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1. Introduction. An inequality for Lebesgue area which corresponds to the fact that the measure of a quadrilateral is not less than the product of the distances between the two pairs of opposite sides may sometimes be useful for the study of this area. This inequality is an extension of a result of Besicovitch [2].

The important results of Cesari [4] and Federer [8] showing the equivalence of Geöcze and Lebesgue area will be used to show that several other 'areas' are equivalent to these two.

This paper depends upon definitions and results of [11] and [12]. In particular we shall use the area defined in [11] which agrees with Lebesgue area for surfaces in Euclidean space.

Let Q be the square $0 \leqq u, v \leqq 1$ having consecutive sides a, b, c, and d. The set of continuous functions on Q into m, the space of bounded sequences [1], will be denoted by C, and the family of homeomorphisms of Q into Q by H.

Let $x, y \in C$. Then $\tilde{x}$ is defined on $Q \times Q$ to the nonnegative real numbers by

$$\tilde{x}(p, q) = \|x(p) - x(q)\|$$

for $(p, q) \in Q \times Q$. If there exists a positive real number M such that

$$\tilde{x}(p, q) \leqq M\|p - q\|$$

for all $(p, q) \in Q \times Q$ then x is *Lipschitzian*. If $\tilde{x}(p, q) \leqq \tilde{y}(p, q)$ for all $(p, q) \in Q \times Q$, then we shall write $\tilde{x} \leqq \tilde{y}$. The Lebesgue area of x is denoted by $L(x)$. If $\tilde{x} \leqq \tilde{y}$ then $L(x) \leqq L(y)$ [Kolmogoroff's principle].

If i and k are distinct positive integers, let π_{ik} be the plane in m consisting of those points all of whose components, except the ith and kth, are zero. The set of all planes π_{ik} is Π. Let E^2 be the Euclidean plane provided with a cartesian coordinate system, and let T_{ik} be the homeomorphism of E^2 onto π_{ik} defined by

$$T_{ik}(s, t) = \{w^j\} ,$$

where $(s, t) \in E^2$, $w^i = s$, $w^k = t$, and $w^j = 0$ for $i \neq j \neq k$. If $E \subset E^2$, and E is Lebesgue measurable, then the measure of $T_{ik}(E)$, $|T_{ik}(E)|$, is the

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Lebesgue measure of E.

If $x, x_n \in C$, $n=1, 2, \cdots$, we write $x_n \to x$ for uniform convergence of x_n to x. If f and g are two functions with range $f \subset$ domain g, then gf is the composition of f and g.

In [12] we were interested in geodetic properties of Lebesgue area, and definitions were framed accordingly. Now we are interested in properties which are analogous, equivalent for light mappings, but possibly different in general. The geodetic distance between two points in the parameter square Q is obtained by considering curves in Q joining these two points. In this paper we shall define a pseudo-geodetic distance by considering curves in a "middle-space". Every curve in Q corresponds to a curve in the middle-space, but not conversely. Consequently the pseudo-geodetic distance may be less than the geodetic distance, and the results of [12] will necessarily hold with the modified definition.

If f is continuous on $[0, 1]$ into m let $\rho(f)=$diameter range f. If $x \in C$, define $\tilde{x}_\mu$ on $Q \times Q$ by

$$\tilde{x}_\mu(p, q) = \inf_g \mu(xg) \qquad (p, q) \in Q \times Q$$

for all continuous functions g on $[0, 1]$ into Q with $g(0)=p$ and $g(1)=q$. In [12] a function $x_\mu \in C$ was constructed such that $\widetilde{(x_\mu)}=\tilde{x}_\mu$. Furthermore, if $x_k \to x$ then $x_{k\mu} \to x_\mu$. Finally, x_μ is the monotone factor in a monotone-light factorization of x and has the same Lebesgue area as x.

