Pacific Journal of Mathematics

QUOTIENT ALGEBRA OF A FINITE AW^*-ALGEBRA

TI YEN

Vol. 6, No. 2 December 1956

QUOTIENT ALGEBRA OF A FINITE AW*-ALGEBRA

TI YEN

1. Introduction. In a recent paper [5] Wright proves that *if A is an AW^*-algebra* [2] *having a trace and if M is a maximal ideal of A, then A/M is an AW^*-factor* (that is, *an AW^*-algebra whose center consists of complex numbers*) *having a trace.* The trace enters into his argument in the characterization [5, Theorem 3.1] of the one-to-one correspondence between maximal ideals of A and those of its center Z. This is, in turn, used to verify that A/M satisfies the countable chain condition, namely: every set of mutually orthogonal projections is at most countable, which is crucial to prove that every set of mutually orthogonal projections has a least upper bound (LUB). It is the purpose of this paper to prove the following.

THEOREM. *Let A be a finite AW^*-algebra, and M a maximal ideal of A. Then A/M is a finite AW^*-factor.*

It is not known whether a finite AW^*-factor always has a trace. Since [3] a finite AW^*-algebra of type I always has a trace, our result adds nothing new in this case, and we shall be solely concerned with algebras of type II_1.

Our terminology is that of [2]. We assume familiarity with [2] and [1] (especially [1, pp. 234–242]).

2. Maximal ideal M. We begin with a slightly sharpened version of [5, Theorem 2.5] on p-ideals. A set P of projections is called a p-ideal if

(1) P contains $e\vee f$ whenever it contains e and f

(2) P contains f whenever it contains an $e\succ f$.

It follows from (1) that $e_1\vee\cdots\vee e_n$ is in P if $e_1, \cdots, e_n$ are in P. For any set S of A let S_p denote the set of projections contained in S.

LEMMA 1. *Let A be an AW^*-algebra. The closed linear subspace M generated by a p-ideal P is an ideal with $M_p=P$. Conversely an ideal M of A is the closed linear subspace generated by the p-ideal M_p.*

Proof. Let P be a p-ideal and M the closed linear subspace generated by P. For M to be an ideal we need to prove that M contains xe for any $x\in A$ and $e\in P$. The left projection [2, p. 244] f of xe, being $\prec e$, is contained in P. Hence P contains $g=e\vee f$. $xe\in gAg\subset M$,

Received February 1, 1955.

as gAg is the closed linear subspace generated by all projections $\leqq g$.

Let M_0 denote the linear subspace algebraically generated by P; the elements of M_0 are of the form $x=\sum_{i=1}^n \lambda_i e_i$ (λ_i complex numbers, $e_i \in P$). As P contains $e_1 \vee \cdots \vee e_n$, the left and right projections of x are in P. Take an f in M_p, and an $\varepsilon>0$. There is an $x \in M_0$ with $\|f-x\| <\varepsilon$. The left projection h of fx, being $\prec$ the right projection of x, is in P. We have $h \leqq f$ and $\|f-h\|=\|(f-h)(f-fx)\| \leqq \|f-fx\|<\varepsilon$. Hence $f=h$. This proves that $M_p=P$.

Assume now that M is an ideal. M_p is [5, Lemma 2.1] a p-ideal. Let M' denote the closed linear subspace generated by M_p. We wish to prove that $M=M'$. Take $x \in M$ and $\varepsilon>0$. There is[1] [2, Lemma 2.1] a projection e, which is a multiple of x, such that $\|x-ex\|<\varepsilon$. Since $ex \in M'$ and M' is closed, $x \in M'$.

Let now A be an AW^*-algebra of type II_1, Z its center. Then [2, p. 247] A admits a dimension function D defined on A_p with values in Z. D has the following properties:

(1) $0 \leqq D(e) \leqq 1$ for every e,
(2) $D(e)=e$ if $e \in Z$,
(3) $D(e)=D(f)$ if and and only if $e \sim f$,
(4) $D(\sum e_i)=\sum D(e_i)$ if the e_i's are mutually orthogonal [1, Lemma 6.13].

