

Pacific Journal of Mathematics

ON THE SPECTRA OF LINKED OPERATORS

CHARLES JOHN AUGUST HALBERG, JR. AND ANGUS E. TAYLOR

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1. Introduction. Let X, Y be complex linear spaces, and Z a non-void complex linear space contained in both X and Y . Let X be a Banach space X_1 , Y a Banach space Y_2 under the norms n_1, n_2 respectively. Let Z be a Banach space Z_N under the norm N defined by $N(z) = \max [n_1(z), n_2(z)]$. (This is equivalent to saying that if $\{z_n\}$ is any sequence with $z_n \in Z$, such that $z_n \rightarrow x$ in the topology of X_1 and $z_n \rightarrow y$ in the topology of Y_2 , then $x = y \in Z$. Our particular method of stating this here will be useful for later purposes.) With the usual uniform norms let T_1, T_2 be bounded distributive operators on X_1, Y_2 respectively, such that $T_1 z = T_2 z \in Z$ when $z \in Z$. Operators satisfying these conditions will be said to be "linked". If, in addition, it is assumed that Z is dense in X_1 , T_1 and T_2 will be said to be "linked densely relative to X_1 ".

We are interested in relationships between the spectra of linked operators. That there are linked, and densely linked operators with different spectra will be shown in § 3. The main result of this paper is the demonstration that, if T_1 and T_2 are linked densely relative to X_1 , under certain circumstances any component of the spectrum of T_1 has a non-void intersection with the spectrum of T_2 . Sufficient conditions are that if λ belongs to the intersection of the resolvent sets of T_1 and T_2 and $z \in Z$, then $(\lambda I - T_1)^{-1} z = (\lambda I - T_2)^{-1} z \in Z$. With this result we obtain some interesting consequences in the special case where the Banach spaces considered are the sequence spaces l_p .

2. Preliminary definitions and notation. Supposing X to be a complex linear space such that under a norm n_a , ($x \in X, n_a(x) = \|x\|_a$), X becomes a complex Banach space X_a , we let $[X_a]$ denote the set of all operators T that are bounded under the induced norm

$$\|T\|_a = \sup \|Tx\|_a \text{ (for all } x \in X_a, \|x\|_a = 1).$$

Such a T will be denoted by T_a when considered as an element of the algebra $[X_a]$. If $T_a \in [X_a]$ we classify all complex numbers into two sets:

(1) The resolvent set $\rho(T_a)$, consisting of all λ such that $\lambda I - T_a$ defines a one-to-one correspondence of X_a onto X_a .

(2) The spectrum $\sigma(T_a)$, consisting of all λ not in $\rho(T_a)$.

The spectrum is divided into three parts:

- (1) The point spectrum $p(T_a)$, consisting of those λ for which $(\lambda I - T_a)^{-1}$ does not exist.
- (2) The continuous spectrum $c(T_a)$, consisting of those λ not in $\rho(T_a)$ or $p(T_a)$ for which the range of $\lambda I - T_a$ is dense in X_a ; and
- (3) The residual spectrum $r(T_a)$, consisting of those λ not in $\rho(T_a)$, $p(T_a)$ or $c(T_a)$.

We shall also have occasion to refer to the so-called "approximate point spectrum," consisting of those λ for which $(\lambda I - T_a)^{-1}$ is not bounded. It is well known that $\sigma(T_a)$ is closed, bounded and nonempty. It is also well known that $R_\lambda(T_a) \equiv (\lambda I - T_a)^{-1}$ is analytic in $\rho(T_a)$ as a function with values in $[X_a]$.

3. An example of linked operators with different spectra. Consider the well known sequence spaces l_1 and l_2 . Let T_1 and T_2 be the operation defined as elements of $[l_1]$ and $[l_2]$ respectively by the infinite matrix (t_{ij})

$$t_{ij} = \begin{cases} \frac{j}{(i-1)i} & \text{if } i > j \\ 0 & \text{if } i \leq j. \end{cases}$$

