DIFFERENTIABLE POINTS OF ARCS IN CONFORMAL n-SPACE

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Introduction. This paper is a generalization to n dimensions of the classification of the differentiable points in the conformal plane [2], and in conformal 3-space [3]. In the present paper, this classification depends on the intersection and support properties of certain families of tangent \((n-1)\)-spheres, and on the nature of the osculating \(m\)-spheres at such a point \((m=1, 2, \cdots, n-1)\).

The discussion is also related to the classification [4] of the differentiable points of arcs in projective \((n+1)\)-space, since conformal \(n\)-space can be represented on the surface of an \(n\)-sphere in projective \((n+1)\)-space.

1. Pencils of \(m\)-spheres. \(p, t, P, P_1, \cdots,\) will denote points of conformal \(n\)-space and \(S^{(m)}\) will denote an \(m\)-sphere. When there is no ambiguity, the superscript \((n-1)\) will be omitted in the case of \(S^{(n-1)}\); thus an \((n-1)\)-sphere \(S^{(n-1)}\) will usually be denoted by \(S\) alone. Such an \((n-1)\)-sphere \(S\) decomposes the \(n\)-space into two open regions, its interior \(S_i\), and its exterior \(S_e\). If \(P \not\in S\), the interior of \(S\) may be defined as the set of all points which do not lie on \(S\) and which are not separated from \(P\) by \(S\); the exterior of \(S\) is then defined as the set of all points which are separated from \(P\) by \(S\). An \(m\)-sphere through an \((m-1)\)-sphere \(S^{(m-1)}\) and a point \(P \not\in S^{(m-1)}\) will be denoted by \(S^{(m)}(P; S^{(m-1)})\). The \(m\)-sphere through \((m+2)\)-points \(P_0, P_1, \cdots, P_{m+1}\), not all lying on the same \((m-1)\)-sphere, will occasionally be denoted by \(S^{(m)}(P_0, P_1, \cdots, P_{m+1})\). Such a set of points is said to be independent. Most of the following discussion will involve the use of pencils \(\pi^{(m)}\) of \(m\)-spheres determined by certain incidence and tangency conditions. An \((m-1)\)-sphere which is common to all the \(m\)-spheres of a pencil \(\pi^{(m)}\) is called fundamental \((m-1)\)-sphere of \(\pi^{(m)}\). In the pencil \(\pi^{(m)}\) through a fundamental \((m-1)\)-sphere \(S^{(m-1)}\) there is one and only one \(m\)-sphere \(S^{(m)}(P, \pi^{(m)})\) of \(\pi^{(m)}\) through each point \(P\) which does not lie on \(S^{(m-1)}\). Similarly, in the pencil \(\pi^{(m)}\) of all the \(m\)-spheres which touch a given \(m\)-sphere at a given point \(Q\), there is one and only one \(m\)-sphere \(S^{(m)}(P, \pi^{(m)})\) through each point \(P \not\in Q\). The fundamental point \(Q\) is regarded as a point \(m\)-sphere belonging to \(\pi^{(m)}\).

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2. Convergence. We call a sequence of points $P_1, P_2, \ldots$, convergent to $P$ if to every $(n-1)$-sphere $S$ with $P \subset S$, there corresponds a positive integer $N = N(S)$ such that $P \subset S$ if $\lambda > N$. We define the convergence of $m$-spheres to a point in a similar fashion.

We call a sequence of $(n-1)$-spheres $S_1, S_2, \ldots$, convergent to $S$ if to every pair of points $P \subset S$ and $Q \subset S$ there corresponds a positive integer $N = N(P, Q)$ such that $P \subset S$ and $Q \subset S$ for every $\lambda > N$.

Finally, a sequence of $m$-spheres $S_1^{(m)}, S_2^{(m)}, \ldots$, will be called convergent to an $m$-sphere $S^{(m)}$ if to every $S^{(n-m-1)}$ which links $P$ with $S^{(m)}$ there exists a positive integer $N = N(S^{(n-m-1)})$ such that $S^{(m)}$ links with $S^{(n-m-1)}$ whenever $\lambda > N$, $(m=1, 2, \ldots, n-2)$.

3. Arcs. An arc $A$ is the continuous image of a real interval. The images of distinct points of this parameter interval are considered to be different points of $A$ even though they may coincide in space. The notation $t \neq p$ will indicate that the points $t$ and $p$ do not coincide. If a sequence of points of the parameter interval converges to a point $p$, we define the corresponding sequence of image points on the arc $A$ to be convergent to the image of $p$. We shall use the same small italics $p, t, \ldots$, to denote both the points of the parameter interval and their image points on $A$. The end- (interior) points of $A$ are the images of the end- (interior) points of the parameter interval. A neighbourhood of $p$ on $A$ is the image of a neighbourhood of the parameter on the parameter interval. If $p$ is an interior point of $A$, this neighbourhood is decomposed by $p$ into two (open) one-sided neighbourhoods.

4. Differentiability. Let $p$ be a fixed point of an arc $A$, and let $t$ be a variable point of $A$. Let $1 \leq m < n$. If $p, P_1, \ldots, P_{m+1}$ do not lie on the same $(m-1)$-sphere, then there exists a unique $m$-sphere $S^{(m)}(P_1, \ldots, P_{m+1}, p)$ through these points. It is convenient to denote this $m$-sphere by the symbol $S^{(m)}_0(P_1, \ldots, P_{m+1}; \tau_0)$; here $\tau_0$ indicates that this $m$-sphere passes through $p$. In the following, the $m$-sphere $S^{(m)}(P_1, \ldots, P_{m+1}; \tau_r)$ is defined inductively by means of the conditions $\Gamma^{(m)}_r$ given below (the $\tau_r$ in the symbol $S^{(m)}(P_1, \ldots, P_{m+1}; \tau_r)$ indicates that this sphere is a tangent sphere of the arc $A$ at the point $p$ meeting $A$ $(r+1)$-times at $p$). We call $A$ $(m+1)$-times differentiable at $p$ if the following sequence of conditions is satisfied.

