)+r(\xi)] \exp[-\frac{1}{2}|\sigma|(x-\xi)] \sinh R(\xi, x)}{4[r(x)r(\xi)]^{3/2}} g(\xi)Z_1(\xi)d\xi
 \end{aligned}$$

where  $g'(\xi) = \frac{d}{d\lambda}g(\xi)$ . Now  $\lambda[r(x)r(\xi)]^{-1/2}$  and hence  $\lambda[r(x)+r(\xi)][r(x)r(\xi)]^{-3/2}$  are both bounded by some constant  $C$  as  $|\lambda| \rightarrow \infty$ . Also,  $\exp[-\frac{1}{2}|\sigma|(x-\xi)] \times \cosh R(\xi, x)$ , and  $\exp[-\frac{1}{2}|\sigma|(x-\xi)] \sinh R(\xi, x)$  are both bounded by some constant  $C$  as  $|\lambda| \rightarrow \infty$  and  $0 \leq \xi \leq x$ . Using these results we obtain from equation (17) that

$$\begin{aligned}
 (18) \quad |\rho(x)| \leq & 2C^2 + \frac{C^2}{|\lambda|} \int_0^x |g(\xi)\rho(\xi)| d\xi + \frac{C^2}{|\lambda|} \int_0^x |g'(\xi)Z_1(\xi)| d\xi \\
 & + \frac{C^2}{|\lambda|} \int_0^x \frac{1}{2}|x-\xi| |g(\xi)Z_1(\xi)| d\xi + \frac{C^2}{|\lambda|} \int_0^x |g(\xi)Z_1(\xi)| d\xi.
 \end{aligned}$$

If we set  $\mu(\lambda)$  equal to the maximum of  $|\rho(x)|$  in  $0 \leq x \leq 1$ , then we certainly have that

$$\mu \leq \frac{2C^2}{1 - \frac{C^2}{|\lambda|} \int_0^x |g(\xi)| d\xi} + \frac{\frac{C^2}{|\lambda|} \int_0^x (|g'(\xi)Z_1(\xi)| + \frac{1}{2}|x-\xi| |g(\xi)Z_1(\xi)| + |g(\xi)Z_1(\xi)|) d\xi}{1 - \frac{C^2}{|\lambda|} \int_0^x |g(\xi)| d\xi}.$$

Therefore,  $\mu$ , and consequently  $\rho(x)$  are bounded as  $|\lambda| \rightarrow \infty$ . Rewrite equation (15) as follows:

$$\begin{aligned}
 (19) \quad w_1(x, \lambda) = & \frac{x \cosh R(0, x)}{2[r(x)r(0)]^{1/2}} - \frac{[r(x)+r(0)] \sinh R(0, x)}{4[r(x)r(0)]^{3/2}} \\
 & - \lambda^{-1} \exp(\frac{1}{2}|\sigma|x) \int_0^x \frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)] \sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g(\xi)\rho(\xi)d\xi \\
 & - \lambda^{-1} \exp(\frac{1}{2}|\sigma|x) \int_0^x \frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)] \sinh R(\xi, x)}{[r(x)r(\xi)]^{1/2}} g(\xi)Z_1(\xi)d\xi \\
 & - \lambda^{-1} \exp(\frac{1}{2}|\sigma|x) \int_0^x (x-\xi) \frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)] \sinh R(\xi, x)}{2[r(x)r(\xi)]^{1/2}} g(\xi)Z_1(\xi)d\xi \\
 & + \lambda^{-1} \exp(\frac{1}{2}|\sigma|x) \int_0^x \frac{\exp[-\frac{1}{2}|\sigma|(x-\xi)][r(x)+r(\xi)] \sinh R(\xi, x)}{4[r(x)r(\xi)]^{3/2}} g(\xi)Z_1(\xi)d\xi.
 \end{aligned}$$