The uniform norm for the operator T defined by such a matrix, when considered as an operator on l_1 , can be shown to be the supremum of the l_1 norms of the column sequences of the matrix (t_{ij}) :

$$\|T_1\|_1 = \sup_j \sum_{i=1}^{\infty} |t_{ij}|$$

([1, pp. 696-697]). From this it is easy to see that $\|T_1\|_1 = 1$. In fact

$$\sum_{i=1}^{\infty} |t_{ij}| = j \sum_{i=j+1}^{\infty} \frac{1}{(i-1)i} = j \cdot \frac{1}{j} = 1,$$

the sum being independent of j . Next, considering the powers of T :

$$T^n = (t_{ij})^n = (t_{ij}^{(n)}),$$

we see that

$$\|T_1^n\|_1 \equiv \sup_j \sum_{i=1}^{\infty} |t_{ij}^{(2)}| = \sup_j \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |t_{ik} t_{kj}| = \sup_j \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} |t_{ik}| \right) t_{kj} = \sup_j \sum_{k=1}^{\infty} t_{kj} = 1.$$

By induction it is easy to show that $\sum_{i=1}^{\infty} |t_{ij}^{(n)}| = 1$ for any j , and hence $\|T_1^n\|_1 = 1$. Now it is well known that the spectral radius of T_1 ,

$$|\sigma(T_1)| \equiv \sup |\lambda|, \text{ (for } \lambda \in \sigma(T_1)\text{)},$$

is given by the formula

$$|\sigma(T_1)| = \lim_{n \rightarrow \infty} (\|T_1^n\|_1)^{1/n};$$

hence the spectral radius of T_1 is 1.

On the other hand, by making use of an inequality due to Schur [2, p. 6], we can estimate the norm of T as an operator on l_2 :

$$\|T_2\|_2 \leq [(\sup_i \sum_{j=1}^{\infty} |t_{ij}|)(\sup_j \sum_{i=1}^{\infty} |t_{ij}|)]^{\frac{1}{2}}.$$

In this way we see that

$$\|T_2\|_2 \leq \sqrt{1 \cdot \frac{1}{2}} < 1,$$

since

$$\sum_{j=1}^{\infty} |t_{ij}| = \sum_{j=1}^{i-1} \frac{j}{(i-1)i} = \frac{1}{2},$$

the sum being independent of i . Since it is always true that $|\sigma(T_2)| \leq \|T_2\|_2$, it is now clear that $|\sigma(T_2)| < |\sigma(T_1)|$, whence we immediately infer that there exists a λ such that $\lambda \in \sigma(T_1)$ and $\lambda \notin \sigma(T_2)$.

4. The projection corresponding to a spectral set. For the proof of our main theorem we need the concepts of spectral set and the projection associated with a spectral set. For this purpose we introduce the following definitions.

Suppose X is a complex Banach space, and T an element of $[X]$. A set σ in the complex plane is called a spectral set of T if $\sigma \subset \sigma(T)$ and if σ is both open and closed in the relative topology of $\sigma(T)$.

If σ is a spectral set of T , the corresponding projection is the operator defined by

$$E_\sigma[T] = \frac{1}{2\pi i} \int R_\lambda(T) d\lambda,$$

the integral being extended in the positive sense around the boundary of a suitable bounded open set D such that $\sigma \subset D$ and the closure of D does not intersect the rest of $\sigma(T)$. It is easy to see that if Δ is a closed set which does not intersect σ , the set D may be chosen to satisfy the additional requirement that its closure does not intersect Δ .

We now proceed to the proof of our main theorem.

5. Relations between the spectra of linked operators. Let X and Y be complex linear spaces such that X becomes a Banach space X_1 and Y becomes a Banach space Y_2 under the norms n_1 and n_2 , respectively.

THEOREM. *Let $T_1 \in [X_1]$ and $T_2 \in [Y_2]$ be linked densely relative to X_1 and let $Z \subset X \cap Y$ be a complex linear space that becomes a Banach space Z_N under the norm N defined by $N(z) = \max [n_1(z), n_2(z)]$. Let $R_\lambda(T_1)z = R_\lambda(T_2)z \in Z$ for every $z \in Z$, provided that $\lambda \in \rho(T_1) \cap \rho(T_2)$. Then if C is any component of $\sigma(T_1)$, $C \cap \sigma(T_2)$ is non-void.*

Proof. We shall first prove that if σ is any non-void spectral set of $\sigma(T_1)$, then $\sigma \cap \sigma(T_2)$ is non-void.