2. An inequality for Lebesgue area. In this section the fundamental inequality of the paper is proved.

LEMMA 1. *Let $\pi \in \Pi$. If $f \in C$ is Lipschitzian, and range $f \subset \pi$, then $L(f) \geqq |f(Q)|$.*

Proof. Let J be the Jacobian of f, and let R be the set on which J is defined. Then $|Q-R|=0$ and, consequently, $|f(Q)|=|f(R)|$. Now, by Federer's Theorem [6],

$$L(f)=\iint_Q |J(u, v)|\,du\,dv=\iint_R |J(u, v)\,du\,dv$$
$$=\iint_\pi N((s, t), f, R)\,ds\,dt \geqq |f(R)|=|f(Q)|,$$

where $N((s, t), f, R)$ is the number, possibly infinite, of points $(u, v) \in Q$ such that $f(u, v)=(s, t)$.

If c is an oriented simple closed curve in the plane, and f is continuous on some region containing c into R^2 (the plane of ordered pairs of real numbers with the topology of the Euclidean plane), then for each

$p \in R^2$ let $\mu(p, f, c)$ be the index of p relative to f and c [9, II. 4.34].

LEMMA 2. *Let f be continuous from Q into R^2. If $\mu(p, f, Q^*) \neq 0$, where Q^* is the boundary of Q, then $p \in f(Q)$.*

This lemma is proved in [9, IV. 1.4–26].

LEMMA 3. *Let $f=(f^1, f^2)$ be continuous from Q into R^2. Suppose*

$$p \in a \Longrightarrow f^1(p)=0\,, \qquad p \in b \Longrightarrow f^2(p)=0\,,$$
$$p \in c \Longrightarrow f^1(p) \geqq r > 0\,, \qquad p \in d \Longrightarrow f^2(p) \geqq s > 0\,.$$

If y and z are real numbers with $0 < y < r$ and $0 < z < s$, then $(y, z) \in f(Q)$.

Proof. If we use the notation of Lemma 2 and the result of [9, II. 4.35] we see that $\mu(p, f, Q^*) \neq 0$ if $p=(y, z)$. An application of Lemma 2 completes the proof.

Let $x \in C$ and $x=lM$ be a monotone-light factorization of x. If $(p, q) \in Q \times Q$, and G is the set of continuous functions g defined on $[0, 1]$ into range M with $g(0)=M(p)$ and $g(1)=M(q)$ then define

$$\hat{x}_G(p, q) = \inf \{\text{length } lg\} \qquad \text{for all } g \in G.$$

This definition is independent of the particular factorization of x. Finally, if $x=lx_\mu$ is the indicated monotone-light factorization of x, and $g \in G$ (for $M=x_\mu$), then

$$\text{length } lg \leqq \text{length } g\,.$$

From this it follows that

$$\hat{x}_G \leqq (\hat{x}_\mu)_G\,.$$

DEFINITION 1. If $x \in C$ define

$$\alpha(x) = \min_{p \in a,\ q \in c} \tilde{x}(p, q) \qquad \text{and} \qquad A(x) = \inf_{p \in a,\ q \in c} \hat{x}_G(p, q)\,.$$

Define $\beta(x)$ and $B(x)$ by replacing a by b and c by d. Let

$$\gamma_{ik} = \{\min_{0 \leqq v,\ w \leqq 1} |x^i(1, v) - x^i(0, w)|\}\{\min_{0 \leqq u,\ w \leqq 1} |x^k(u, 1) - x^k(w, 0)|\}\,.$$

LEMMA 4. *If $x_n \to x$ in C then $A(x_\mu) \leqq \liminf_{n \to \infty} A(x_{n\mu})$.*