$\Gamma^{(m)}_r[r=1, 2, \ldots, m+1]$: If the parameter $t$ is sufficiently close to, but different from, the parameter $p$, then the $m$-sphere $S^{(m)}(P_1, \ldots, P_{m+1}; t; \tau_{r-1})$ is uniquely defined. It converges if $t$ tends to $p$. Thus its limit sphere, which will be denoted by
will be independent of the way $t$ converges to $p$ [condition $\Gamma_{m+1}^j$ reads: $S_{m+1}^{(m)}(t; \tau_m)$ exists and converges to $S_{m+1}^{(m)}(\tau_{m+1})$].

It is convenient to use the symbols $S_0^{(j)}$ to denote pairs of points $P, p$, and $S_0^{(j)}$ to denote the point pair $p, p$ (or the point $p$).

We call $A$ once differentiable at $p$ if $\Gamma_{1}^{(m)}$ is satisfied. The point $p$ is called a differentiable point of $A$ if $A$ is $n$-times differentiable at $p$.

Let $\tau_{r}^{(m)}$ denote the family of all the $S_r^{(m)}$'s. Thus $\tau_{r}^{(m)}$ consists only of $S_{m+1}^{(r)}$, the osculating $m$-sphere of $A$ at $p$.

5. The structure of the families $\tau_{r}^{(m)}$ of $m$-spheres $S_r^{(m)}$ through $p$.

**Theorem 1.** Suppose $A$ satisfies condition $\Gamma_{1}^{(m)}$ at $p$. Let $S_{m+1}^{(m-1)}$ be any $(m-1)$-sphere. Then there is a neighbourhood $N$ of $p$ on $A$ such that if $t \in N, t \neq p$, then $t \not\subset S_{m+1}^{(m-1)}, (m=1, 2, \ldots, n-1)$.

**Proof.** The assertion is evidently true if $p \not\subset S_{m+1}^{(m-1)}$. Suppose $p \subset S_{m+1}^{(m-1)}$. Choose points $P_1, \ldots, P_m$ on $S_{m+1}^{(m-1)}$ such that $p, P_1, \ldots, P_m$ are independent. If the parameter $t$ is sufficiently close to, but different from, the parameter $p$, condition $\Gamma_{1}^{(m)}$ implies that $S_{m+1}^{(m)}(P_1, \ldots, P_m; t; \tau_0)$ is uniquely defined. Thus $t \not\subset S_{m+1}^{(m-1)}(P_1, \ldots, P_m; \tau_0)=S_{m+1}^{(m-1)}$.

**Corollary.** If $A$ satisfies condition $\Gamma_{1}^{(m)}$ at $p$, and $S^{(k)}$ is any $k$-sphere, then $t \not\subset S^{(k)}$ when the parameter $t$ is sufficiently close to, but different from, the parameter $p$ ($k=0, 1, \ldots, m-1$).

In particular, this holds when $m=n-1$.

**Theorem 2.** Let $1 \leq m \leq n$; $1 \leq k \leq m$. If $A$ satisfies $\Gamma_{1}^{(m)}, \ldots, \Gamma_{k}^{(m)}$ at $p$, then $\Gamma_{1}^{(m-k)}, \ldots, \Gamma_{k}^{(m-k)}$ will hold there and

$$S_{m-k}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k) = \prod_{r=1}^{k} S_{m-r}^{(m)}(P_1, \ldots, P_{m-r}, P_r).$$

Conversely, let $A$ satisfy $\Gamma_{1}^{(m-k)}, \ldots, \Gamma_{k}^{(m-k)}$ at $p$, and let $S_{m-k}^{(m-k)} \not\subset p$ if $k=m$. If $P_{m-k+1} \not\subset S_{m-k}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k)$, then $\Gamma_{k}^{(m)}$ will hold for the points $P_1, \ldots, P_{m-k+1}$ and

$$S_{m}^{(m)}(P_1, \ldots, P_{m-k+1}; \tau_k) = S_{m}^{(m)}[P_{m-k+1}; S_{m-1}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k)]$$

$$r=1, \ldots, k).$$

**Remark.** In general, $\Gamma_{1}^{(m-k)}, \ldots, \Gamma_{k}^{(m-k)}$ do not imply $\Gamma_{1}^{(m)}, \ldots, \Gamma_{k}^{(m)}$ (see [3], §7).

