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ON EMBEDDING UNIFORM AND TOPOLOGICAL SPACES

RICHARD ARENS AND JAMES EELLS, JR.

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In this note we prove the following.

THEOREM. *Every space with separated uniform structure can be embedded as a closed subset of a separated convex linear space.*

Every metric space can be isometrically embedded as a closed subset of a normed linear space.

These statements follow at once from the theorem of § 3. Such an embedding is known for any *complete* metric space; and it is also known that any metric space is isometric which a relatively closed subset of a convex subset of a Banach space.

We also describe an embedding of an arbitrary T_1 space as a closed subset of a special homogeneous space.

1. Preliminaries.

(A) A *semi-metric* on a set X is a real non-negative function ρ on $X \times X$ such that $\rho(x, x) = 0$, $\rho(x, y) = \rho(y, x)$, and $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$. A semi-metric is a metric if and only if $\rho(x, y) = 0$ implies $x = y$.

A collection of semi-metrics $(\rho_\alpha)_{\alpha \in A}$ on X indexed by a set A defines a uniform structure (and a topology) on X , generated by sets $U_{\alpha a} = \{(x, y) : \rho_\alpha(x, y) < a\}$, where $a > 0$ and $\alpha \in A$. Conversely, every uniform structure can be defined by a family of semi-metrics; see Bourbaki [1]. We will say that the uniform structure is *separated* if for every pair $x, y \in X$ there is a ρ_α such that $\rho_\alpha(x, y) \neq 0$.

(B) If X is a real linear space, a *semi-norm* on X is a real non-negative function s on X such that $s(\lambda x) = |\lambda|s(x)$ and $s(x + y) \leq s(x) + s(y)$ for all $x, y \in X$ and for all real numbers λ . A semi-norm is a norm if and only if $s(x) = 0$ implies $x = 0$.

A collection of semi-norms $(s_\alpha)_{\alpha \in A}$ on X indexed by a set A defines a (locally) convex topology (and a uniform structure) compatible with the algebraic operations in X . Conversely, every convex topology can be described by a family of semi-norms; see Bourbaki [2]. We will say that the convex topology is *separated* if for every $x \neq 0$ in X there is an s_α such that $s_\alpha(x) \neq 0$.

(C) **REMARK.** Let X and X' be two sets with uniform structures

given by the semi-metrics $(\rho_\alpha)_{\alpha \in A}$ and $(\rho'_\alpha)_{\alpha \in A}$ indexed by the same set A . If $\phi: X \rightarrow X$ is a one-one correspondence such that for all $\alpha \in A$ we have $\rho_\alpha(x, y) = \rho'_\alpha(\phi(x), \phi(y))$, then ϕ preserves the uniform structure and topology.

2. The space of molecules.

(A) A *molecule* of a set X is a real-valued function m on X which is zero except (perhaps) at a finite number of points x_1, \dots, x_k of X and which satisfies $\sum_{i=1}^k m(x_i) = 0$. Setting $\lambda_i = m(x_i)$, we will represent m as a linear combination $m = \sum_i \lambda_i x_i$ with $\sum_i \lambda_i = 0$. The totality of molecules forms a real linear space $M(X)$ with algebraic operations defined pointwise.

(B) Suppose that X has a uniform structure defined by the semi-metrics $(\rho_\alpha)_{\alpha \in A}$. Then for each $\alpha \in A$ we define the semi-norms s_α on $M(X)$ by

$$(1) \quad s_\alpha(m) = \inf \left\{ \sum_j |\mu_j| \rho_\alpha(y_j, z_j) \right\},$$

the infimum being taken over all representations of $m = \sum_i \lambda_i x_i$ as $m = \sum_j \mu_j (y_j - z_j)$; the condition $\sum_i \lambda_i = 0$ insures that such representations of m do exist.

It follows from the definition (1) that for all $x, y \in X$ and for any $\alpha \in A$,

$$(2) \quad s_\alpha(x - y) \leq \rho_\alpha(x, y).$$

In fact, it is easily seen that s_α is the largest semi-norm on $M(X)$ satisfying (2); that is, given any such semi-norm s , we have $s(m) \leq s_\alpha(m)$ for all $m \in M(X)$.

(C) Let us fix a "base point" $x_0 \in X$. We then note that the set of all elements of the form $x - x_0$ with $x_0 \neq x \in X$ forms a *base* for the linear space $M(X)$. Also, any linear functional F on $M(X)$ defines a real function f on X by

$$(3) \quad f(x) = F(x - x_0);$$

conversely, any real function f on X such that $f(x_0) = 0$ defines a linear functional F on $M(X)$ by $F(m) = \sum_i \lambda_i f(x_i)$, and the relation (3) holds.