The above four integrals are all at least  $O(\lambda^{-2} \exp[\frac{1}{2}|\sigma|x])$ . Also,

$$[r(x)r(0)]^{-1/2} = \lambda^{-1} + O(\lambda^{-2}),$$

and

$$[r(x)]^{-1/2}[r'(0)]^{-3/2} = \lambda^{-2} + O(\lambda^{-3}).$$

So it follows that

$$(20) \quad w'_1(1, \lambda) = \lambda^{-1} \cosh\left(\int_0^1 r(t) dt\right) + O(\lambda^{-2} \exp[\frac{1}{2}|\sigma|]) - \frac{\lambda^{-2}}{2} \sinh\left(\int_0^1 r(t) dt\right) + O(\lambda^{-3} \exp[\frac{1}{2}|\sigma|]).$$

Using this result for  $w'_1(1, \lambda)$  in equation (9), we have, for  $\mathcal{R}\lambda > 0$ ,

$$(21) \quad C'(\lambda) = \lambda^{-1} \left[ \exp\left(-\lambda + \int_0^1 p(t) dt\right) - \frac{1}{2} \lambda^{-2} \left[ \exp\left(-\int_0^1 p(t) dt\right) - \exp\left(-\lambda + \int_0^1 p(t) dt\right) \right] \right] + O(\lambda^{-3}) + O(\lambda^{-2}),$$

and for  $\mathcal{R}\lambda < 0$ ,

$$(22) \quad C'(\lambda) = \lambda^{-1} \left[ \exp\left(-\lambda + \int_0^1 p(t) dt\right) - \frac{1}{2} \lambda^{-2} \left[ \exp\left(-\int_0^1 p(t) dt\right) - \exp\left(-\lambda + \int_0^1 p(t) dt\right) \right] \right] + O(\lambda^{-3} e^\lambda).$$

**3. Distribution of the eigenvalues.** Since by [2]

$$C(\lambda) = \lambda^{-1} \exp[-\lambda a] \left( \exp\left[-\int_a^b p(t) dt\right] - \exp\left[-\lambda(b-a) + \int_a^b p(t) dt\right] + O(\lambda^{-1}) \right) = \lambda^{-1} \exp[-\lambda a] C_1(\lambda) \quad \text{for } \mathcal{R}\lambda \geq 0,$$

$$C(\lambda) = \lambda^{-1} \exp[-\lambda b] \left( \exp\left[-\lambda(a-b) - \int_a^b p(t) dt\right] - \exp\left[\int_a^b p(t) dt\right] + O(\lambda^{-1}) \right) = \lambda^{-1} \exp[-\lambda b] C_2(\lambda) \quad \text{for } \mathcal{R}\lambda \leq 0,$$

and where  $a$  and  $b$  equal 0 and 1 respectively.

The condition that  $\lambda$  be an eigenvalue is that  $C(\lambda)$  and hence either  $C_1(\lambda)$  or  $C_2(\lambda)$  be zero. Equating  $C_1(\lambda)$  to zero we obtain

$$(23) \quad \exp\left[-\lambda(b-a) + \int_a^b p(t) dt\right] = \exp\left[-\int_a^b p(t) dt\right] + O(\lambda^{-1}) = \exp\left[-\int_a^b p(t) dt\right] (1 + O(\lambda^{-1})).$$

By taking the logarithm of both sides of the above equation (23) and expanding the term  $\log(1 + O(\lambda^{-1}))$  we obtain that the large eigenvalues satisfy the equation

$$-\lambda_n = -2 \int_0^1 p(t)dt + 2n\pi i + O(\lambda_n^{-1}), \quad n = \pm N, \pm N+1, \dots$$

Hence the eigenvalues with positive real parts, if they exist, are given by

$$(24) \quad \lambda_n = 2n\pi i + 2 \int_0^1 p(t)dt + O\left(\frac{1}{n}\right).$$

The equation  $C_2(\lambda) = 0$  leads to the same result for those eigenvalues with negative real parts. Consequently, all the eigenvalues are represented by equation (24).