**Proof.** (by induction with respect to $k$): Suppose $k=1$; $1 \leq m \leq n$. 


Let $\Gamma_{[m]}$ hold. If $P_1, \ldots, P_{m-1}, P, p$ are independent points, $S^{(m)}(P_1, \ldots, P_{m-1}, t; \tau_0)$ exists when $t$ is sufficiently close to $p$, $t \neq p$, $t \in A$. Thus $P_1, \ldots, P_{m-1}, P, t, p$ are also independent, $S^{(m-1)}(P_1, \ldots, P_{m-1}, t; \tau_0)$ exists, and

$$S^{(m-1)}(P_1, \ldots, P_{m-1}, t; \tau_0) = \prod_P S^{(m)}(P_1, \ldots, P_{m-1}, P, t; \tau_0).$$

If $t \to p$, $S^{(m)}(P_1, \ldots, P_{m-1}, P, t; \tau_0)$ converges, and hence $S^{(m-1)}(P_1, \ldots, P_{m-1}, t; \tau_0)$ also converges, $\Gamma_{[m]}$ is satisfied, and

$$S^{(m-1)}(P_1, \ldots, P_{m-1}; \tau_0) = \prod_P S^{(m)}(P_1, \ldots, P_{m-1}; P; \tau_0).$$

Next, suppose that $\Gamma_{[m]}$ is satisfied, and $P_m \not\subset S^{(m-1)}(P_1, \ldots, P_{m-1}; \tau_0)$. Then $P_m \not\subset S^{(m-1)}(P_1, \ldots, P_{m-1}, t; \tau_0)$ when $t$ is sufficiently close to $p$, $t \in A$, $t \neq p$, and

$$S^{(m)}(P_1, \ldots, P_m, t; \tau_0) = S^{(m)}[P_m, S^{(m-1)}(P_1, \ldots, P_{m-1}, t; \tau_0)]$$

exists. Hence when $t \to p$, $S^{(m)}(P_1, \ldots, P_m, t; \tau_0)$ converges, $\Gamma_{[m]}$ is satisfied relative to the points $P_1, \ldots, P_m$, and

$$S^{(m)}(P_1, \ldots, P_m; \tau_0) = S^{(m)}[P_m; S^{(m-1)}(P_1, \ldots, P_{m-1}; \tau_0)].$$

Thus Theorem 2 is satisfied when $h=1$.

Assume that Theorem 2 holds when $k$ is replaced by $1, 2, \ldots, h$, where $1 \leq h < k \leq m$.

Let $\Gamma_{[m]}$, $\Gamma_{[h+1]}$ hold. Then $S^{(m)}(P_1, \ldots, P_{m-h-1}, P, t; \tau_h)$ exists when $t$ is sufficiently close to $p$, $t \neq p$, $t \in A$. Now $\Gamma_{[m]}$, $\ldots, \Gamma_{[h]}$ imply $\Gamma_{[m-1]}$, $\ldots, \Gamma_{[h-1]}$. If $h=m-1$, $\Gamma_{[h]} = \Gamma_{[m-1]}$ implies that $S^{(m-1)}(t; \tau_{m-1})$ exists, if $t \neq p$. If $h < m-1$, $\Gamma_{[m-1]}, \ldots, \Gamma_{[h]}$ imply $\Gamma_{[m-2]}$, $\ldots, \Gamma_{[h-2]}$. Thus $S^{(m-2)}(P_1, \ldots, P_{m-h-1}; \tau_h)$ exists. Furthermore, $\Gamma_{[m-1]}$ and Theorem 1 imply that $t \not\subset S^{(m-2)}(P_1, \ldots, P_{m-h-1}; \tau_h)$. But then Theorem 2, equation (2), with $k$ replaced by $h$, implies that

$$S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h) = S^{(m-1)}[t; S^{(m-2)}(P_1, \ldots, P_{m-h-1}; \tau_h)].$$

exists. By Theorem 2, equation (1), with $k$ replaced by $h$,

$$S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h) = \prod_P S^{(m)}(P_1, \ldots, P_{m-h-1}, P, t; \tau_h).$$

When $t \to p$, $S^{(m)}(P_1, \ldots, P_{m-h-1}, P, t; \tau_h)$ converges, hence $S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h)$ also converges, $\Gamma_{[h+1]}$ is satisfied, and

$$S^{(m-1)}(P_1, \ldots, P_{m-h-1}; \tau_{h+1}) = \prod P S^{(m)}(P_1, \ldots, P_{m-h-1}; P; \tau_{h+1}).$$

Next, suppose $\Gamma_{[m]}$, $\ldots, \Gamma_{[h+1]}$ hold, and let $P_{m-h} \not\subset S^{(m-1)}(P_1, \ldots,$
Then \( P_{m-h} \not\subset S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h) \) if \( t \) is sufficiently close to \( p, t \in A, t \neq p \). But Theorem 2, with \( k \) replaced by \( h \), then implies that

\[
S^{(m)}(P_1, \ldots, P_{m-h-1}, P_{m-h}, t; \tau_h) = S^{(m)}[P_{m-h}; S^{(m-1)}(P_1, \ldots, P_{m-h-1}, t; \tau_h)]
\]
exists. Hence when \( t \to p \), \( S^{(m)}(P_1, \ldots, P_{m-h}, t; \tau_h) \) converges, \( \Gamma^{(m)}_{h+1} \) is satisfied for \( P_1, \ldots, P_{m-h} \), and

\[
S^{(m)}(P_1, \ldots, P_{m-h}; \tau_{h+1}) = S^{(m)}[P_{m-h}; S^{(m-1)}(P_1, \ldots, P_{m-h-1}; \tau_{h+1})].
\]

**Corollary 1.** Let \( 1 \leq m < n \). If \( A \) is \((m+1)\)-times differentiable at \( p \) then it is \( m \)-times differentiable there.

**Corollary 2.** If \( A \) satisfies \( \Gamma^{(n-1)}_1, \ldots, \Gamma^{(n-1)}_{m+1} \) at \( p \), then it is \((m+1)\)-times differentiable there \((0 \leq m < n)\).

