

Pacific Journal of Mathematics

NON-RECURRENT RANDOM WALKS

KAI LAI CHUNG AND CYRUS DERMAN

NON-RECURRENT RANDOM WALKS

K. L. CHUNG AND C. DERMAN¹

Introduction and Summary. Let $\{X_i\}$ $i=1, 2, \dots$ be a sequence of independent and identically distributed integral valued random variables such that 1 is the absolute value of the greatest common divisor of all values of x for which $P(X_i=x) > 0$. Define

$$S_n = \sum_{i=1}^n X_i.$$

Chung and Fuchs [5] showed that if x is any integer, $S_n=x$ infinitely often or finitely often with probability 1 according as $EX_i=0$ or $\neq 0$, provided that $E|X_i| < \infty$. Let $0 < EX_i < \infty$, and A denote a set of integers containing an infinite number of positive integers. It will be shown that any such set A will be visited infinitely often with probability 1 by the sequence $\{S_n\}$ $n=1, 2, \dots$. Conditions are given so that similar results hold for the case where X_i has a continuous distribution and the set A is a Lebesgue measurable set whose intersection with the positive real numbers has infinite Lebesgue measure.

A Theorem about Markov Chains. Let $\{Z_n\}$, $n=0, 1, \dots$ denote a Markov chain with stationary transition probabilities where each Z_n takes on values in an abstract state space X . The distribution of Z_0 is given but arbitrary. Let Ω denote the space of all possible sample sequences w , P the probability measure over Ω and $P(\cdot|\cdot)$ the conditional probability. The following theorem appears in [4].

THEOREM 1. *Let A be any event in X . A sufficient condition that*

$$(1) \quad P(Z_n \in A \text{ infinitely often}) = 1$$

is

$$(2) \quad \inf_{z \in X} P(Z_n \in A \text{ for some } n | Z_0 = z) > 0.$$

Since [4] is not readily accessible, we shall prove the theorem here.

*Proof.*² We have with probability 1 that for $j \geq N$

Received April 23, 1955 and in revised form August 1, 1955. Work supported in part by the United States Air Force through the office of Scientific Research of the Air Research and Development Command under contract AF 18 (600)-760.

¹ Now at Columbia University.

² The proof given here is a modification of one suggested by J. Wolfowitz.

$$\begin{aligned}
 (3) \quad & P(Z_n \in A \text{ for some } n \geq N | Z_0 = z_0, \dots, Z_j = z_j) \\
 & \geq P(Z_n \in A \text{ for some } n > j | Z_0 = z_0, \dots, Z_j = z_j) \\
 & = P(Z_n \in A \text{ for some } n | Z_0 = z_j)
 \end{aligned}$$

using the Markovian and stationarity properties. As $j \rightarrow \infty$ the left member of (3) approaches with probability 1 the characteristic function b_N of the event

$$B_N = \{Z_n \in A \text{ for some } n \geq N\}$$

(see Doob [8, p. 332]). The right member of (3) is bounded below by a positive number on account of (2). Hence $b_N = 1$ with probability 1; that is, $P(B_N) = 1$. This being true for all N we have

$$P(\lim_{N \rightarrow \infty} B_N) = \lim_{N \rightarrow \infty} P(B_N) = 1.$$

But $\lim_{N \rightarrow \infty} B_N$ is the event that $Z_n \in A$ infinitely often. This proves the theorem.

If X has only a denumerable number of states and if all the states belong to the same class (that is, for every pair of states i and j there exists integers n_1 and n_2 such that $P(Z_{n_1} = j | Z_0 = i)P(Z_{n_2} = i | Z_0 = j) > 0$) it can be easily seen that (2) is both a necessary and sufficient condition for (1). In fact, the probability in (2) must be 1 for all states z .³

Sums of lattice random variables. Let $\{X_i\}$ $i = 1, 2, \dots$ be a sequence of independent and identically distributed integral valued random variables such that 1 is the absolute value of the greatest common divisor of all values of x for which $P(X_i = x) > 0$. Consider the sequence $\{S_n\}$ $n = 0, 1, \dots$, where we set $S_0 = 0$ with probability 1 and

$$S_n = S_0 + \sum_{i=1}^n X_i.$$

The sequence $\{S_n\}$ is then a Markov chain with stationary transition probabilities and a denumerable state space. Because the transition probabilities are stationary, we shall simply write

$$P(S_{n+m} = i | S_n = j) = P(S_m = i | S_0 = j)$$

even though $S_0 = 0$ with probability 1.