Proof. We may suppose that for some $M > 0$, $A(x_{n\mu}) < M$ for all n. Hence there exist continuous functions g_n on $[0, 1]$ into range $x_{n\mu}$ with $g_n(0) \in x_{n\mu}(a)$ and $g_n(1) \in x_{n\mu}(c)$, such that

$$\text{length } g_n < A(x_{n\mu}) + 1/n < M + 2 \qquad \text{for all } n.$$

We may now use Hilbert's Theorem to conclude that a subsequence (still denoted by the same subscript) of the g_n converges in the sense of Fréchet to a continuous function g. Let $p \in$ range g. There exist Fréchet equivalent representations γ_n of g_n, and a point $t \in [0, 1]$, such that $\gamma_n(t) \to p$. Let

$$P_n \in x_{n\mu}^{-1}[\gamma_n(t)] .$$

A subsequence of P_n (still denoted by the same subscript) will converge to a point $P \in Q$. Then

$$\begin{aligned} ||p - x_\mu(P)|| \leqq & ||p - x_{n\mu}(P_n)|| + ||x_{n\mu}(P_n) - x_\mu(P_n)|| \\ & + ||x_\mu(P_n) - x_\mu(P)|| . \end{aligned}$$

Thus range g is contained in range x_μ. Also $g(0) \in x_\mu(a)$ and $g(1) \in x_\mu(b)$. Hence

$$A(x_\mu) \leqq \text{length } g \leqq \liminf_{n \to \infty} \text{ length } g_n = \liminf_{n \to \infty} A(x_{n\mu}) .$$

If $z \in C$ is quasilinear (Q may be subdivided into a finite number of triangles on each of which z is linear) then there exists a $z_G \in C$ such that z_G is Lipschitzian, $\tilde{z}_G = (\widehat{z_\mu})_G = \hat{z}_G$, and $L(z) = L(z_G)$ [12].

LEMMA 5. *If $z \in C$ is quasilinear* then

$$L(z) \geqq A(z_\mu) B(z_\nu)$$

Proof. It is sufficient to show that

$$L(z_G) \geqq \alpha(z_G) \beta(z_G) .$$

Let f be the flat transformation from Q to π_{12} defined by

$$f(p) = \{f^j(p)\} \qquad (p \in Q) ,$$

where

$$f^1(p) = \min_{q \in a} \tilde{z}_G(p, q), \quad f^2(p) = \min_{q \in b} \tilde{z}_G(p, q), \quad f^j(p) = 0 \qquad (j > 2) .$$

If $p, q, p' \in Q$, then

$$\begin{aligned} & \tilde{z}_G(p, q) \leqq \tilde{z}_G(p, p') + \tilde{z}_G(p', q) , \\ & \min_{q \in a} \tilde{z}_G(p, q) \leqq \tilde{z}_G(p, p') + \min_{q \in a} \tilde{z}_G(p' q) , \\ & \min_{q \in a} \tilde{z}_G(p, q) - \min_{q \in a} \tilde{z}_G(p', q) \leqq \tilde{z}_G(p, p') . \end{aligned}$$

The interchange of p and p', and of a and b, shows that

$$||f(p) - f(p')|| \leqq \tilde{z}_G(p, p') .$$

Then by Kolmogoroff's principle and Lemmas 1 and 3,

$$L(z_G) \geq L(f) \geq |f(Q)| \geq \alpha(z_G)\beta(z_G) .$$

We now make the convention that $0 \cdot \infty = 0$.

THEOREM 1. $L(x) \geq A(x_\mu)B(x_\mu) \geq A(x)B(x)$.

Proof. Let $\{z_n\}$ be a sequence of quasilinear functions in C such that $z_n \to x$ and $L(z_n) \to L(x)$. Then

$$\begin{aligned} L(x) &= \lim_{n\to\infty} L(z_n) \geq \limsup_{n\to\infty} A(z_{n\mu})B(z_{n\mu}) \\ &\geq \{\liminf_{n\to\infty} A(z_{n\mu})\}\{\liminf_{n\to\infty} B(z_{n\mu})\} \\ &\geq A(x_\mu)B(x_\mu) \geq A(x)B(x) . \end{aligned}$$

3. Functionals related to Lebesgue area. The alternative definitions of Lebesgue area given in this section may be of some interest to the reader.