**Corollary 3.**

\[
S^{(m-1)}_m \subset S^{(m)}_{m+1} \quad (m=1, 2, \ldots, n-1).
\]

**Proof.** By (1),

\[
S^{(m)}(t; \tau_m) \supset \bigcup_P S^{(m)}(P; \tau_m) = S^{(m-1)}_m.
\]
Hence \( S^{(m)}_{m+1} \supset S^{(m-1)}_m \).

The last remark implies the following.

**Corollary 4.** Let \( 1 \leq m < n \). If \( S^{(m)}_{m+1} = p \), then \( S^{(r)}_{r+1} = p \) \((r=0, 1, \ldots, m-1)\). Thus there is an index \( i \), where \( 1 \leq i \leq n \) such that \( S^{(r)}_{r+1} = p \) for \( r=0, 1, \ldots, i-1 \), but \( S^{(r)}_{r+1} \neq p \), if \( r \geq i \).

**Corollary 5.** Let \( 1 \leq m < n; 1 \leq r \leq m \). Then

\[
S^{(m)}(P_1, \ldots, P_{m+1-r}; \tau_r) \supset S^{(m-1)}(P_1, \ldots, P_{m+1-r}; \tau_{r-1})\).
\]

**Proof.**

\[
S^{(m)}(P_1, \ldots, P_{m+1-r}; \tau_r) = \lim_{t \to p} S^{(m)}(P_1, \ldots, P_{m+1-r}; t; \tau_{r-1})
\]

\[
\supset S^{(m-1)}(P_1, \ldots, P_{m+1-r}; \tau_{r-1})\).
\]

From Corollary 5, we get the following.

**Corollary 6.** Let \( 1 \leq m < n; 1 \leq r \leq m \). If \( P_{m+2-r} \subset S^{(m)}(P_1, \ldots, P_{m+1-r}; \tau_r) \) and \( P_{m+2-r} \not\subset S^{(m-1)}(P_1, \ldots, P_{m+1-r}; \tau_{r-1}) \) then
THEOREM 3. Let $1 \leq r \leq m < n$. Suppose $\Gamma_{\{m\}} \cdots , \Gamma_{\{r\}}$ are satisfied at $p$.

(i) If $S_{\{r\}} \neq p$, $\tau_{\{r\}}^{(m)}$ consists of all the $m$-spheres through $S_{\{r\}}^{(r-1)}$.

(ii) Let $S_{\{r\}}^{(r-1)} = p$. Choose any $S_{\{r\}}^{(r)} \in \tau_{\{r\}}^{(r)}$. Then $\tau_{\{r\}}^{(m)}$ is the set of all the $m$-spheres which touch $S_{\{r\}}^{(r)}$ at $p$.

Proof of (i). By Theorem 2, equation (1),

$$S^{(m)}(P_1, \cdots , P_{m+1-r}; \tau_r) \supset S^{(m-1)}(P_1, \cdots , P_{m-r}; \tau_r) \supset \cdots \supset S^{(r)}(P_1; \tau_r) \supset S_{\{r\}}^{(r-1)}.$$ 

Let $S^{(m)}$ be any $m$-sphere through $S_{\{r\}}^{(r-1)}$. By Theorem 2, if $P_1 \subseteq S^{(m)}$, $P_1 \not\subseteq S_{\{r\}}^{(r-1)}$,

$$S^{(r)}(P_1; S_{\{r\}}^{(r-1)}) = S^{(r)}(P_1; \tau_r) \subseteq S^{(m)}.$$ 

Suppose $S^{(k)}(P_1, \cdots , P_{r+k-2-r}; \tau_r) \subseteq S^{(m)}$, $(r \leq k < m)$. Choose $P_{k+2-r} \subseteq S^{(m)}$, $P_{k+2-r} \not\subseteq S^{(k)}(P_1, \cdots , P_{k+1-r}; \tau_r)$. Then by Theorem 2,

$$S^{(k+1)}(P_1, \cdots , P_{r+k+1-r}; \tau_r) = S^{(k)}(P_{k+2-r}; S^{(k)}(P_1, \cdots , P_{k+1-r}; \tau_r)) \subseteq S^{(m)}.$$ 

For $k = m-1$, this yields $S^{(m)}(P_1, \cdots , P_{m+1-r}; \tau_r) = S^{(m)}$. Thus $S^{(m)} \in \tau_{\{r\}}^{(m)}$.

Proof of (ii). Suppose $S_{\{r\}}^{(r-1)} = p$. As above, we have

$$S_{\{r\}}^{(m)} = S^{(m)}(P_1, \cdots , P_{m+1-r}; \tau_r) \supset \cdots \supset S^{(r)}(P_1; \tau_r).$$ 

Let $S^{(r)}(Q; \tau_r)$ be any $S^{(r)} \in \tau_{\{r\}}^{(r)}$. By Theorem 2, equation (1),

$$S^{(r)}(P, t; \tau_{r-1}) \cap S^{(r)}(Q, t; \tau_{r-1}) \supset S^{(r-1)}(t; \tau_{r-1}).$$ 

Let $P$ and $Q$ be variable points and let $S^{(r-1)}$ be a variable $(r-1)$-sphere converging to a fixed point. Suppose there is an $(n-1)$-sphere which separates this point from $P$ and $Q$. Then

$$\lim_{t \to p} [S^{(r)}(P; S^{(r-1)}), S^{(r)}(Q; S^{(r-1)})] = 0$$

whether or not the spheres $S^{(r)}(P; S^{(r-1)})$ and $S^{(r)}(Q; S^{(r-1)})$ themselves converge. In particular,

$$\lim_{t \to p} [S^{(r)}(P, t; \tau_{r-1}), S^{(r)}(Q, t; \tau_{r-1})] = 0.$$ 

Thus $S^{(r)}(P; \tau_r)$ touches $S^{(r)}(Q; \tau_r)$ at $p$. Furthermore, if $S^{(r)}(P; \tau_r)$ and $S^{(r)}(Q; \tau_r)$ have a point $\neq p$ in common, they coincide. Thus $\tau_{\{r\}}^{(r)}$ consists of the family of $r$-spheres which touch $S^{(r)}(Q; \tau_r)$ at $p$.