We now state as lemmas some known results to be used below.

LEMMA 1. *Let $\{Z_n\}$ $n = 0, 1, \dots$ be a Markov chain with a denumerable state space. If $\sum_{n=1}^{\infty} P(Z_n = j | Z_0 = i) < \infty$ for all i and j , then*

³ We are indebted to J. Wolfowitz for this remark.

$$(4) \quad P(Z_n=j \text{ for some } n | Z_0=i) = \frac{\sum_{n=1}^{\infty} P(Z_n=j | Z_0=i)}{1 + \sum_{n=1}^{\infty} P(Z_n=j | Z_0=j)}$$

When $EX_i = \mu > 0$, a result of Chung and Fuchs [5] implies that

$$(5) \quad \sum_{n=1}^{\infty} P(S_n=j | S_0=i) < \infty$$

for all i and j . Therefore, on replacing Z_n by S_n in (4) and noting that $P(S_n=j | S_0=j) = P(S_n=0 | S_0=0)$ we have

$$(4') \quad P(S_n=j \text{ for some } n | S_0=i) = \frac{\sum_{n=1}^{\infty} P(S_n=j | S_0=i)}{1 + \sum_{n=1}^{\infty} P(S_n=0 | S_0=0)}$$

Lemma 1 is a special case of a relation given by Doebelin [7] (see Chung [3]). However, we shall sketch a direct proof.

Proof. We define $P(Z_0=j | Z_0=j) = 1$. Then we have

$$(6) \quad P(Z_n=j | Z_0=i) = \sum_{m=1}^n P(Z_m=j, Z_r \neq j \text{ for } 1 \leq r < m | Z_0=i) P(Z_n=j | Z_m=j) \\ = \sum_{m=1}^n P(Z_m=j, Z_r \neq j \text{ for } 1 \leq r < m | Z_0=i) P(Z_{n-m}=j | Z_0=j)$$

On summing over n in (6) and interchanging summations on the right we get

$$(7) \quad \sum_{n=1}^{\infty} P(Z_n=j | Z_0=i) = \sum_{m=1}^{\infty} P(Z_m=j, Z_r \neq j \text{ for } 1 \leq r < m) (1 + \sum_{n=1}^{\infty} P(Z_n=j | Z_0=j)) \\ = P(Z_n=j \text{ for some } n) (1 + \sum_{n=1}^{\infty} P(Z_n=j | Z_0=j)),$$

the relation (4).

LEMMA 2. If $EX_i = \mu > 0$, then

$$(8) \quad \lim_{j \rightarrow \infty} \sum_{n=1}^{\infty} P(S_n=j | S_0=i) = \begin{cases} \frac{1}{\mu} > 0, & \mu < \infty \\ = 0, & \mu = +\infty. \end{cases}$$

Lemma 2 is due to Chung and Wolfowitz [6]. We now prove the following.

THEOREM 2. (i) If $0 < EX_i = \mu < \infty$ and A is any set containing an infinite number of positive integers, then $S_n \in A$ infinitely often with probability 1.

(ii) If $EX_i = +\infty$, then there exists a set A containing an infinite number of positive integers such that $S_n \in A$ only finitely often with probability 1.

Proof of (i). Since $0 < \mu < \infty$, by (8) there exists a constant $c > 0$, independent of i , and an integer $J(i)$ such that for all $j > J(i)$

$$(9) \quad \sum_{n=1}^{\infty} P(S_n = j | S_0 = i) > c.$$

Therefore by (4') and (5)

$$(10) \quad P(S_n = j \text{ for some } n | S_0 = i) > \frac{c}{1+c}, \quad j > J(i)$$

where $c' = \sum_{n=1}^{\infty} P(S_n = 0 | S_0 = 0) < \infty$. Since A contains infinitely many positive integers, it always contains an integer greater than $J(i)$ for every i . Therefore (2) holds and part (i) of Theorem 2 follows from Theorem 1.