For convenience we reserve the letter F to denote a finite subset of H such that ranges of distinct elements of F have no interior points in common.

DEFINITION 2. The functionals ψ^*, φ, and Φ are defined on C by

$$\begin{aligned} \psi^*(x) &= \sup_F \sum_{h\in F} \sup_{i,k} \gamma_{ik}(xh) , \\ \varphi(x) &= \sup_F \sum_{h\in F} \alpha(xh)\beta(xh) \\ \Phi(x) &= \sup_F \sum_{h\in F} A(xh)B(xh) . \end{aligned}$$

THEOREM 2. *If $x \in C$ is light and if $\Phi(x) < \infty$ then there exists an arc g connecting a and c such that xg is rectifiable.*

Proof. If x is light then x_μ is a homeomorphism and $B(x_\mu) \neq 0$. Since $A(x_\mu)B(x_\mu) < \infty$ we can conclude that $A(x) \leq A(x_\mu) < \infty$, from which the theorem follows.

Let Ω be any of the three functionals ψ^*, φ, or Φ. A familiar argument gives the following.

LEMMA 6. $\Omega(x) \geq \sum_{h\in F} \Omega(xh)$.

LEMMA 7. $\psi^* \leq \varphi \leq \Phi \leq L$.

Proof. It is sufficient to recall that $L(x) \geq \sum_{h\in F} L(xh)$.

After we show that $\psi^* = L$ for flat transformations we can use the

results of Cesari and Federer to conclude that the inequalities in the lemma are actually equalities. It will be more convenient for us to use the equality of Peano and Lebesgue rather than that of Geöcze and Lebesgue area. If $x \in C$, then $P(x)$ is the Peano area of x. If $L(x) < \infty$ then it is known that $P(x) = L(x)$ [11].

If N is a positive integer let π^N be the transformation on m into itself determined by

$$\pi^N(t) = \{s^j\}$$

where

$$s^j = \begin{cases} t^j & j \leq N, \\ 0 & j > N. \end{cases}$$

LEMMA 8. $P(\pi^N x) = L(\pi^N x)$.

Proof. Let T map range π^N into E^N by

$$T(\{s^j\}) = \{s\}.$$

Then T and T^{-1} are both Lipschitzian with constants not exceeding $N^{1/2}$ and 1, respectively. Thus if $z \in C$ is quasilinear we can use the definition of area in m [11] to conclude that

$$(1/N) \text{ area } T\pi^N z \leq \text{area } \pi^N z \leq \text{area } T\pi^N z.$$

Thus

$$(1/N) L'(T\pi^N x) \leq L(\pi^N x) \leq L'(T\pi^N x)$$

where L' is Lebesgue area in E^N, and an analogous inequality relates P and P', where P' is Peano area in E^N. If $P(\pi^N x)$ is finite then $P'(T\pi^N x) = L'(T\pi^N x)$ are also, and furthermore, so is $L(\pi^N x)$. From what has already been said we can conclude that $P(\pi^N x) = L(\pi^N x)$.

Let f be continuous from Q into π_{12}. If $p \in \pi_{12}$ and $\mu(p, f, c) \neq 0$ for a simple closed curve c bounding a Jordan region J contained in Q, then there are open oriented rectangles (sides parallel to the coordinate axes) R' and R with $p \in R' \subset \text{closure } R' \subset R \subset \text{closure } R$ and $\mu(q, f, c) \neq 0$ for all $q \in R$.