**4. On the uniform convergence of series (3).** Consider equation (4). In [2] it was shown that

$$(25) \quad B^* V_1(x) = \exp \left[ -\lambda x + \int_0^x p(t)dt \right] + \Omega_1,$$

where

$$\Omega_1 = \begin{cases} O(\lambda^{-2}) + O(\lambda^{-1} \exp[-\lambda x]) & \text{for } \Re \lambda \geq 0 \\ O(\lambda^{-1} \exp[-\lambda x]) & \text{for } \Re \lambda \leq 0. \end{cases}$$

Similarly,

$$(26) \quad B^* V_2(x) = \exp \left[ -\lambda x + \int_1^x p(t)dt \right] + \Omega_2,$$

where

$$\Omega_2 = \begin{cases} O(\lambda^{-1} \exp[-\lambda x]) & \text{for } \Re \lambda \geq 0 \\ O(\lambda^{-1} \exp[-\lambda x]) + O(\lambda^{-1} \exp[-\lambda b]) & \text{for } \Re \lambda \leq 0. \end{cases}$$

Also from [2], we have that

$$(27) \quad u_1(x) = \lambda^{-1} \left\{ \exp \left[ \lambda x - \int_0^x p(t)dt \right] - \exp \int_0^x p(t)dt \right\} + O(\lambda^{-2} e^{\lambda x})$$

and

$$(28) \quad u_2(x) = \lambda^{-1} \left\{ \exp \left[ \lambda(x-1) - \int_1^x p(t)dt \right] - \exp \int_1^x p(t)dt \right\} + O(\lambda^{-2}).$$

Using equations (26) and (27), for  $\Re \lambda > 0$ , we have

$$\begin{aligned}
 a_n u_1 &= u_1 \int_0^1 F(\xi) \frac{B^* V_2(\xi)}{C'(\lambda_n)} d\xi \\
 &= O\left(\frac{e^{\lambda_n x}}{\lambda_n}\right) \int_0^1 \frac{F(\xi) \left[ \exp\left(-\lambda_n \xi + \int_1^\xi p(t) dt\right) \right]}{C'(\lambda_n)} d\xi \\
 &\quad + O(\lambda_n^{-1} e^{\lambda_n x}) \int_0^1 \frac{F(\xi) O(\lambda_n^{-1} e^{-\lambda_n \xi})}{C'(\lambda_n)} d\xi \\
 &= A \int_0^1 \frac{F(\xi) \exp\left(-\lambda_n \xi + \int_1^\xi p(t) dt\right)}{C'(\lambda_n)} d\xi + A \int_0^1 \frac{F(\xi) O(\lambda_n^{-1} e^{-\lambda_n \xi})}{C'(\lambda_n)} d\xi
 \end{aligned}$$

where  $A$  is bounded.

By equation (22),  $C'(\lambda_n) = O(\lambda_n^{-1})$ , therefore  $\frac{1}{C'(\lambda_n)} = O(\lambda_n)$ .

Hence

$$(29) \quad a_n u_1 = B_n \int_0^1 F(\xi) \exp\left(-\lambda_n \xi + \int_0^\xi p(t) dt\right) d\xi + B_n \int_0^1 F(\xi) O(\lambda_n^{-1} e^{-\lambda_n \xi}) d\xi$$

where  $B_n = \lambda^{-1} O(\lambda_n) A$  is also bounded. Using equation (26) for  $\mathcal{R} \lambda > 0$ , and observing that  $O(\lambda^{-1} \exp(-\lambda_n \xi))$  is the indefinite integral of a bounded function, it can be easily shown that

$$(30) \quad \int_0^1 F(\xi) O(\lambda_n^{-1} e^{-\lambda_n \xi}) d\xi = O(\lambda_n^{-2}).$$