Suppose $r < m$ and an $m$-sphere $S_{\{r\}}^{(m)} = S^{(m)}(P_1, \cdots , P_{m+1-r}; \tau_r)$ of $\tau_{\{r\}}^{(m)}$ has a point $R \neq p$ in common with $S^{(m)}(Q; \tau_r)$. From the above,
If \( R \subset S^{(r)}(P; \tau_r) \) we have
\[
S^{(m)}_r \supset S^{(r)}(P; \tau_r) = S^{(r)}(R; \tau_r) = S^{(r)}(Q; \tau_r)
\]
while if \( R \not\subset S^{(r)}(P; \tau_r) \), we have, by Theorem 2,
\[
S^{(m)}_r \supset S^{(r+1)}[P; R; \tau_r] = S^{(r+1)}[P_1; S^{(r)}(R; \tau_r)] \supset S^{(r)}(R; \tau_r) = S^{(r)}(Q; \tau_r).
\]

On the other hand, suppose an \( m \)-sphere \( S^{(m)} \) touches \( S^{(r)} = S^{(r)}(Q; \tau_r) \) at \( p \). If \( S^{(m)} \supset S^{(r)} \) it follows, as in the proof of part (i), that \( S^{(m)} \in \tau^{(m)}_r \). Suppose \( S^{(m)} \cap S^{(r)} = p \). Choose an \( S^{(r)} \subset S^{(m)} \) such that \( S^{(r)} \) touches \( S^{(r)}(Q; \tau_r) \) at \( p \). Thus \( S^{(r)} \subset \tau^{(r)}_r \). It again follows that \( S^{(m)} \in \tau^{(m)}_r \).

**Corollary 1.** Let \( I^{(r-1)}, \ldots, I^{(r-1)} \) hold and let \( S^{(r-1)} = p \). Suppose \( \lim_{t \to p} S^{(r)}(P, t; \tau_{r-1}) \) exists for a single point \( P, P \neq p \). Then \( I^{(r)} \) holds at \( p \) \((1 < r < n)\).

**Proof.** This follows from equation (3).

**Corollary 2.** There is only one \( S^{(m)}_r \) of the pencil \( \tau^{(m)}_r \) which contains \((m+1-r)\) points which do not lie on the same \( S^{(m-1)}_r \).

**Proof.** Such an \( S^{(m)}_r \) can be uniquely constructed as in the proof of (i), Theorem 3.

**Corollary 3.** If two \( S^{(m)}_r \)'s intersect in an \( S^{(m-1)}_r \) then this \( S^{(m-1)}_r \) \( \in \tau^{(m-1)}_r \).

**Proof.** The \( S^{(m)}_r \)'s and hence also \( S^{(m-1)}_r \) contain \( S^{(r-1)}_r \). In case \( S^{(r-1)}_r = p \), let \( R \subset S^{(m-1)}_r \), \( R \neq p \). Then each of the \( S^{(m)}_r \)'s and hence also \( S^{(m-1)}_r \) contains \( S^{(r)}(R; \tau_r) \).

**Corollary 4.**
\[
\tau^{(m)}_0 \supset \tau^{(m)}_1 \supset \cdots \supset \tau^{(m)}_{m+1}.
\]

**Proof.** When \( k < m \), or when \( k = m \) and \( S^{(m-1)}_m \neq p \), Theorem 3 implies that \( \tau^{(m)}_k \) is the set of all the \( m \)-spheres through \( S^{(k-1)}_k \). Hence \( S^{(m+1)}_k \), being the limit of a sequence of such \( m \)-spheres, must itself contain \( S^{(k-1)}_k \), and by Theorem 3, \( S^{(m)}_{k+1} \in \tau^{(m)}_k \). Suppose \( k = m \) and \( S^{(m-1)}_m = p \). By Theorem 3, \( \tau^{(m)}_m \) is the set of all the \( m \)-spheres which touch a given \( m \)-sphere \( S^{(m)}_m \neq p \) of \( \tau^{(m)}_m \) at \( p \). Hence \( S^{(m)}_{m+1} \), being the limit of a sequence of such \( m \)-spheres, must itself touch \( S^{(m)}_m \) at \( p \), and, again by
Theorem 3, \( S_m^{(m)} \subseteq \tau_m^{(m)} \).

**Theorem 4.** Let \( 1 < m < n; 1 \leq k \leq m \), and suppose that \( S_{m-1}^{(m-1)} \neq p \) if \( k = m \). If the conditions \( \Gamma_1^{(m)}, \ldots, \Gamma_k^{(m)} \) hold at \( p \), then \( \Gamma_{k+1}^{(m)} \) also holds there.