Proof of (ii). If $\mu = +\infty$, then from (8) there exists an increasing subsequence $\{i_j\}$ of positive integers such that

$$(11) \quad \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} P(S_n = i_j | S_0 = 0) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} P(S_n = i_j | S_0 = 0) < \infty.$$

Let $A = \{i_j\}$. Now (11) is the expected number of n such that $S_n \in A$. Since this expectation is finite it follows that the number of n such that $S_n \in A$ is finite with probability 1. This completes the proof of the theorem.

Random variables with continuous distribution functions. Consider now a sequence $\{X_i\}$ $i=1, 2, \dots$ of independent, identically distributed random variables possessing a common density function $f(x)$. Again let $\{S_n\}$ $n=0, 1, \dots$ denote the cumulative sums $S_n = S_0 + \sum_{i=1}^n X_i$ where $S_0 = 0$ with probability 1. Our previous remark pertaining to the notation $P(\dots | S_0 = x)$ applies here also. Suppose $EX_i = \mu > 0$. Then a result of Chung and Fuchs [5] implies that $H(x) = \sum_{n=1}^{\infty} P(S_n \leq x) < \infty$ for all x . Since $H(x)$ is non-decreasing, $H'(x)$ exists everywhere except on a set N_0 of Lebesgue measure zero. Let

$$\begin{aligned} h(x) &= H(x) & x \notin N, \\ &= 1, \text{ say,} & x \in N, x \geq 0 \\ &= 0 & x \in N, x < 0 \end{aligned}$$

We shall say that $f(x)$ satisfies condition I if there exist constants K_1 and K_2 such that

$$(12) \quad 0 < K_1 \leq \liminf_{x \rightarrow \infty} h(x) \leq \overline{\lim}_{x \rightarrow \infty} h(x) \leq K_2 < \infty$$

and if

$$(13) \quad \lim_{x \rightarrow -\infty} h(x) = 0$$

The behavior of $h(x)$ for large $|x|$ has been investigated in various papers on renewal theory. Smith [10], for example, has shown that if $f(x) = 0$ for $x < 0$, $f(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and $f(x) \in L_{1+\delta}$ for some $\delta > 0$, then

$$\begin{aligned} \lim_{x \rightarrow \infty} h(x) &= \frac{1}{\mu}, & \mu < \infty \\ &= 0, & \mu = +\infty \end{aligned}$$

More recently, Smith⁴ has shown that the condition that $f(x) = 0$ for $x < 0$ may be dropped, and furthermore (13) holds. We now prove the following.

LEMMA 3. *If $EX_i = \mu < \infty$, $f(x)$ satisfies condition I , A is any Lebesgue measurable set of positive real numbers having infinite measure, then*

$$(14) \quad \inf_{-\infty < x < \infty} P(S_n \in A \text{ for some } n | S_0 = x) > 0.$$

Proof. For every x , let A_x be a measurable subset of A with $0 < c_1 < m(A_x) < c_2 < \infty$ and such that for a given number L_1 all points in A_x exceed x by at least L_1 . Such a set exists since $m(A) = \infty$. For any $\epsilon > 0$ it follows from (12) that there exists an $L_1 = L_1(\epsilon)$ such that

$$(15) \quad 0 < (1 - \epsilon)K_1c_1 < \sum_{n=1}^{\infty} P(S_n \in A_x | S_0 = x) < (1 + \epsilon)K_2c_2 < \infty.$$

Let A'_x be any measurable set with $m(A'_x) \leq c_2$ and such that for a given L_2 all points in A'_x are exceeded by x by at least L_2 . By (13)⁵ there exists an $L_2 = L_2(\epsilon)$ such that

$$(16) \quad \sum_{n=1}^{\infty} P(S_n \in A'_x | S_0 = x) < \epsilon.$$

⁴ Communication by letter.

⁵ Added in proof: Condition (13) can be dropped; (16) follows from the fact that $\lim_{x \rightarrow -\infty} H(x) = 0$ whether (13) holds or not.

Let $L = \max(L_1, L_2)$. For a given $y \in A_x$ let $A_{xy}^1 = A_x \cap [y - L, y + L)$, $A_{xy}^2 = A_x \cap [y + L, \infty)$ and $A_{xy}^3 = A_x \cap (-\infty, y - L)$.