LEMMA 9. *Under the conditions of the preceding paragraph there is a simple closed curve B with $f(B) \subset R - \text{closure } R'$ and such that $\mu(q, f, B) \neq 0$ for all $q \in R'$.*

Proof. There are only a finite number of components A_j, $j = 1, 2, \cdots, n$ of $f^{-1}(\text{closure } R')$ whose intersection with $f^{-1}(p)$ is not empty. The minimum distance between any two of these components is positive

as is the distance between any of these components and the complement of $f^{-1}(R)$. Hence there exist simple closed curves c_j bounding Jordan regions J_j with $A_j \subset J_j \subset f^{-1}(R)$. Some of the curves will not be contained in Jordan regions bounded by any of the other curves. Let d_i, $i=1, 2, \cdots, m$, be this set of curves and denote the regions they bound by K_i. Suppose that $\mu(p, f, d_i)=0$ for all i. By introducing $(m+1)$ arcs joining c to d_1, d_i to d_{i+1} for $1 \leqq i \leqq m-1$ and d_m to c, we can decompose $J - \bigcup K_i$ into Jordan regions L_1 and L_2 with bounding curves e_1 and e_2. Furthermore

$$0 \neq \mu(p, f, c) - \sum \mu(p, f, d) = \mu(p, f, e_1) + \mu(p, f, e_2).$$

Hence either $\mu(p, f, e_1) \neq 0$ or $\mu(p, f, e_2) \neq 0$, which implies that

$$0 = \bigcup K_i \cap (J - \bigcup K_i) \supset f^{-1}(R') \cap (J - \bigcup K_i) \supset f^{-1}(p) \cap (J - \bigcup K_i) \neq 0.$$

Therefore $\mu(p, f, d_i) \neq 0$ for some d_i. Let B be this d_i.

LEMMA 10. *Let $h \in H$ with $h(Q^*)=B$. Then $\psi^*(fh) \geqq |R'|$.*

Proof. Choose $e>0$ and less than (width R')/10. Take R so that the distance between the boundary of R' and the boundary of R is less than e. Let the sides of R' be X, Y, Z, and T. Since $\mu(p, f, B) \neq 0$ there is a point $q \in B$ such that the distance between $f(q)$ and one side, say X, of R' is less than e. Start from q and traverse B in a positive sense (with respect to the Jordan region J which it bounds). There will be a point $q_1 \in B$ such that if r is between q and q_1 (going from q to q_1 on B) the distance from $f(r)$ to X does not exceed $2e$, and no point beyond q_1 has this property. Now start from q_1 and obtain q_2 such that for any point s between q_1 and q_2 the distance between $f(s)$ and Y, a side of R' adjoining X, does not exceed $2e$ but that no point beyond q_2 has this property. (It may be, of course, that points beyond q_2 have images close to X.) In this manner B is divided into a finite number N of closed intervals having only endpoints in common such that the image of a single interval will be within distance $2e$ of one of the sides of R'. Since $\mu(p, f, B) \neq 0$, it is necessary that $N>3$.

If $N=4$ take $h \in H$ so that the images of the sides of Q are the four intervals into which B is divided. Then

$$\gamma_{12}(fh) \geqq (A-4e)(B-4e)$$

where A and B are the lengths of adjoing sides of R'.

If $N>4$ let W be a strip of width e whose centerline w is parallel to one of the coodinate axes and which divides R into congruent rectang-

les S_1 and S_2. There exists an open rectangle S'' which contains the closure of S_1 and for which $\mu(p, f, c)\neq 0$ if $p\in S''$. Thus there is a Jordan curve B'' with $f(B'')\subset S''-S_1$ and $\mu(p, f, B'')\neq 0$ for $p\in S$. Let m be a component of $B''-B$ which contains a point whose image is within a distance e of the center of R. If m were all of B'' we could conclude that $\mu(p, f, B)\neq 0$ for all $p\in f(B'')$, but this is false for any $p\in S_1-R'$. Thus m connects two points of B whose images lie near opposite sides of R' and thus divides J into two Jordan regions J_1 and J_2 with bounding curves B_1 and B_2. The image of B_1 is contained in $R-R'-W$ as is the image of B_2.