**Proof.** By Theorem 2, \( \Gamma_1^{(m-1)}, \ldots, \Gamma_k^{(m-1)} \) hold at \( p \). Hence if \( p, P_1, \ldots, P_{m-k} \) are independent points \( S_{m-1}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k) \) is defined. Furthermore, by Theorem 1, we can assume that \( t \not\in S_{m-1}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k) \) and by Theorem 2 again,

\[
S_m^{(m)}(P_1, \ldots, P_{m-k}, t; \tau_k) = S_m^{(m-1)}[t; S_{m-1}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k)].
\]

Thus \( S_m^{(m)}(P_1, \ldots, P_{m-k}, t; \tau_k) \) exists when \( t \) is close to \( p \), \( t \in A, t \neq p \). Choose \( P_{m+1-k} \subseteq S_{m-2}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k) \), \( P_{m+1-k} \not\subseteq S_{m-2}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k) \). Then Theorem 2 implies that

\[
S_{m-1}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k) = S_{m-1}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k) - 1
\]

when \( k < m \), or \( k = m \) and \( S_{m-1}^{(m-1)} \neq p \); if \( k = m \) and \( S_{m-1}^{(m-1)} = p \), this equation follows from Theorem 3, Corollary 4. Hence

\[
\lim_{t \to p} S_m^{(m)}(P_1, \ldots, P_{m-k}, t; \tau_k) = \lim_{t \to p} S_m^{(m-1)}[t; S_{m-1}^{(m-1)}(P_1, \ldots, P_{m-k}; \tau_k)]
\]

Thus \( \Gamma_{k+1}^{(m)} \) holds at \( p \) and

\[
S_m^{(m)}(P_1, \ldots, P_{m-k}; \tau_{k+1}) = S_m^{(m)}(P_1, \ldots, P_{m-k}; \tau_k).
\]

**Corollary 1.** If \( \Gamma_1^{(m)} \) holds at \( p \), then \( \Gamma_r^{(m)} \) holds there, \( r = 1, 2, \ldots, m \). Furthermore, if \( S_{m-1}^{(m-1)} \neq p \), \( A \) is \( m+1 \) times differentiable at \( p \).

**Corollary 2.** If \( \Gamma_1^{(m-1)} \) holds at \( p \), then \( p \) is a differentiable point of \( A \) if and only if \( \lim_{t \to p} S_{m-1}^{(m-1)}(t; \tau_{m-1}) \) exists and converges if \( t \) tends to \( p \).

**Corollary 3.** If \( \Gamma_1^{(m-1)} \) holds at \( p \), and \( S_{m-2}^{(m-2)} \neq p \), then \( p \) is a differentiable point of \( A \).

**Corollary 4.** If \( \Gamma_1^{(m)} \) holds at \( p \), all the conditions \( \Gamma_k^{(r)} \), except possibly \( \Gamma_{m+1}^{(m)} \), automatically hold at \( p \) (\( 1 \leq k \leq r + 1 \leq m + 1 \)).

Let \( p \) be a differentiable point of \( A \). We define the index \( i \) of \( p \) as in Theorem 2, Corollary 4. Let \( P \subseteq S_m^{(i)} \), \( P \neq p \). Let \( S_m^{(m)} = S_m^{(m)}(P; \tau_m) \), \( m = 0, 1, \ldots, i \). Then the set of \( \tau_{m}^{(m)} \)'s is completely determined by
the sequence
\[ S_0^{(n-1)} \subset S_1^{(n-1)} \subset \cdots \subset S_{r+1}^{(n-1)} \subset \cdots \subset S_{n-1}^{(n-1)}. \]
Its structure is determined by the single index \( i \).

### 6. Support and intersection

Let \( p \) be an interior point of \( A \). Then we call \( p \) a point of support (intersection) with respect to an \((n-1)\)-sphere \( S \) if a sufficiently small neighbourhood of \( p \) is decomposed by \( p \) into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by \( S \). \( S \) is then called a supporting (intersecting) \((n-1)\)-sphere of \( A \) at \( p \). Thus \( S \) supports \( A \) at \( p \) if \( p \not\subset S \).

By definition, the point \((n-1)\)-sphere \( p \) always supports \( A \) at \( p \). It is possible for an \((n-1)\)-sphere to have points \( \neq p \) in common with every neighbourhood of \( p \) on \( A \). In this case, \( S \) neither supports nor intersects \( A \) at \( p \).

### 7. Support and intersection properties of \( \tau_r^{(n-1)} - \tau_{r+1}^{(n-1)} \)

Let \( p \) be a differentiable interior point of \( A \). In the following,
\[ \tau_r^{(n-1)} - \tau_{r+1}^{(n-1)} \]
will denote the family of those \((n-1)\)-spheres of \( \tau_r^{(n-1)} \) which do not belong to \( \tau_{r+1}^{(n-1)} \) (cf. Theorem 3, Corollary 4). Our classification of the differentiable points \( p \) of \( A \) will be based on the index \( i \) of \( p \), and on the support and intersection properties of \( S_{n-1}^{(n-1)} \) and the families \( \tau_r^{(n-1)} - \tau_{r+1}^{(n-1)} \), \( r=0, 1, \ldots, n-1 \). We shall omit the superscript \((n-1)\) of \( \tau_r^{(n-1)} \) when there is no ambiguity; thus \( \tau_r = \tau_r^{(n-1)} \).