Suppose that $\sigma \cap \sigma(T_2)$ is void. Let $E_\sigma[T_1]$ be the projection in $[X_1]$ associated with σ , that is

$$E_\sigma[T_1] = \frac{1}{2\pi i} \int_{+B(D)} R_\lambda(T_1) d\lambda,$$

where $B(D)$ is the boundary of a bounded Cauchy domain such that $\sigma \subset D$ while the closure of D intersects neither $\sigma(T_2)$ nor the rest of $\sigma(T_1)$. We know that $E_\sigma[T_1] \neq 0$ by a theorem [3, p. 210] which states that the spectral set σ is empty if and only if $E_\sigma[T_1] = 0$. Now consider the operator (an element of $[Y_2]$)

$$F = \frac{1}{2\pi i} \int_{+B(D)} R_\lambda(T_2) d\lambda.$$

Since D and $B(D)$ lie in $\rho(T_2)$, $R_\lambda(T_2)$ is analytic inside and on $B(D)$; therefore the integral defining F is the zero element of $[Y_2]$, by Cauchy's theorem.

If $\lambda \in \rho(T_1) \cap \rho(T_2)$, then by hypothesis $R_\lambda(T_1)z = R_\lambda(T_2)z$ for $z \in Z$, and from this we see that

$$Fz = E_\sigma[T_1]z \text{ for } z \in Z,$$

since the integrals defining Fz and $E_\sigma[T_1]z$ can be regarded as limits, in Y_2 and X_1 respectively, of the same sequence in Z . However, since $E_\sigma[T_1] \neq 0$ and is continuous, and Z is dense in X_1 , there exists a z , $z \in Z$, such that $E_\sigma[T_1]z \neq 0$. But $Fz = 0$, which is a contradiction. Thus any non-void spectral set of $\sigma(T_1)$ has a non-void intersection with $\sigma(T_2)$.

Let C be any component of $\sigma(T_1)$. To show that $C \cap \sigma(T_2)$ is non-void we will need the following theorem [4, p. 15]: *If A and B are disjoint closed subsets of a compact set K such that no component of K intersects both A and B , there exists a separation $K = K_1 \cup K_2$, where K_1 and K_2 are disjoint compact sets containing A and B respectively.* Now suppose that $C \cap (\sigma(T_1) \cap \sigma(T_2))$ is void. Then, since C and $\sigma(T_1) \cap \sigma(T_2)$ are non-void disjoint closed subsets in $\sigma(T_1)$ and as the only component of $\sigma(T_1)$ intersecting C is C itself, we have $\sigma(T_1) = K_1 \cup K_2$, where $K_1 \supset C$, $K_2 \supset \sigma(T_1) \cap \sigma(T_2)$, and K_1, K_2 are disjoint compact sets. But K_1 is closed, being compact, and also relatively open, since it is the rela-

tive complement of the closed set K_2 . Thus K_1 is a spectral set of $\sigma(T_1)$, and $K_1 \cap (\sigma(T_1) \cap \sigma(T_2))$ is void, which is in contradiction to what we have shown above. Thus if C is any component of $\sigma(T_1)$, then $C \cap \sigma(T_2)$ is non-void, as was to be proved.

We note that if in the hypotheses of the theorem we only require T_2 to be a closed distributive operator on Y_2 , such that $\sigma(T_2)$ is non-void, the conclusion and proof of the theorem will be unaltered. Also, if we replace the hypotheses that $T_1 \in [X_1]$ and $T_2 \in [Y_2]$ by “ T_1 and T_2 are closed distributive operators on X_1 and Y_2 respectively, such that $\sigma(T_2)$ is nonvoid”, and retain the remaining hypotheses, we can conclude, using the same reasoning as before, that any non-void bounded spectral set of $\sigma(T_1)$ has a non-void intersection with $\sigma(T_2)$.

A very special case of our theorem, but one of considerable practical importance, is given in the following corollary.

COROLLARY 1. *In addition to the hypotheses of the preceding theorem let Z be dense in Y_2 , and let $\sigma(T_1)$ and $\sigma(T_2)$ be such that all of their components are single points. Then $\sigma(T_1) = \sigma(T_2)$.*

In the special case where $X \subset Y$, the operators $T_1 \in [X_1]$, $T_2 \in [Y_2]$ are linked, and X plays the role of Z , we have the following two corollaries.

COROLLARY 2. *If C is any component of $\sigma(T_1)$, then $C \cap \sigma(T_2)$ is non-void.*

Proof. This follows from the theorem, since if $\lambda \in \rho(T_1) \cap \rho(T_2)$, then $R_\lambda(T_1)x = R_\lambda(T_2)x$ for $x \in X$.