Moreover, D is uniquely determined by these properties. It is an immediate consequence of (4) that given $0<\lambda<1$ there is a projection e with $D(e)=\lambda$.

Let C be a commutative AW^*-subalgebra [3] of A. C is the closed linear subspace generated by C_p. We shall extend D to a linear transformation T_C of C into Z. First define T_C on the linear combinations of projections by setting

$$T_C\Big(\sum_{i=1}^n \lambda_i e_i\Big)=\sum_{i=1}^n \lambda_i D(e_i).$$

We must show that T_C is uniquely defined, i.e., if $x=y$ then $T_C(x)= T_C(y)$. If $x=\sum_{i=1}^n \lambda_i e_i$, there are orthogonal projections $f_1, \cdots, f_m$ such that each e_i is a sum of the f's:

$$e_i=\sum_{j=1}^m \alpha_{ij} f_j \quad \text{where} \quad \alpha_{ij}=\begin{cases}1 & \text{if} \quad e_i f_j=f_j \\ 0 & \text{if} \quad e_i f_j=0.\end{cases}$$

$$x=\sum_{i=1}^n \lambda_i e_i=\sum_{j=1}^m\Big(\sum_{i=1}^n \lambda_i \alpha_{ij}\Big) f_j.$$

[1] To use [2, Lemma 2.1] we first imbed xx^* in a maximal commutative self-adjoint subalgebra of A. Working in this subalgebra we get a projection e with $\|xx^*-exx^*\|<\varepsilon^2$. Then $\|x-ex\|=\|(x-ex)(x-ex)^*\|^{1/2}=\|xx^*-exx^*\|^{1/2}<\varepsilon$.

It follows from $D(e_i)=\sum_{j=1}^{m}\alpha_{ij}D(f_j)$ that

$$T_C(\sum_{i=1}^{n}\lambda_i e_i)=T_C(\sum_{i,j}\lambda_i\alpha_{ij}f_j)\,.$$

Hence to prove the uniqueness of T_C we may restrict ourselves to the linear combinations of mutually orthogonal projection, $\sum_{i=1}^{n}\lambda_i e_i$. Moreover, as D is additive on orthogonal projections, we may assume that all the coefficients λ_i are unequal. Suppose therefore

$$x=\sum_{i=1}^{n}\lambda_i e_i=\sum_{j=1}^{m}\mu_i f_j\,,$$

where the e's and f's are mutually orthogonal and the λ's and μ's are all different. Then $xf_j=\mu_j f_j=(\sum_{i=1}^{n}\lambda_i e_i)f_j$. Since the λ's are all different, to each j there is exactly one i such that $e_i f_j=f_j$ and $\lambda_i=\mu_j$. By symmetry $e_i f_j=e_i$. Hence $\sum_{j=1}^{m}\mu_j f_j$ is merely a rearrangement of $\sum_{i=1}^{n}\lambda_i e_i$. This proves the uniqueness of T_C. If $x=\sum_{i=1}^{n}\lambda_i e_i$ where the e's are mutually orthogonal and $\lambda_i>0$, then

$$\begin{aligned}\|T_C(x)\|=\|\sum_{i=1}^{n}\lambda_i D(e_i)\|&\leqq\max\lambda_i\|\sum_{i=1}^{n}D(e_i)\|\\&\leqq\max\lambda_i=\|x\|\,.\end{aligned}$$

Hence T_C is a bounded linear operator defined on a dense subset of C, therefore can be extended to all of C. T_C is positive because D is. If $x\in A$ is normal, x can be imbedded in a maximal commutative self-adjoint subalgebra C' of A. Let C be the intersection of all such C', C and C' are AW^*-subalgebras of A. As x can be approximated within both C and C', $T_{C'}(x)=T_C(x)$. Let $T(x)$ denote their common value. T is unitarily invariant (i.e. $T(uxu^{-1})=T(x)$ for every unitary u), because if $\sum\lambda_i e_i$ is an approximation of x then $\sum\lambda_i ue_i u^{-1}$ is one of uxu^{-1} and D is unitarily invariant. T is also linear on each commutative AW^*-subalgebra of A. We shall use this T to play the role of trace.

