NON-RECURRENT RANDOM WALKS

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Introduction and Summary. Let \( \{X_i\} \) \( i = 1, 2, \cdots \) be a sequence of independent and identically distributed integral valued random variables such that \( 1 \) is the absolute value of the greatest common divisor of all values of \( x \) for which \( P(X_i = x) > 0 \). Define

\[
S_n = \sum_{i=1}^{n} X_i.
\]

Chung and Fuchs [5] showed that if \( x \) is any integer, \( S_n = x \) infinitely often or finitely often with probability 1 according as \( E X_i = 0 \) or \( \neq 0 \), provided that \( E|X_i| < \infty \). Let \( 0 < \mathbb{E}X_i < \infty \), and \( A \) denote a set of integers containing an infinite number of positive integers. It will be shown that any such set \( A \) will be visited infinitely often with probability 1 by the sequence \( \{S_n\} \) \( n = 1, 2, \cdots \). Conditions are given so that similar results hold for the case where \( X_i \) has a continuous distribution and the set \( A \) is a Lebesgue measurable set whose intersection with the positive real numbers has infinite Lebesgue measure.

A Theorem about Markov Chains. Let \( \{Z_n\}, n = 0, 1, \cdots \) denote a Markov chain with stationary transition probabilities where each \( Z_n \) takes on values in an abstract state space \( X \). The distribution of \( Z_0 \) is given but arbitrary. Let \( \Omega \) denote the space of all possible sample sequences \( w \), \( P \) the probability measure over \( \Omega \) and \( P(\cdot | \cdot) \) the conditional probability. The following theorem appears in [4].

**Theorem 1.** Let \( A \) be any event in \( X \). A sufficient condition that

\[
P(Z_n \in A \text{ infinitely often}) = 1
\]

is

\[
\inf_{z \in X} P(Z_n \in A \text{ for some } n | Z_0 = z) > 0.
\]

Since [4] is not readily accessible, we shall prove the theorem here.

*Proof.* We have with probability 1 that for \( j \geq N \)
using the Markovian and stationarity properties. As \( j \to \infty \) the left member of (3) approaches with probability 1 the characteristic function \( b_N \) of the event

\[
B_N = \{Z_n \in A \text{ for some } n \geq N \}
\]

(see Doob [8, p. 332]). The right member of (3) is bounded below by a positive number on account of (2). Hence \( b_N = 1 \) with probability 1; that is, \( P(B_N) = 1 \). This being true for all \( N \) we have

\[
P(\lim_{N \to \infty} B_N) = \lim_{N \to \infty} P(B_N) = 1.
\]

But \( B_N \) is the event that \( Z_n \in A \) infinitely often. This proves the theorem.

If \( X \) has only a denumerable number of states and if all the states belong to the same class (that is, for every pair of states \( i \) and \( j \) there exists integers \( n_1 \) and \( n_2 \) such that \( P(Z_{n_1} = j | Z_0 = i)P(Z_{n_2} = i | Z_0 = j) > 0 \)) it can be easily seen that (2) is both a necessary and sufficient condition for (1). In fact, the probability in (2) must be 1 for all states \( z \).\(^3\)

**Sums of lattice random variables.** Let \( \{X_i\} \ i = 1, 2, \ldots \) be a sequence of independent and identically distributed integral valued random variables such that 1 is the absolute value of the greatest common divisor of all values of \( x \) for which \( P(X_i = x) > 0 \). Consider the sequence \( \{S_n\} \ n = 0, 1, \ldots \), where we set \( S_0 = 0 \) with probability 1 and

\[
S_n = S_0 + \sum_{i=1}^{n} X_i.
\]

The sequence \( \{S_n\} \) is then a Markov chain with stationary transition probabilities and a denumerable state space. Because the transition probabilities are stationary, we shall simply write

\[
P(S_{n+m} = i | S_n = j) = P(S_m = i | S_0 = j)
\]

even though \( S_0 = 0 \) with probability 1.

We now state as lemmas some known results to be used below.