With that identification of functionals, we have the following.

PROPOSITION. *The linear functionals F on $M(X)$ which are continuous in the topology of the semi-norms $(s_\alpha)_{\alpha \in A}$ correspond to those real*

functions f on X vanishing at x_0 and satisfying

$$(4) \quad |f(x) - f(y)| \leq K\rho_\alpha(x, y)$$

for some constant K and semi-norm ρ_α , both depending on f . If X is a metric space then the continuous linear functionals correspond to the Lipschitz functions on X vanishing at x_0 .

Note that the functions f are uniformly continuous on X .

Proof. The functional F is continuous on $M(X)$ if and only if it is bounded (by some K) for some semi-norm s_α ; thus if F is continuous and defines f as in (3), $|f(x) - f(y)| = |F(x - y)| \leq Ks_\alpha(x - y) \leq K\rho_\alpha(x, y)$ by (2). Conversely, if f is a function such that $f(x_0) = 0$ and which satisfies (4) for some K and ρ_α , then for any $m \in M(X)$ and $\varepsilon > 0$ we can choose a representation $m = \sum_j \mu_j(y_j - z_j)$ such that $\sum_j |\mu_j| \rho_\alpha(y_j, z_j) \leq s_\alpha(m) + \varepsilon$. Then

$$|F(m)| \leq \sum_j |\mu_j| |f(y_j) - f(z_j)| \leq K \sum_j |\mu_j| \rho_\alpha(y_j, z_j) \leq K[s_\alpha(m) + \varepsilon].$$

Since that is true for all $\varepsilon > 0$, we have $F(m) \leq Ks_\alpha(m)$, whence F is continuous on $M(X)$.

Relation (2) is, in fact, always an equality:

PROPOSITION. For any $x, y \in X$ and $\alpha \in A$ we have

$$(5) \quad s_\alpha(x - y) = \rho_\alpha(x, y).$$

Proof. The function $f(z) = \rho_\alpha(z, y)$ clearly satisfies $f(y) = 0$ and also (4) with $K = 1$; let F be the corresponding linear functional with x_0 replaced by y . Given any representation of the molecule $x - y = \sum_j \mu_j(y_j - z_j)$, we have $\rho_\alpha(x, y) = f(x) = F(x - y) = \sum_j \mu_j F(y_j - z_j)$, whence $\rho_\alpha(x, y) \leq \sum_j |\mu_j| |\rho_\alpha(y_j, y) - \rho_\alpha(z_j, y)| \leq \sum_j |\mu_j| \rho_\alpha(y_j, z_j)$. Taking the infimum over all such representations of $x - y$, we have $\rho_\alpha(x, y) \leq s_\alpha(x - y)$, proving (5).

The following two statements (and their converses) are easy consequences of (5).

PROPOSITION. If the uniform structure on X is separated, then so is the induced convex topology on $M(X)$.

If the uniform structure on X is given by a single metric (or is metrizable), then the induced convex topology on $M(X)$ is normed (is normable).

(D) REMARK. There are many interesting variants of the semi-norms (1). For instance, suppose we let $\tilde{M}(X)$ denote the linear space of all $m = \sum_i \lambda_i x_i$, with no additional conditions on the λ_i ; then by choosing a

base point $x_0 \in X$ we can define the semi-norm \tilde{s}_α corresponding to the semi-metric ρ_α by

$$(6) \quad \tilde{s}_\alpha(m) = \inf \left\{ \sum_k |\nu_k| \rho_\alpha(w_k, x_0) + \sum_j |\mu_j| \rho_\alpha(y_j, z_j) \right\},$$

the infimum being taken over all representations of m as a sum $m = m_1 + m_2$, where $m_1 = \sum_k \nu_k w_k$ and $m_2 = \sum_j \mu_j (y_j - z_j)$. It can be shown that for all $\alpha \in A$ the semi-norm (6) is equal to the semi-norm (1) on the subspace $M(X)$ of $\tilde{M}(X)$.

Semi-norms related to those of type (6) have been studied (in quite a different connection) by H. Whitney; see [4, p. 249].

3. Embedding a uniform space. Take a base point $x_0 \in X$, and then define the transformation $\phi: X \rightarrow M(X)$ by $\phi(x) = x - x_0$. Then ϕ is clearly one-one, and by (5) we have $s_\alpha(\phi(x)) = \rho_\alpha(x, x_0)$.