THEOREM 1. *Let A be an AW^*-algebra of type II_1, Z its center. Let N be a maximal ideal of Z. Then the unique maximal ideal M of A containing N is that generated by the p-ideal P consisting of all projections e with $T(e)\in N$. Or, equivalently, M is the set of elements x with $T(x^*x)\in N$.*

Proof. Consider Z as functions on its structure space of maximal ideals. Then N contains $b\geqq 0$ whenever it contains $a\geqq b$; therefore P

satisfies (2) of a p-ideal. (1) follows from $T(e\vee f)=T(e)+T(f)-T(e\wedge f)$ because [2, Theorem 5.4] $e\vee f-e\sim f-e\wedge f$. Thus M is an ideal by Lemma 1. Moreover $M\neq A$ as $1\notin P$. Let M' be a maximal ideal containing M. Then M is maximal if and only if $M_p=M'_p$. Take an $e\in M'_p$. If $e\notin P$ then $T(e)\equiv\lambda$ (mod N) with $\lambda>0$. Choose an integer n and a projection f such that $T(f)=1/n<\lambda$. f is a simple projection with central carrier 1, that is, there exist mutually orthogonal projections $f=f_1, \cdots, f_n$ with $f_1+\cdots+f_n=1$. Compare e and f; there exists [2, Theorem 5.6] a central projection g with $ge\succ gf$ and $(1-g)e\prec(1-g)f$. Then gf and, therefore, g are in M'. As

$$0\leqq T((1-g)f)-T((1-g)e)\equiv(1/n-\lambda)(1-g) \pmod N$$

and $1/n-\lambda<0$, $1-g$ is also in M'. Hence $1\in M'$, contradicting the choice of M'. Hence $e\in P$ and $M=M'$ is maximal. The uniqueness follows from [5, Theorem 2.5].

Finally we assert that $x\in M$ if and only if $T(x^*x)\in N$. It is well known that $x\in M$ if and only if $x^*x\in M$. Thus we need only to prove that $0<x\in M$ if and only if $T(x)\in N$. Suppose $0<x\in M$. Given $\varepsilon>0$ there is a projection e, which is a multiple of x, such that $||x-ex||<\varepsilon$. $T(e)\in N$ because $e\in M$. Then $T(ex)\leqq||x||T(e)$ is also in N. Therefore $T(x)\in N$. Conversely, assume $T(x)\in N$, $x>0$. Imbed x in a maximal communtative subalgebra C. Given $\varepsilon>0$ there are projections $e_1, \cdots, e_n$ in C and positive real numbers $\lambda_1, \cdots, \lambda_n$ such that

$$0\leqq x-\sum_{i=1}^{n}\lambda_i e_i<\varepsilon .$$

$T(e_i)\in N$ $(i=1, \cdots, n)$ for $\lambda_i T(e_i)\leqq T(x)$. Hence $e_i\in M$ $(i=1, \cdots, n)$, and $x\in M$.

3. The quotient algebra A/M.

LEMMA 2. *Let $\bar{e}_1, \bar{e}_2, \cdots$ be a countable set of mutually orthogonal projections in A/M. There exist mutually orthogonal projections $e_1, e_2, \cdots$ in A such that $\bar{e}_n=e_n+M$, $(n=1, 2, \cdots)$.*

Proof. By [5, Theorem 3.2] we can find a projection e_1 representing $\bar{e}_1$. If x is a representative of $\bar{e}_2$, so is $(1-e_1)x(1-e_1)$. Hence the proof of [5, Theorem 3.2] shows that $\bar{e}_2$ admits a projection representative e_2 orthogonal to e_1. A straight forward induction yields Lemma 2.