COROLLARY 3. *If T_1 and T_2 are linked densely relative to Y_2 and C is any component of $\sigma(T_2)$, then $C \cap \sigma(T_1)$ is non-void.*

This should be clear from the proof of the theorem in view of the remark following the statement of Corollary 2.

DEFINITION. *If A, B, C are sets such that any component of C has a non-void intersection with both A and B we shall say that A and B are “linked by C ”. If in addition every component of A has a non-void intersection with C we shall say that A is “totally linked to B by C ”.*

Now suppose that neither X nor Y is necessarily contained in the other and let $T \in [Z_N]$ be the operator defined by $Tz = T_1z$ for $z \in Z$. Then we have the following results for $T_1 \in [X_1]$ and $T_2 \in [Y_2]$.

COROLLARY 4. *If T_1 and T_2 are linked (not necessarily densely linked), then $\sigma(T_1)$ and $\sigma(T_2)$ are linked by $\sigma(T)$.*

This follows immediately from Corollary 2.

COROLLARY 5. *If T_1 and T_2 are linked densely relative to X_1 , then $\sigma(T_1)$ is totally linked to $\sigma(T_2)$ by $\sigma(T)$.*

This follows from Corollary 3.

COROLLARY 6. *If T_1 and T_2 are linked, then*

$$\sigma(T) - (\sigma(T_1) \cup \sigma(T_2))$$

is contained in that portion of the residual spectrum of T for which $(\lambda I - T)^{-1}$ is bounded.

Proof. Clearly $p(T)$ is contained in both $p(T_1)$ and $p(T_2)$. If λ belongs to the approximate point spectrum of T then there exists a sequence $\{z_n\}$, $z_n \in Z$, such that

$$\lim_{n \rightarrow \infty} \|(\lambda I - T)z_n\|_N = 0 \text{ and } \|z_n\|_N = 1.$$

But either 1°: Infinitely many z_n are such that $\|z_n\|_{n_1} = 1$, or 2°: Infinitely many z_n are such that $\|z_n\|_{n_2} = 1$. If 1° holds there exists a subsequence $\{x_n\}$ of $\{z_n\}$ such that

$$\lim_{n \rightarrow \infty} \|(\lambda I - T)x_n\|_{n_1} = 0 \text{ and } \|x_n\|_{n_1} = 1,$$

and thus λ belongs to the approximate point spectrum of T_1 . If 2° holds similar reasoning shows that λ belongs to the approximate point spectrum of T_2 . From these results it follows that the only possibility for an element λ , $\lambda \in \sigma(T)$, to be such that $\lambda \notin \sigma(T_1) \cup \sigma(T_2)$ is for λ to be an element of the residual spectrum of T with $(\lambda I - T)^{-1}$ bounded.

The following is a corollary concerning the sequence spaces l_n , which we considered earlier.

COROLLARY 7. *Suppose that $1 \leq r < s$, and suppose that the infinite matrix (t_{ij}) defines operators T_r and T_s on l_r and l_s , respectively, such that $T_r \in [l_r]$ and $T_s \in [l_s]$. Then $C \cap \sigma(T_s)$ is non-void for any component C of $\sigma(T_r)$. Moreover, $C \cap \sigma(T_r)$ is non-void for any component C of $\sigma(T_s)$.*

Proof. These are special cases of Corollaries 2 and 3, for it is well known that, for the classes l_r and l_s , we have $l_r \subset l_s$; that $\|x\|_s \leq \|x\|_r$ for $x \in l_r$; and that l_r is dense in l_s .

Corollary 7 is true even if $s = \infty$. (We recall that l_∞ is the set of all sequences $x = \{\xi_i\}$ such that $\sup |\xi_i| < \infty$, and such that if $x \in l_\infty$, $\|x\|_\infty = \sup |\xi_i|$.) For, although in this case it is not true that l_r is dense in l_∞ , the following is true: if an element of $[l_\infty]$ is defined by an infinite matrix, and if the operator is 0 when restricted to l_r , then it is the zero operator on l_∞ . The reasoning of the main theorem now applies with only slight modifications for the case in which $X_1 = l_\infty$, $Y_2 = l_r (1 \leq r < \infty)$, $Z = l_r$ and T_1 and T_2 are defined by the same matrix.