THEOREM. *The transformation ϕ is a uniformly bi-continuous homeomorphism of X into $M(X)$. If the uniform structure of X is separated, then ϕ maps X onto a closed subset of $M(X)$.*

If X is a metric space, then ϕ is an isometric map of X onto a closed subset of $M(X)$.

Proof. As we have remarked in §1C, such a ϕ is a uniformly continuous homeomorphism and an isometry if X is metric.

Supposing that the uniform structure of X is separated, we will now show that $\phi(X)$ is closed in $M(X)$. Given $m \in M(X)$ not belonging to $\phi(X)$, we will construct a neighborhood of m not meeting $\phi(X)$. Suppose first of all that m has the form $\lambda(y - x)$; since $m \notin \phi(X)$, we have $y \neq z$, $\lambda \neq 0$.

In case $z \neq x_0$, there is a semi-metric ρ and a constant $a > 0$ such that $\rho(y, z) \geq a$, $\rho(x_0, z) \geq a$; in fact, ρ can be defined as the sum of two suitably chosen semi-metrics of the separating family $(\rho_\alpha)_{\alpha \in A}$. Let s_ρ be the semi-norm defined by (1) using ρ . Set $f(x) = \max\{a - \rho(x, z), 0\}$, and let F be the corresponding continuous linear functional as in §2C; we note that $|F(n)| \leq s_\rho(n)$ for all $n \in M(X)$. Then for any $m_0 = x - x_0$ in $\phi(X)$, we have

$$F(m_0 - m) = f(x) - f(x_0) - |\lambda|f(y) + |\lambda|f(z) = f(x) + |\lambda|a,$$

whence $s_\rho(m_0 - m) \geq |\lambda|a$.

In case $z = x_0$, we have $\lambda \neq 1$ since $m \notin \phi(X)$. As before, take a semi-metric ρ such that $\rho(y, x_0) > 2a$. Then for any $m_0 = x - x_0$ in $\phi(X)$, either $\rho(x, x_0) > a$ or $\rho(x, y) > a$. In the former event define $f(z) = \max\{a - \rho(z, x_0), 0\}$; then $s_\rho(m_0 - m) \geq |F(m_0 - m)| = |\lambda| - 1|a$. In the latter

event define $f(z) = \max\{a - \rho(z, y), 0\}$; then $s_\rho(m_0 - m) \geq |\lambda|$.

Thus in any case $s_\rho(m_0 - m)$ exceeds some positive constant independent of m_0 ; thus if $m = \lambda(y - z) \notin \phi(X)$, then m has a neighborhood not meeting $\phi(X)$. In general, let $m = \sum_{i=1}^k \lambda_i x_i$ with $k > 2$; we can suppose that the x_i are distinct and that $|\lambda_i| \geq b > 0$ for all i . As usual, take a semi-norm ρ on X such that $\rho(x_i, x_j) \geq 2c$ for some $c > 0$ and for all pairs i, j with $i \neq j$. Now suppose $m' = \sum_j \lambda'_j x'_j$ is a molecule with less than k points. Then there is an i such that $\rho(x'_j, x_i) \geq c$ for all j . Let $f(x) = \max\{c - \rho(x, x_i), 0\}$. Then $s_\rho(m - m') \geq |F(m) - F(m')| = |F(m)| \geq bc$. Thus if m' satisfies $s_\rho(m - m') < bc$, then m' has at least as many points as m . Since every element of $\phi(X)$ has the form $x - x_0$, it follows that we can construct a neighborhood of $m = \sum_{i=1}^k \lambda_i x_i$ which does not intersect $\phi(X)$. The proof of the theorem is now complete.

4. Embedding topological spaces.

(A) M. Shimrat [3] has shown that every topological space X can be embedded in a homogeneous space X^* (a space X^* is *homogeneous* if for every two points $x, y \in X^*$ there is a homeomorphism h of X^* into itself such that $h(x) = y$); furthermore, if X is T_1 , then so is X^* and the image of X is closed in X^* . In the following theorem we shall show that any T_1 space X can be embedded as a closed subset of a T_1 space X^* such that for any two points $x, y \in X^*$ there is a homeomorphism of period two interchanging the points.

However, Shimrat manages to prove that if X has stronger separation properties (for example, X is Hausdorff, regular, normal), then X^* has these same properties. No such conclusion can be drawn for our X^* . Shimrat also produces a variant construction embedding a metric space X as a closed set in a metrically homogeneous space X^* ; his X^* (as he points out) is not necessarily locally connected, whereas our embedding space $X^* = M(X)$ in § 3 is (being a normed linear space).