**Lemma 1.** Let \( \{Z_n\} \ n = 0, 1, \ldots \) be a Markov chain with a denumerable state space. If \( \sum_{n=1}^{\infty} P(Z_n = j | Z_0 = i) < \infty \) for all \( i \) and \( j \), then

\(^3\) We are indebted to J. Wolfowitz for this remark.
(4) \[ P(Z_n=j \text{ for some } n \ | Z_0=i) = \frac{\sum_{n=1}^{\infty} P(Z_n=j \ | Z_0=i)}{1 + \sum_{n=1}^{\infty} P(Z_n=j \ | Z_0=j)} . \]

When \( EX_i = \mu > 0 \), a result of Chung and Fuchs [5] implies that

(5) \[ \sum_{n=1}^{\infty} P(S_n=j \ | S_0=i) < \infty \]

for all \( i \) and \( j \). Therefore, on replacing \( Z_n \) by \( S_n \) in (4) and noting that \( P(S_n=j \ | S_0=j) = P(S_n=0 \ | S_0=0) \) we have

(4') \[ P(S_n=j \text{ for some } n \ | S_0=i) = \frac{\sum_{n=1}^{\infty} P(S_n=j \ | S_0=i)}{1 + \sum_{n=1}^{\infty} P(S_n=0 \ | S_0=0)} \]

Lemma 1 is a special case of a relation given by Doeblin [7] (see Chung [3]). However, we shall sketch a direct proof.

**Proof.** We define \( P(Z_0=j \ | Z_0=j)=1 \). Then we have

(6) \[ P(Z_n=j \ | Z_0=i) = \sum_{m=1}^{\infty} P(Z_m=j, Z_r \neq j \text{ for } 1 \leq r < m \ | Z_0=i) P(Z_n=j \ | Z_0=i, Z_0=j) \]

On summing over \( n \) in (6) and interchanging summations on the right we get

(7) \[ \sum_{n=1}^{\infty} P(Z_n=j \ | Z_0=i) = \sum_{m=1}^{\infty} P(Z_m=j, Z_r \neq j \text{ for } 1 \leq r < m \ | Z_0=i) (1 + \sum_{n=1}^{\infty} P(Z_n=j \ | Z_0=i)) \]

the relation (4).

**Lemma 2.** If \( EX_i = \mu > 0 \), then

(8) \[ \lim_{j \to \infty} \sum_{n=1}^{\infty} P(S_n=j \ | S_0=i) = \frac{1}{\mu} > 0, \quad \mu < \infty \]

\[ = 0, \quad \mu = +\infty \ . \]

Lemma 2 is due to Chung and Wolfowitz [6]. We now prove the following.
**THEOREM 2.** (i) If $0 < \mu < \infty$ and $A$ is any set containing an infinite number of positive integers, then $S_n \in A$ infinitely often with probability 1.

(ii) If $EX_i = +\infty$, then there exists a set $A$ containing an infinite number of positive integers such that $S_n \in A$ only finitely often with probability 1.

*Proof of (i).* Since $0 < \mu < \infty$, by (8) there exists a constant $c > 0$, independent of $i$, and an integer $J(i)$ such that for all $j > J(i)$

\[
\sum_{n=1}^{\infty} P(S_n = j | S_0 = i) > c.
\]

Therefore by (4') and (5)

\[
P(S_n = j \text{ for some } n | S_0 = i) > \frac{c}{1 + c'}, \quad j > J(i)
\]

where $c' = \sum_{n=1}^{\infty} P(S_n = 0 | S_0 = 0) < \infty$. Since $A$ contains infinitely many positive integers, it always contains an integer greater than $J(i)$ for every $i$. Therefore (2) holds and part (i) of Theorem 2 follows from Theorem 1.