Before stating the final corollary we recall the following facts.

If $1 < p \leq \infty$ and $1/p + 1/p' = 1$ (with $p' = 1$ if $p = \infty$), we can identify the conjugate space $(l_{p'})^*$ with l_p . If (t_{ij}) is an infinite matrix defining a bounded linear operator T on $l_{p'}$, we can identify the adjoint operator T^* with the bounded linear operator T^t defined on l_p by the transposed matrix (t_{ij}^t) , where $t_{ij}^t = t_{ji}$. Since $\sigma(T) = \sigma(T^*)$, as is well known [5, pp. 304 and 306], we have $\sigma(T_{p'}) = \sigma(T_p^t)$, where the subscripts serve to remind us on what space the operator is defined.

COROLLARY 8. *Suppose the matrix (t_{ij}) defines $T_p \in [l_p]$ and $T_{p'} \in [l_{p'}]$, where $1 < p \leq \infty$. Then $C \cap \sigma(T_{p'})$ is non-void for any component C of $\sigma(T_p)$, and $C \cap \sigma(T_p)$ is non-void for any component C of $\sigma(T_{p'}^t)$.*

Proof. This follows from Corollary 7 and the foregoing remarks, by taking p and p' to be r and s or s and r , depending on whether $p \leq 2$ or $2 < p$.

6. Further comments. The referee made some suggestions concerning the condition which was imposed in the main theorem of § 5, namely that

$$(R) \quad R_\lambda(T_1)z = R_\lambda(T_2)z \in Z \text{ if } z \in Z \text{ and } \lambda \in \rho(T_1) \cap \rho(T_2).$$

We shall refer to this as Condition (R). We add some discussion of this condition, guided in part by the suggestions of the referee.

As in § 5, let us denote by T the member of $[Z_N]$ defined by $Tz = T_1z = T_2z$ when $z \in Z$. It is then easy to see that $R_\lambda(T)z = R_\lambda(T_k)z \in Z$ if $z \in Z$ and $\lambda \in \rho(T) \cap \rho(T_k)$, $k = 1, 2$. Consequently $R_\lambda(T_1)z = R_\lambda(T_2)z \in Z$ if $z \in \rho(T) \cap \rho(T_1) \cap \rho(T_2)$. The intersection of these three resolvent sets certainly contains all sufficiently large values of λ . Now let D be the set of those $\lambda \in \rho(T_1) \cap \rho(T_2)$ for which $R_\lambda(T_1)z = R_\lambda(T_2)z \in Z$ if $z \in Z$. This set is evidently closed relative to $\rho(T_1) \cap \rho(T_2)$ (by the continuity of the resolvents and the way in which the norm of Z is defined). It is also open relative to $\rho(T_1) \cap \rho(T_2)$, as we may see by using the expansion

$$R_\lambda = \sum_{n=0}^{\infty} (\mu - \lambda)^n R_\mu^{n+1}$$

for the resolvent of an operator in the neighborhood of a point μ in the resolvent set. Consequently D contains all of any particular component of $\rho(T_1) \cap \rho(T_2)$ if it contains any point of that component. In particular D contains all of the unbounded component of $\rho(T_1) \cap \rho(T_2)$. This shows that we can omit the Condition (R) if $\rho(T_1) \cap \rho(T_2)$ has only one component. In particular this will be true if $\sigma(T_1)$ and $\sigma(T_2)$ are totally disconnected. From what was said previously it is clear that $\rho(T_1) \cap \rho(T_2) - D$ lies in $\sigma(T) - (\sigma(T_1) \cup \sigma(T_2))$, and hence, by Corollary 6, in that part of $\sigma(T)$ for which $(\lambda I - T)^{-1}$ exists and is bounded. It is not very difficult to prove that a point of this latter kind is not in the closure of $\rho(T)$. (The argument uses the functional equation of the resolvent, $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$, to show that if $\alpha \in \overline{\rho(T)}$ then $\lim_{\lambda \rightarrow \alpha} R_\lambda$ exists and is necessarily R_α .) Consequently we see that D contains the set $\overline{\rho(T)} \cap \rho(T_1) \cap \rho(T_2)$. This shows, for example, that Condition (R) is superfluous if $\rho(T)$ is everywhere dense in the plane.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, c/o University of California Press, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

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