Then from (15) and (16)

$$\begin{aligned}
 (17) \quad \sum_{n=1}^{\infty} P(S_n \in A_x | S_0 = y) &= \sum_{n=1}^{\infty} P(S_n \in A_{xy}^1 | S_0 = y) \\
 &+ \sum_{n=1}^{\infty} P(S_n \in A_{xy}^2 | S_0 = y) + \sum_{n=1}^{\infty} P(S_n \in A_{xy}^3 | S_0 = y) \\
 &\leq \sum_{n=1}^{\infty} P(-L < S_n < L | S_0 = 0) + K_2 c_2 (1 + \varepsilon) + \varepsilon.
 \end{aligned}$$

The first term on the right of (17) is finite by the result of Chung and Fuchs [5]. Therefore, since (17) is true for all $y \in A_x$ we have

$$(18) \quad \sup_{y \in A_x} \sum_{n=1}^{\infty} P(S_n \in A_x | S_0 = y) < c_3 < \infty$$

Let $F_x^{(v)}(B) = P(S_{v'} \in B, S_{v'} \notin A_x \text{ for } 1 \leq v' < v | S_0 = x)$ where B is any measurable subset of A_x . Define $P(S_0 \in A_x | S_0 = y) = 1$ if $y \in A_x$ and $= 0$ otherwise. Then we have

$$\begin{aligned}
 \sum_{n=1}^N P(S_n \in A_x | S_0 = x) &= \sum_{n=1}^N \sum_{v=1}^n \int_{A_x} P(S_n \in A_x | S_v = y) F_x^{(v)}(dy) \\
 &= \sum_{v=1}^N \int_{A_x} \sum_{n=v}^N P(S_n \in A_x | S_v = y) F_x^{(v)}(dy) \\
 &\leq \sum_{v=1}^N \int_{A_x} \sum_{n=0}^{\infty} P(S_n \in A_x | S_0 = y) F_x^{(v)}(dy) \\
 &\leq \sum_{v=1}^N F_x^{(v)}(A_x) \sup_{y \in A_x} \sum_{n=0}^{\infty} P(S_n \in A_x | S_0 = y) \\
 &\leq P(S_n \in A_x \text{ for some } n | S_0 = x) (1 + c_3).
 \end{aligned}$$

This being true for all N the lemma follows on account of (15).

We now state the following.

THEOREM 3. (i) *If $0 < EX_i = \mu < \infty$, Condition I is satisfied, and A is any Lebesgue measurable subset of the positive real numbers, then $S_n \in A$ infinitely often or finitely often with probability 1 according as $m(A) = \infty$ or $< \infty$.*

(ii) *If $\mu = \infty$, then there exists a measurable subset A of the positive real numbers with $m(A) = \infty$ such that $S_n \in A$ for only finitely many n with probability 1.*

Proof of (i). If $m(A) = \infty$, the result follows from Theorem 1 and Lemma 3. If $m(A) < \infty$ it follows from (15) that $\sum_{n=1}^{\infty} P(S_n \in A) < \infty$.

Since that is the expected number of n such that $S_n \in A$, the assertion follows immediately.

Proof of (ii). A result due to Blackwell [1] asserts that for any fixed $d > 0$.

$$\lim_{y \rightarrow \infty} \sum_{n=1}^{\infty} P(y \leq S_n \leq y+d) = 0.$$

Using this result the rest of the proof is similar to that of part (ii) Theorem 2.

Unsolved problems. Let $\{X_i\}$ be a sequence of independent and identically distributed r -dimensional random vectors, $S_n = \sum_{i=1}^n X_i$, B be any Borel set in the r -dimensional Euclidean space R^r . It has been recently proved by Hewitt and Savage [9] (in the lattice case also by Blackwell [2]) that the probability that $S_n \in B$ infinitely often is necessarily either 0 or 1. It would be of interest to determine for which sets the probability is 0, and for which the probability is 1. Our results give a criterion for this dichotomy in certain cases in R^1 , namely in the lattice case where EX_i exists and is finite (Theorem 2) and in the continuous case under more restrictive conditions (Theorem 3).

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, c/o University of California Press, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

* During the absence of E. G. Straus.

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