**Theorem 5.** Every \((n-1)\)-sphere \( \neq S_{n-1}^{(n-1)} \) either supports or intersects \( A \) at \( p \).

**Proof.** If an \((n-1)\)-sphere \( S \) neither supports nor intersects \( A \) at \( p \), then \( p \subset S \) and there exists a sequence of points \( t \to p, t \subset A \cap S, t \neq p \). Suppose \( p, P_1, \ldots, P_n \) are independent points on \( S \). Suppose that for some \( r, 0 \leq r < n-1 \), \( S = S^{(n-1)}(P_1, \ldots, P_{n-r}; \tau_r) \). By Theorem 2, equation (1),
\[ S^{(n-1)}(P_1, \ldots, P_{n-r}; \tau_r) \supseteq S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \tau_r). \]
By Theorem 1, \( t \subset S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \tau_r) \) and again by Theorem 2, equation (2),
\[ S = S^{(n-1)}(t; S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \tau_r)) = S^{(n-1)}(P_1, \ldots, P_{n-r-1}, t; \tau_r) \]
for each \( t \). Condition \( I_r^{(n-1)} \) now implies that
Thus we get, in this way,

\[ S = S^{(n-1)}(P_1, \cdots, P_{n-1}; \tau_{r+1}) \cdot \]

By Theorem 2, \( S \supset S^{(n-2)}_{n-1} \), and by Theorem 1, \( t \subset S^{(n-2)}_{n-1} \) when the parameter \( t \) is close to, but different from, the parameter \( p \). If \( S^{(n-2)}_{n-1} \neq p \), Theorem 2, equation 2, implies that \( S = S^{(n-1)}_{n-1}[t; S^{(n-2)}_{n-1}] = S^{(n-1)}[t; \tau_{n-1}] \), while if \( S^{(n-2)}_{n-1} = p \), Theorem 3 implies that \( S = S^{(n-1)}(t; \tau_{n-1}) \). Applying condition \( \Gamma_n^{(n-1)} \), we are led to the conclusion \( S = S^{(n-1)}_n \).

**Theorem 6.** If \( S^{(n-1)}_n = p \), then the \((n-1)\)-spheres of \( \tau_{n-1} - \tau_n \) all intersect \( A \) at \( p \), or they all support.

**Proof.** Let \( S \) and \( S' \) be two distinct \((n-1)\)-spheres of \( \tau_{n-1} - \tau_n \). Since \( S^{(n-1)}_n = p \), Theorem 2, Corollary 4 implies that \( S^{(n-2)}_{n-1} = p \), and Theorem 3 implies that \( S \) and \( S' \) touch at \( p \). Thus we may assume that \( S \subset (p \cup S) \) and \( S' \subset (p \cup S') \). Suppose now, for example, that \( S \) supports \( A \) at \( p \) while \( S' \) intersects. Then \( A \cap \overline{S'} \) is not void and \( A \subset (p \cup \overline{S}) \). Let \( t \to p \) in \( A \cap S' \). Hence \( S^{(n-1)}(t; \tau_{n-1}) \subset (S' \cap \overline{S}) \cup p \). Consequently, \( S(t; \tau_{n-1}) \) can not converge to \( S^{(n-1)}_n = p \), as \( t \) tends to \( p \). Thus \( S \) and \( S' \) must both support, or both intersect \( A \) at \( p \).

**Theorem 7.** If \( S^{(r)}_{r+1} \neq p \) while \( S^{(r-1)}_r = p \), then every \((n-1)\)-sphere of \( \tau_r - \tau_{r+1} \) supports \( A \) at \( p \) (1 \( \leq r \leq n-1 \)).

**Proof.** Suppose \( S^{(r-1)}_r = p \), so that by Theorem 3, the \( r \)-spheres of \( \tau_r' \) all touch any \((n-1)\)-sphere of \( \tau_r \). Let \( S \in \tau_r - \tau_{r+1} \), \( S \neq p \). If a sequence of points \( t \) exists such that \( t \subset A \cap \overline{S} \), \( t \to p \), then each \( S^{(r)}(t; \tau_r') \) lies in the closure of \( \overline{S} \). Hence \( S^{(r)}_{r+1} \) will also lie in the same closed domain. Since \( S^{(r)}_{r+1} \in \tau_r^{(r)} \), either \( S^{(r)}_{r+1} = p \), or it touches \( S \) at \( p \). Since \( S \not\subset \tau_{r+1} \), \( S^{(r)}_{r+1} \) must lie in \( p \cup \overline{S} \). Similarly, the existence of a sequence \( t' \subset S \cap A \), \( t' \to p \), implies that \( S^{(r)}_{r+1} \subset p \cup S \). Thus if \( S \) intersects \( A \) at \( p \), \( S^{(r)}_{r+1} \subset (p \cup \overline{S}) \cap (p \cup S) = p \); that is, \( S^{(r)}_{r+1} = p \).

**Theorem 8.** All the \((n-1)\)-spheres of \( \tau_r - \tau_{r+1} \) support \( A \) at \( p \), or they all intersect; \( r = 0, 1, \cdots, n-1 \).