(B) For any set X let X^* denote the Boolean ring of all finite subsets m of X ; the void set is denoted by 0, and $m + n$ is the symmetric difference of m and n (whence $\{x\} + \{x\} = 0$).

We have a natural one-one transformation $\phi: X \rightarrow X^*$ defined by $\phi(x) = \{x\}$.

THEOREM. *Let X be a T_1 space. Then we can define a topology on X^* for which the additive translations are homeomorphism, and ϕ maps X homeomorphically onto a closed subset of X^* .*

We do not assert that X^* is a topological group under addition. We will show that the transformation $X^* \times X^* \rightarrow X^*$ defined by $(m, n) \rightarrow m+n$ is continuous *in each variable separately*, not that it is simultaneously continuous.

Proof. For every open cover \mathcal{V} of X we define (\mathcal{V}) as the collection of those sets $m \in X^*$ whose points can be listed $x_1, x_2, \dots, x_{2j-1}, x_{2j}$, where the "partners" x_{2j-1}, x_{2j} always lie in one element $V_j \in \mathcal{V}$. Then $0 \in (\mathcal{V})$, and if \mathcal{U} is a common refinement of the open covers \mathcal{V}, \mathcal{W} , we have $(\mathcal{U}) \subset (\mathcal{V}) \cap (\mathcal{W})$.

We take the sets $m+(\mathcal{V})$ as a fundamental system of neighborhoods of $m \in X^*$, and will show that for any open cover \mathcal{V} and any $m \in (\mathcal{V})$ there is an open cover \mathcal{U} such that $n \in (\mathcal{U})$ implies $m+n \in (\mathcal{V})$. It will follow

- 1) that these neighborhoods define a unique topology on X^* , and
- 2) that translation by m is a homeomorphism.

We construct \mathcal{U} as follows: For each $V \in \mathcal{V}$, let V_0 denote the set of points of V not in m ; for each $x_i \in m \cap V$ such that its partner is also in V , we define $U_i = V_0 \cup \{x_i\}$. Thus each U_i is defined by removing a finite number of points from V , and since points of X are closed, it follows that U_i is open. We define the open cover \mathcal{U} of X as the collection of all possible such U_i constructed from all $V \in \mathcal{V}$.

Now take any $n = \{y_1, y_2, \dots, y_{2j-1}, y_{2j}\} \in (\mathcal{U})$, where y_{2j-1}, y_{2j} always belong to some $U \in (\mathcal{U})$; let us suppose all the y 's are distinct. We will arrange $m+n$ into a set of partners which share elements of \mathcal{V} , thus showing that $m+n \in (\mathcal{V})$. If $y_{2j-1}, y_{2j} \in U_i \in \mathcal{U}$, then at most one of them belongs to m , and that one (if any) must be x_i ; we then pair the other y with the partner of x_i , forming a pair not appearing in $m+n$. If neither y belongs to m , we can make them partners of each other. Elements of m not affected by these transactions shall remain partners. That completes the arrangement of $m+n$.

To show that this topology on X^* is itself T_1 , take any $m \neq 0$, and let \mathcal{V} be the set of complements of the sets $m+\{x\}$, where x varies over m . Then \mathcal{V} is an open cover of X , and (\mathcal{V}) is a neighborhood of 0 in X^* which does not contain m .

We will now prove that the map $\phi(x) = \{x\}$ is a homeomorphism of X onto X^* . Given $x \in X$ and a neighborhood (\mathcal{V}) of $\{x\}$, we know there is an open set V such that $x \in V \in \mathcal{V}$; then for any $y \in V$ we have $\{y\} = \{x\} + (\mathcal{V})$, proving that ϕ is continuous. On the other hand, given $\{x\} \in X^*$ and a neighborhood V of x , take the open cover $\mathcal{V} = \{V, X + \{x\}\}$. Then for any $\{y\} \in \{x\} + (\mathcal{V})$, we have $y \in V$ since $x, y \in X + \{x\}$ is impossible; that is, the mapping ϕ^{-1} is continuous.

Finally we will show that $\phi(X)$ is closed in X^* . Take any m with

more than one element, and as above let \mathcal{V} be the set of complements of the sets $m + \{x\}$, where x varies over the elements of m . Then $m + (\mathcal{V})$ does not intersect $\phi(X)$, for if $\{x\} + m \in (\mathcal{V})$, then x has a partner y in m ; that is impossible, for no two elements of m lie in the same $V \in \mathcal{V}$. The proof of the theorem is now complete.

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