Two of the intervals into which B was divided, which correspond to opposite sides of R', will each have been divided into two smaller intervals by the arc m. One of each of these pairs of smaller intervals will belong to B_1 and the other to B_2.

The appropriate intervals for B_i are all those contained in B_i which originally belonged to B plus the arc m, $i=1, 2$. Thus

$$N_1+N_2=N+4$$

where N_i is the number of intervals into which B_i is divided.

For the proof of Lemma 11 we shall wish to remember that J_1 and J_2 have no interior points in common.

If $N_1=4$ and $N_2=4$ then h_1 and h_2 can be defined as was h in the case $N=4$. If W were chosen so as to make $A/2$ and B the width and length of the rectangles into which R' was divided then

$$\gamma_{12}(fh_1)+\gamma_{12}(fh_2)>(A-5e)(B-4e)\,.$$

If either $N_1>4$ or $N_2>4$, the preceding procedure can be repeated using a sufficiently narrow dividing strip. Since N is finite we can conclude that $\psi^*(fh)\geqq|R'|$.

LEMMA 11. *If* $h\in H$, $B=h(Q^*)$, $G=\{p|\mu(p, f, B)\neq 0\}$, *then* $\psi^*(fh)\geqq|G|$.

Proof. Choose $e>0$. Let R_i, $i=1, 2, \cdots, N$, be a finite set of non-overlapping open (oriented) rectangles contained in G with $|\cup R_i|>|G|-e$. For each i take another such rectangle R_i' whose closure is contained in R_i and such that $|R_i'|>|R_i|-e/N$. By Lemmas 9 and 10 there exist $h_i\in H$ with $\psi^*(fh_i)\geqq|R_i'|$ and $B_m\cap B_n=0$ for $m\neq n$ where $B_i=h_i(Q^*)$. Let $J_i=h_i(Q)$. If we knew that $J_m\cap J_n=0$ the lemma would follow since we could take F to be this collection of h.

Suppose that $J_m\cap J_n\neq 0$. Since $B_m\cap B_n=0$, we may assume that $J_m\subset J_n$. Fix $w>0$ and use the method of the proof of Lemma 10 to bisect R_n into rectangles S_1 and S_2 containing rectangles S_1' and S_2' respec-

tively such that

(i) $|S_1'|+|S_2'|>|R_n'|-w$,

(ii) If K_1 and K_2 are appropriate Jordan regions for S_1 and S_2 then $K_1 \cup K_2 = J_n$ and K_1 and K_2 have no interior points in common.

In a similar manner we can subdivide S_1' and S_2'. Take j so that $|R_n|/2^j < w$. The area of a rectangle of the collection $\mathscr{S}$ into which R_n is divided will be less than w. Let $\mathscr{S}'$ be a collection of smaller rectangles, one contained in each element of $\mathscr{S}$. Let $\mathscr{K}$ be the collection of associated Jordan regions. We can take $\mathscr{S}'$ and $\mathscr{K}$ so that

(i) $|\bigcup_{S' \in \mathscr{S}'} S'|>|R_n'|-w$,

(ii) distinct elements of $\mathscr{K}$ have no interior points in common and $\bigcup_{K \in \mathscr{K}} K = J_n$.

Let $H_n=\{h \in H | \text{ range } h \in \mathscr{K}\}$. For some $h \in H_n$, $J_m \subset$ range h. Let h^* be one such h. Now put $\mathscr{H}=H_n-\{h^*\}$. Then

$$\psi^*(fh_n) \geqq \psi^*(fh_m)+\sum_{h \in \mathscr{H}} \psi^*(fh) \geqq |R_m'|+|R_n'|-w-|R_n|/2^j .$$