LEMMA 3. $e\equiv f$ (mod M) *if and only if* $T(e)\equiv T(f)\equiv T(efe)\equiv T(fef)$ (mod $N=M\cap Z$). *Consequently A/M satisfies the countable chain condition.*

Proof. If $e\equiv f$ (mod M) then $0\leqq e-efe\in M$. Hence $T(e)\equiv T(efe)$

(mod N). Similarly $T(f)\equiv T(fef)$ (mod N). But [6, Corollary to Lemma 2.1] $efe=u(fef)u^*$ for some unitary u. Hence $T(e)\equiv T(f)\equiv T(efe)\equiv T(fef)$ (mod N). Conversely, if $T(e)\equiv T(f)\equiv T(efe)\equiv T(fef)$, then $e\equiv efe$ and $f\equiv fef$ (mod M). As $(e-fe)^*(e-fe)=e-efe\equiv 0$ (mod M) we have, $e\equiv fe$ and $e\equiv f$ (mod M). The above result permits us to define an "additive" function $\bar{D}$ on the projections of A/M by setting $\bar{D}(\bar{e})$ to be the common value of $T(e)$ at N where $e+M=\bar{e}$. $D(\bar{e})\neq 0$ if $\bar{e}\neq 0$. Hence A/M satisfies the countable chain condition.

LEMMA 4. *Any set of mutually orthogonal projections in A/M has a least upper bound.*

Proof. By Lemma 3 such a set is countable. Let $\bar{e}_1, \bar{e}_2, \cdots$ be mutually orthogonal projections. We first prove a sharpened version of Lemma 2:

(*) *there exist mutually orthogonal projections $e_1, e_2, \cdots$ representing $\bar{e}_1, \bar{e}_2, \cdots$, respectively, such that $T(e_n)=\bar{D}(\bar{e}_n)$ for $n=1, 2, \cdots$.*

Let f_1 be a projection representing $\bar{e}_1$ and g_1 a projection with $T(g_1)=\bar{D}(\bar{e}_1)$. Compare f_1 and g_1; there is a central projection h_1 such that $h_1g_1\succ h_1f_1$ and $(1-h_1)g_1\prec(1-h_1)f_1$. There are projections e_1' and e_1'' such that $h_1g_1\sim e_1'\geqq h_1f_1$ and $(1-h_1)g_1\sim e_1''\leqq(1-h_1)f_1$. From

$$0\leqq T(e_1'-h_1f_1)=T(e_1')-T(h_1f_1)=h_1(T(g_1)-T(f_1))\equiv 0 \pmod{M\cap Z}$$

it follows that $e_1'\equiv h_1f_1$ (mod M). Similarly $e_1''\equiv(1-h_1)f_1$ (mod M). Hence $e_1=e_1'+e_1''\equiv f_1$ (mod M) and $T(e_1)=\bar{D}(\bar{e}_1)$. Next let f_2, g_2 be projections $<1-e_1$ and be such that $\bar{e}_2=f_2+M$ and $T(g_2)=\bar{D}(\bar{e}_2)$. Repeat the argument applied to f_1 and g_1 we can find the desired $e_2=e_2'+e_2''$; (Since $1-e_1>h_2g_2$ and h_2g_2, e_2' can be taken inside $1-e_1$. So is e_2'', therefore $e_1e_2=0$). A simple induction yields (*).

Let $e=\mathrm{LUB}_n e_n$. We wish to prove that $\bar{e}=e+M$ is the LUB of $\bar{e}_n$. Or, equivalently, $\bar{f}\bar{e}=0$ if $\bar{f}\bar{e}_n=0$ for all n. Choose representatives f, f_1, f_2, $\cdots$ of $\bar{f}$ so that $f_ne_m=0$ for all $m\leqq n$. Consider efe. We have

$$efe\equiv ef_nfe\equiv g_nf_nfe\equiv g_nfe\equiv g_nefe \pmod M$$

where $g_n=e-e_1-\cdots-e_n$. Imbed efe in a maximal commutative self-adjoint subalgebra C and apply [2, Lemma 2.1] which states (in C): given $\varepsilon>0$ there exists a projection h, which is a multiple of efe, such that $\|efe-hefe\|<\varepsilon$. efe is in M if all such h's are.