*Proof of (ii).* If $\mu = +\infty$, then from (8) there exists an increasing subsequence $\{i_1, i_2, \ldots\}$ of positive integers such that

\[
\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} P(S_n = i_j | S_0 = 0) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} P(S_n = i_j | S_0 = 0) < \infty.
\]

Let $A = \{i_j\}$. Now (11) is the expected number of $n$ such that $S_n \in A$. Since this expectation is finite it follows that the number of $n$ such that $S_n \in A$ is finite with probability 1. This completes the proof of the theorem.

**Random variables with continuous distribution functions.** Consider now a sequence $\{X_i\}$ $i = 1, 2, \ldots$ of independent, identically distributed random variables possessing a common density function $f(x)$. Again let $\{S_n\}$ $n = 0, 1, \ldots$ denote the cumulative sums $S_n = S_0 + \sum_{i=1}^{n} X_i$ where $S_0 = 0$ with probability 1. Our previous remark pertaining to the notation $P(\cdots | S_0 = x)$ applies here also. Suppose $EX_i = \mu > 0$. Then a result of Chung and Fuchs [5] implies that $H(x) = \sum_{n=1}^{\infty} P(S_n \leq x) < \infty$ for all $x$. Since $H(x)$ is non-decreasing, $H'(x)$ exists everywhere except on a set $N_0$ of Lebesque measure zero. Let
We shall say that \( f(x) \) satisfies condition I if there exist constants \( K_1 \) and \( K_2 \) such that
\[
0 < K_1 \leq \lim_{x \to -\infty} h(x) \leq \lim_{x \to -\infty} h(x) \leq K_2 < \infty
\]
and if
\[
\lim_{x \to -\infty} h(x) = 0
\]
The behavior of \( h(x) \) for large \( |x| \) has been investigated in various papers on renewal theory. Smith [10], for example, has shown that if \( f(x) = 0 \) for \( x < 0 \), \( f(x) \to 0 \) as \( |x| \to \infty \) and \( f(x) \in L_{1+\delta} \) for some \( \delta > 0 \), then
\[
\lim_{x \to -\infty} h(x) = \frac{1}{\mu} , \quad \mu < \infty
\]
\[
= 0 , \quad \mu = + \infty
\]
More recently, Smith' has shown that the condition that \( f(x) = 0 \) for \( x < 0 \) may be dropped, and furthermore (13) holds. We now prove the following.

**Lemma 3.** If \( \mathbb{E}X_i = \mu < \infty \), \( f(x) \) satisfies condition I, \( A \) is any Lebesgue measurable set of positive real numbers having infinite measure, then
\[
\inf_{-\infty < x < \infty} P(S_n \in A \text{ for some } n \mid S_0 = x) > 0 .
\]

**Proof.** For every \( x \), let \( A_x \) be a measurable subset of \( A \) with \( 0 < c_1 < m(A_x) < c_2 < \infty \) and such that for a given number \( L_1 \) all points in \( A_x \) exceed \( x \) by at least \( L_1 \). Such a set exists since \( m(A) = \infty \). For any \( \varepsilon > 0 \) it follows from (12) that there exists an \( L_1 = L_1(\varepsilon) \) such that
\[
0 < (1 - \varepsilon)K_1c_1 \leq \sum_{n=1}^{\infty} P(S_n \in A_x \mid S_0 = x) < 1 + \varepsilon K_2c_2 < \infty .
\]
Let \( A_x' \) be any measurable set with \( m(A_x') \leq c_2 \) and such that for a given \( L_2 \) all points in \( A_x' \) are exceeded by \( x \) by at least \( L_2 \). By (13)\(^5\) there exists an \( L_2 = L_2(\varepsilon) \) such that
\[
\sum_{n=1}^{\infty} P(S_n \in A_x' \mid S_0 = x) < \varepsilon .
\]

\(^4\) Communication by letter.

\(^5\) Added in proof: Condition (13) can be dropped; (16) follows from the fact that \( \lim_{x \to -\infty} H(x) = 0 \) whether (13) holds or not.
Let $L = \max (L_1, L_2)$. For a given $y \in A_x$ let $A^1_{xy} = A_x \cap [y-L, y+L)$, $A^2_{xy} = A_x \cap [y+L, \infty)$ and $A^3_{xy} = A_x \cap (-\infty, y-L)$.