Thus

$$\psi^*(fh_n) \geqq |R_m'|+|R_n'| \text{ if } J_m \subset J_n .$$

In a similar manner we can conclude that if

$$h_{m_j}(Q) \subset h_n(Q) \qquad j=1, 2. \cdots, M$$

then

$$\psi^*(fh_n) \geqq \sum_{j=1}^{M} |R_{m_j}'|+|R_n'| .$$

Now let F consist of those h_m for which range h_m is not contained in range h_n for $m \neq n$. Then

$$\psi^*(f) \geqq \sum_{h \in F} \psi^*(fh) \geqq \sum |R_i'| > |G|-2e .$$

LEMMA 12. $\psi^*(f)=L(f)$

Proof. By a result of Cesari [5], similar to results of Rado and Reichelderfer [10] and Federer [6],

$$L(f)=\sup_F \sum_{h \in F} |G_h|$$

where

$$G_h=\{p | \mu(p, f, h(Q^*)) \neq 0\} .$$

Thus, by the preceding lemma,

$$\psi^*(f)\leqq L(f)=\sup_F \sum_{h\in F}|G_h|\leqq \sup_F \sum_{h\in F}\psi^*(fh)\leqq\psi^*(f).$$

THEOREM 3. $P=\psi^*=\varphi=\Phi=L$.

Proof. We saw in [11] that $L(x)=\lim_{N\to\infty} L(\pi^N x)$ and $P(x)=\lim_{N\to\infty} P(\pi^N x)$. By Lemma 9 we can conclude that $P=L$. Since $\psi^*=L=P$ for flat transformations, we have that $P(x)\leqq\psi^*(x)$ for all $x\in C$. Thus

$$L=P\leqq\psi^*\leqq\varphi\leqq\Phi\leqq L.$$

Let $\tilde{x}_G(p, q)$ be the geodetic distance between p and q, that is, $\tilde{x}_G(p, q)=\inf$ length xg for all g continuous on $[0, 1]$ into Q with $g(0)=p$ and $g(1)=q$. According to Busemann [3] an area S is *intrinsic* if $S(x)=S(y)$ whenever $\tilde{x}_G=\tilde{y}_G$. Let $C_l=\{x\in C|x$ is light$\}$. If we use the definition of Φ and the fact that x_μ is a homeomorphism if $x\in C_l$ we see that Φ, and therefore L, is intrinsic on C_l.

Choquet suggested an inequality stronger than that the author was able to prove. His inequality would be that of Theorem 1 if geodetic rather than pseudo-geodetic distances were used.

REFERENCES

1. S. Banach, *Theorie des operations lineaires*, Warsaw, 1932.
2. A. S. Besicovitch, *On two problems of Loewner*, J. London Math. Soc., **27** (1952), 141-144.
3. H. Busemann, *Intrinsic areas*, Ann. of Math., **48** (1947), 234-267.
4. L. Cesari, *Frechet surfaces of finite Lebesgue area in E_n*, Abstract 604, Bull. Amer. Math. Soc., **59** (1953), 534.
5. ———, *Sui fondamenti geometrici dell'integrale classico per l'area delle superficie in forma parametrica*, Atti della R. Accademia d'Italia, **13** (1943), 1323-1481.
6. H. Federer, *Surface area* II, Trans. Amer. Math. Soc., **55** (1944), 438-456.
7. ———, *Essential multiplicity and Lebesgue area*, Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 611-616.
8. ———, *On Lebesgue area*, Ann. of Math., **61** (1955), 289-353.
9. T. Rado, *Length and area*, Colloquium Publications Amer. Math. Soc., **30**, New York, 1948.
10. T. Rado and P. V. Reichelderfer, *A theory of absolutely continuous transformations in the plane*, Trans. Amer. Math. Soc., **49** (1941), 258-307.
11. E. Silverman, *Definitions of Lebesgue area for surfaces in metric spaces*, Riv. Mat. Univ. Parma, **2** (1951), 47-76.
12. ———, *An intrinsic property of Lebesgue area*, Riv. Mat. Univ. Parma, **2** (1951), 195-201.

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