$$h=efey\equiv g_nefey\equiv g_nh \pmod M.$$

Hence $h\equiv hg_nh$ (mod M) and $T(h)\equiv T(hg_nh)$ (mod $M\cap Z$). But $T(hg_nh)=T(g_nhg_n)\leqq T(g_n)$ and $T(g_n)$ can be made arbitrarily small when n is large enough. Hence $T(h)\equiv 0 \pmod{M \cap Z}$ and $h \in M$. This completes the proof.

THEOREM 2. *A/M is a finite AW^*-factor.*

Proof. To show that A/M is an AW^*-algebra we need to verify two things: (1) every set of mutually orthogonal projections has a LUB and (2) any maximal commutative self-adjoint subalgebra is generated by its projections. (1) is the context of Lemma 4. (2) is equivalent to that every element of A/M has a left and a right projection, or the left (right) annihilator of every element is a principal left (right) ideal generated by a projection. This last can be easily verified following the argument used in [2, Lemmas 2.1, 2.2, and Theorem 2.3]. As A/M is simple it must be factorial. It remains to prove the finitness. This will be the case if we show that $\overline{D}(\overline{e})=\overline{D}(\overline{f})$ if $\overline{e}\sim\overline{f}$, since $\overline{D}$ is non-zero on non-zero projections. This is a consequence of the following lemma, a special case of [4, Proposition 2] if A is a W^*-algebra.

LEMMA 5. *Suppose $\overline{e}\sim\overline{f}$. Then there exists equivalent projections e, f representing $\overline{e}$, $\overline{f}$ respectively.*

Proof. Let $\overline{x}^*\overline{x}=\overline{e}$ and $\overline{x}\overline{x}^*=\overline{f}$. Let x, e_1 and f_1 respectively be the representative of $\overline{x}$, $\overline{e}$ and $\overline{f}$. Then

$$e_1\equiv x^*x\equiv e_1x^*xe_1\equiv e_1x^*(xx^*)xe_1\equiv e_1x^*f_1xe_1\equiv(e_1x^*f_1)(f_1xe_1)\ ,$$

and

$$f_1\equiv xx^*\equiv f_1xx^*f_1\equiv(f_1xe_1)(ex^*f_1)\ .$$

Let e be the left projection of $e_1x^*f_1$ and f the right projection of $e_1x^*f_1$. e and f are the desired projections, for

$$e=ee_1\equiv e(e_1x^*f_1)(f_1xe_1)\equiv(e_1x^*f_1)(f_1xe_1)\equiv e_1$$

and, similarly, $f\equiv f_1$.

REMARK. If an AW^*-factor always possesses a trace, then any AW^*-algebra of type II_1 will admit a trace, for $T(x+y)-T(x)-T(y)$ takes the value 0 at every maximal ideal of Z.

REFERENCES

1. J. Dixmier, *Les anneaux d'opérateurs de classe finie*, Ann. Sci. École Norm. Sup., **66** (1949), 209-261.
2. I. Kaplansky, *Projections in Banach algebras*, Ann. of Math., **53** (1951), 235-249.

3. ———, *Algebras of type I*, Ann. of Math., **56** (1952), 460-472.
4. Y. Misonou, *On a weakly central operator algebra*, Tôhoku Math. J., **4** (1952), 194-202.
5. F. B. Wright, *A reduction for algebras of finite type*, Ann. of Math., **60** (1954), 560-570.
6. T. Yen, *Trace on finite AW*-algebras*, Duke Math. J., **22** (1955), 207-222.

LEHIGH UNIVERSITY

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. Manuscripts intended for the outgoing editors should be sent to their successors. All other communications to the editors should be addressed to the managing editor, Alfred Horn at the University of California, Los Angeles 24, California.

50 reprints of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, c/o University of California Press, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics

Vol. 6, No. 2 December, 1956