**Proof.** Let \( S' \) and \( S'' \) be two distinct \((n-1)\)-spheres of \( \tau_r \). Suppose, for the moment, that the intersection \( S' \cap S'' \) is a proper \((n-2)\)-sphere \( S^{(n-2)}(P_1, \cdots, P_{n-1}; \tau_r) \). Suppose, for example, that \( S' \) intersects, while \( S'' \) supports \( A \) at \( p \). Thus \( A \cap \overline{S'} \) and \( A \cap \overline{S''} \) are not void.
With no loss in generality, we may assume that \( A \subseteq \overline{S'} \cup p \). If \( t \) is close to \( p \), \( t \neq p \), Theorem 1 implies that \( t \subseteq S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \tau_r) \) and Theorem 2, equation 2, implies that
\[
S^{(n-1)}[t; S^{(n-2)}(P_1, \ldots, P_{n-r-1}; \tau_r)] = S^{(n-1)}(P_1, \ldots, P_{n-r-1}, t; \tau_r).
\]
If \( t \subset A \cap S' \), then \( S^{(n-1)}(P_1, \ldots, P_{n-r-1}, t; \tau_r) \) lies in the closure of
\[
(S' \cap \overline{S'}) \cup (S' \cap S'').
\]
Letting \( t \) tend to \( p \), we conclude that \( S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \tau_{r+1}) \) lies in the same closed domain. By letting \( t \) converge to \( p \) through \( S' \cap A \), we obtain symmetrically that \( S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \tau_{r+1}) \) also lies in the closure of
\[
(S' \cap \overline{S'}) \cup (S' \cap S'').
\]
Hence \( S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \tau_{r+1}) \) lies in the intersection \( S' \cup S'' \) of these two domains, that is, \( S^{(n-1)}(P_1, \ldots, P_{n-r-1}; \tau_{r+1}) \) is either \( S' \) or \( S'' \), in other words, one of the \((n-1)\)-spheres \( S' \) and \( S'' \) belongs to \( \tau_{r+1} \). Thus if \( S' \) and \( S'' \) belong to \( \tau_r-\tau_{r+1} \) and have a proper \( S^{(n-2)} \) in common, they both support or both of them intersect.

Suppose now that \( S' \cap S'' = p \). Theorem 3 implies that \( S^{(r-1)}_{\tau_{r+1}} = p \). In view of Theorems 6 and 7, there remain to be considered only the cases where \( r < n-1 \), and, indeed, when \( r \leq n-2 \), we have only to consider those cases for which \( S^{(r-1)}_{\tau_{r+1}} = p \).

By Theorem 3, any \( S^{(n-1)} \) which touches an \( S^{(r)} \), but which does not touch an \( S^{(r+1)} \), belongs to \( \tau_r-\tau_{r+1} \). Hence there exists an \((n-1)\)-sphere \( S \) of \( \tau_r-\tau_{r+1} \) which intersects \( S' \) and \( S'' \) respectively in a proper \((n-2)\)-sphere. From the above, \( S \) and \( S' \), and also \( S \) and \( S'' \) both support or both intersect \( A \) at \( p \). Thus \( S' \) and \( S'' \) both support or both intersect \( A \) at \( p \) in this case also.

8. Characteristic and classification of the differentiable points. The characteristic \((a_0, a_1, \ldots, a_n; i)\) of a differentiable point \( p \) of an arc \( A \) is defined as follows:

- \( a_r = 1 \) or \( 2 \) when \( r < n \); \( a_n = 1, 2, \text{ or } \infty \). The index \( i = 1, 2, \ldots, n \).
- \( a_0 + \cdots + a_r \) is even or odd according as every \( S^{(n-1)} \) of \( \tau_r-\tau_{r+1} \) supports or intersects \( A \) at \( p \); \( r = 0, 1, \ldots, n-1 \).
- \( a_0 + \cdots + a_n \) is even when \( S^{(n-1)} \) supports, odd if \( S^{(n-1)} \) intersects, while \( a_n = \infty \) if \( S^{(n-1)} \) neither supports nor intersects \( A \) at \( p \).
- Finally the characteristic of \( p \) has index \( i \) if and only if \( S^{(i-1)}_i = p \), while \( S^{(i)}_i \neq p \).

Theorem 7, and the convention that \( S^{(n-1)}_n \) supports \( A \) at \( p \) when \( S^{(n-1)}_n = p \), lead to the following restriction on the characteristic \((a_0, a_1, \ldots, a_n; i)\):
As a result of this restriction, the number of types of differentiable points corresponding to each value of \( i < n \) is \( 3(2)^{n-1} \), and there are \( 2^n \) types when \( i = n \). Thus there are \( (3n-1)2^{n-1} \) types altogether.

If we introduce a rectangular Cartesian coordinate system into the conformal \( n \)-space, examples of each of the \( (3n-1)2^{n-1} \) types are given by the curves

\[
(I) \quad x_1 = t^m_1, \quad x_2 = t^m_2, \ldots, \quad x_n = t^m_n
\]

in the cases \( a_n = 1 \) or 2, and

\[
(II) \quad x_1 = t^m_1, \quad x_2 = t^m_2, \ldots, \quad x_n = \begin{cases} t^m_n \sin t^{-1}, & \text{if } 0 < |t| \leq 1 \\ 0, & \text{if } t = 0 \end{cases}
\]

for the cases in which \( a_n = \infty \), all relative to the point \( t = 0 \). The \( m_r \) are positive integers and \( m_1 < m_2 < \cdots < m_n \). The different types are determined by the parities of the \( m_i \) and by the relative magnitudes of the \( m_r \) and \( 2m_1 \). In each of these examples, the \( S_1^{(m)} \) touch the \( x_i \)-axis at the origin; \( m = 1, 2, \ldots, n-1 \).

When \( m_i < 2m_1 < m_{i+1} \), the point \( t = 0 \) has a characteristic of the form \((a_0, a_1, \cdots, a_n; i)\) where \( a_n \) can be 1, 2, or \( \infty \), and \( i < n \).

When \( m_n < 2m_1 \), the point \( t = 0 \) has a characteristic of the form \((a_0, a_1, \cdots, a_n; n)\) where \( a_n \) is either 1 or 2. The following table lists some of the properties of a differentiable point \( p \) having the characteristic \((a_0, a_1, \cdots, a_n; i)\):

\[
(a_0, a_1, \cdots, a_n; i)
\]

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