Then from (15) and (16)

\[
\sum_{n=1}^{\infty} P(S_n \in A_x | S_0 = y) = \sum_{n=1}^{\infty} P(S_n \in A^1_{xy} | S_0 = y) + \sum_{n=1}^{\infty} P(S_n \in A^2_{xy} | S_0 = y) + \sum_{n=1}^{\infty} P(S_n \in A^3_{xy} | S_0 = y) + \sum_{n=1}^{\infty} P\left(-L < S_n < L | S_0 = 0\right) + K_2 c_3 (1 + \epsilon) + \epsilon.
\]

The first term on the right of (17) is finite by the result of Chung and Fuchs [5]. Therefore, since (17) is true for all $y \in A_x$ we have

\[
\sup_{y \in A_x} \sum_{n=1}^{\infty} P(S_n \in A_x | S_0 = y) < c_3 < \infty.
\]

Let $F^{(v)}(B) = P(S_n \in B, S_{v' < v} \in A_x)$ for $1 \leq v' < v | S_0 = x$ where $B$ is any measurable subset of $A_x$. Define $P(S_n \in A_x | S_0 = y) = 1$ if $y \in A_x$ and $= 0$ otherwise. Then we have

\[
\sum_{n=1}^{N} P(S_n \in A_x | S_0 = y) = \sum_{n=1}^{N} \sum_{v=1}^{n} \int_{A_x} P(S_n \in A_x | S_0 = y) F^{(v)}(dy)
\]

\[
\leq \sum_{n=1}^{N} \sum_{v=1}^{n} \sum_{y \in A_x} P(S_n \in A_x | S_0 = y) F^{(v)}(dy)
\]

\[
\leq \sum_{n=1}^{N} F^{(v)}(A_x) \sup_{y \in A_x} \sum_{n=0}^{\infty} P(S_n \in A_x | S_0 = y)
\]

This being true for all $N$ the lemma follows on account of (15).

We now state the following.

**Theorem 3.** (i) If $0 < \mu \leq \infty$, Condition I is satisfied, and $A$ is any Lebesgue measurable subset of the positive real numbers, then $S_n \in A$ infinitely often or finitely often with probability 1 according as $m(A) = \infty$ or $< \infty$.

(ii) If $\mu = \infty$, then there exists a measurable subset $A$ of the positive real numbers with $m(A) = \infty$ such that $S_n \in A$ for only finitely many $n$ with probability 1.

**Proof of (i).** If $m(A) = \infty$, the result follows from Theorem 1 and Lemma 3. If $m(A) < \infty$ it follows from (15) that $\sum_{n=1}^{\infty} P(S_n \in A) < \infty$. 
Since that is the expected number of \( n \) such that \( S_n \in A \), the assertion follows immediately.

**Proof of (ii).** A result due to Blackwell [1] asserts that for any fixed \( d > 0 \):

\[
\lim_{y \to \infty} \sum_{n=1}^{\infty} P(y \leq S_n \leq y + d) = 0.
\]

Using this result the rest of the proof is similar to that of part (ii) Theorem 2.

**Unsolved problems.** Let \( \{X_i\} \) be a sequence of independent and identically distributed \( r \)-dimensional random vectors, \( S_n = \sum_{i=1}^{n} X_i \), \( B \) be any Borel set in the \( r \)-dimensional Euclidean space \( \mathbb{R}^r \). It has been recently proved by Hewitt and Savage [9] (in the lattice case also by Blackwell [2]) that the probability that \( S_n \in B \) infinitely often is necessarily either 0 or 1. It would be of interest to determine for which sets the probability is 0, and for which the probability is 1. Our results give a criterion for this dichotomy in certain cases in \( \mathbb{R}^r \), namely in the lattice case where \( \mathbb{E}X \), exists and is finite (Theorem 2) and in the continuous case under more restrictive conditions (Theorem 3).

**References**


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