Consider now the first integral in equation (29). Setting  $H(\xi) = F(\xi) \exp\left(\int_1^\xi p(t) dt\right)$  and integrating by parts, we obtain

$$\begin{aligned}
 (31) \quad \int_0^1 H(\xi) \exp(-\lambda_n \xi) d\xi &= -\lambda_n^{-1} H(\xi) \exp(-\lambda_n \xi) \Big|_0^1 \\
 &\quad + \int_0^1 \lambda_n^{-1} H'(\xi) \exp(-\lambda_n \xi) d\xi.
 \end{aligned}$$

Since  $F(1) = F(0) = 0$ , then  $H(1) = H(0) = 0$ , and the first term on the right hand side of equation (31) vanishes.

Now

$$(32) \quad H'(\xi) = p(\xi) F(\xi) \exp\left(\int_1^\xi p(t) dt\right) + F'(\xi) \exp\left(\int_1^\xi p(t) dt\right).$$

$F'(\xi)$  is of bounded variation on  $(0, 1)$  and  $p'(\xi)$  is continuous on  $(0, 1)$ . Therefore,  $H'(\xi)$  is of bounded variation on  $(0, 1)$ . Hence,

$$H'(\xi) = \varphi_1(\xi) - \varphi_2(\xi)$$



where  $\varphi_1(\xi)$  and  $\varphi_2(\xi)$  are two bounded, positive, monotone functions, either both nonincreasing or both nondecreasing. Now  $\lambda_n^{-1} \exp(-\lambda_n \xi)$  is bounded and integrable for  $0 \leq \xi \leq 1$ . Assume  $\varphi_1(\xi)$  to be a monotone decreasing function, then

$$(33) \quad \int_0^1 \lambda_n^{-1} H'(\xi) \exp(-\lambda_n \xi) d\xi \\ = \varphi_1(0) \int_0^{\xi_0} \lambda_n^{-1} \exp(-\lambda_n \xi) d\xi - \varphi_2(0) \int_0^{\xi_1} \lambda_n^{-1} \exp(-\lambda_n \xi) d\xi = O(\lambda_n^{-2})$$

where  $\xi_0$  and  $\xi_1$  are on the interval  $(0,1)$ .

Combining the results of (30) and (33) we have

$$\sum_{-\infty}^{\infty} a_n u_n(x) = \sum_{-\infty}^{-(N+1)} O(\lambda_n^{-2}) + \sum_{-N}^{N-1} a_n u_n(x) + \sum_N^{\infty} O(\lambda_n^{-2})$$

where  $\sum_{-N}^{N-1} a_n u_n(x)$  is finite for  $0 < x < 1$ . From (24) it is clear that  $\lambda_n = O(n)$  for  $n = \pm N, \pm N + 1, \dots$  Therefore

$$\sum_{-\infty}^{\infty} a_n u_n(x) = \sum_{-\infty}^{-(N+1)} \frac{O(1)}{n^2} + \sum_{-N}^{N-1} a_n u_n(x) + \sum_N^{\infty} \frac{O(1)}{n^2}$$

where  $O(1)$  is a bounded function.

Since  $\left| \frac{O(1)}{n^2} \right| \leq \frac{M}{n^2}$ ,  $M > 0$  and the series  $M \sum_N^{\infty} \frac{1}{n^2}$  converges, it is clear that  $\sum_{-\infty}^{\infty} a_n u_n(x)$  converges uniformly to  $F(x)$  for  $0 < x < 1$ . And our theorem is proved.

We note, however, that while Theorem 2 is sufficient, it is not a necessary condition for uniform convergence. For suppose  $F(0)$  and  $F(1)$  differ from zero, then by equations (31) and (33) we have  $\sum_{-\infty}^{\infty} a_n u_n = \sum_{-\infty}^{\infty} O\left(\frac{1}{n}\right)$  which may or may not converge uniformly.

In fact, a necessary and sufficient condition for the uniform convergence of this series does not seem to be known.

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