ON SOME SPECIAL SYSTEMS OF EQUATIONS

Harry Herbert Corson, III
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H. H. CORSON

1. Let $F$ be an arbitrary field. Let $S$ be a system of equations which, when solved for two of its variables, takes the following form:

\begin{align*}
  x_1^k &= f(x_3, \ldots, x_n), \\
  x_2^k &= g(x_3, \ldots, x_n),
\end{align*}

(1)

where $f$ and $g$ are arbitrary functions of the indicated variables. Consider also the equation

\begin{equation}
y_1^{k_2} = f^{s_k_2}(y_3, \ldots, y_n)g^{r_k_1}(y_3, \ldots, y_n).
\end{equation}

THEOREM 1. If $(k_1, k_2) = 1$ and $rk_1 + sk_2 = 1$, then the distinct solutions of (1) in $F$ with $x_1x_2 \neq 0$ may be put in one-to-one correspondence with the distinct solutions of (2) in $F$ with $y \neq 0$. Moreover, these solutions of (1), $x_1x_2 \neq 0$, may be determined from the solutions of (2), $y \neq 0$, and conversely, by means of transformations (3) and (4) below.

Proof. Assuming for the rest of this section that $x_1x_2 \neq 0$, $y \neq 0$, we put

\begin{align*}
  x_1 &= y_1^{k_2} \left\{ \frac{f(y_3, \ldots, y_n)}{g(y_3, \ldots, y_n)} \right\}^r, \\
  x_2 &= y_1^{k_1} \left\{ \frac{g(y_3, \ldots, y_n)}{f(y_3, \ldots, y_n)} \right\}^s, \\
  x_i &= y_i \quad \text{for} \quad i = 3, \ldots, n
\end{align*}

(3)

and notice that if $(y, y_3, \ldots, y_n)$ is a solution of (2) then (3) determines a solution of (1). Now let

\begin{align*}
  y &= x_1^r x_2^s, \\
  y_i &= x_i \quad \text{for} \quad i = 3, \ldots, n.
\end{align*}

(4)

It may be verified directly that if $(x_1, x_2, \ldots, x_n)$ is a solution of (1) then (4) determines a solution of (2). Further, given a solution $(x_1, x_2, \ldots, x_n)$ of (1) and a solution $(y, y_3, \ldots, y_n)$ of (2) with $x_i = y_i \ (i = 3, \ldots, n)$, then (3) implies (4) and conversely—which may be verified with the use of the relation $rk_1 + sk_2 = 1$.

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We note that Theorem 1 may be extended by induction to apply to a system like (1) with an arbitrary number of equations, with $z_i$, $z_2^2$, $\ldots$, $z_n^{kn}$ as left members, and with arbitrary functions of $z_{m+1}$, $\ldots$, $z_n$ as right members if $(k_i, k_j)=1, i \neq j$. The argument is the same in going from $n$ to $n+1$ equations, and transformations corresponding to (3) and (4) may be constructed.

Use will also be made of the fact that Theorem 1 is still valid if $x_3, \ldots, x_n$ are restricted to values in $A$, a subset of $F$, as long as $y_3, \ldots, y_n$ are similarly restricted.

2. Let $F$ now be a finite field $GF(q), q=p^k$. Assume $f$ and $g$ to be homogeneous polynomials of degrees $m_1$ and $m_2$ respectively, where $(m_1, k_1)=1$ and $(m_2, k_2)=1$. The solutions of (2) can be determined by the following method used by Hua and Vandiver [1] and Morgan Ward [2].

As $(k_1k_2, sk_1m_1+rk_1m_2)=1$, there are integers $a, b$, and $c$ such that $ak_1k_2+b(sk_1m_1+rk_1m_2)+c(q-1)=1$ with $(a, q-1)=1$. First assuming that $y \neq 0$, set

$$ y = \lambda^a $$

$$ y_i = \lambda^{-b}z_i \quad (i=3, \ldots, n). $$

Equation (2) then assumes the following form:

$$ \lambda = f^s k_2(z_3, \ldots, z_n)g^{r_1}(z_3, \ldots, z_n). $$

Thus every choice of $z_3, \ldots, z_n$ such that $f \neq 0, g \neq 0$ determines a solution of (2).

Now consider the system (1). Determine as above integers $u, v,$ and $w$ such that $uk_1+v=vm_2+w(q-1)=1, (u, q-1)=1$. Assuming $x_3 \neq 0$, set

$$ x_3 = \gamma^u $$

$$ x_i = \gamma^{-v}t_i \quad (i=3, \ldots, n). $$

It is readily seen that all values of $t_3, \ldots, t_n$ such that $f(t_3, \ldots, t_n)=0$ determine solutions of the system (1) whether $g(t_3, \ldots, t_n)=0$ or not.

The same argument is valid if $g$ is assumed zero, which proves the following.

**THEOREM 2.** If $f$ and $g$ are homogeneous polynomials of degrees $m_1$ and $m_2$ respectively, $(m_1, k_1)=1$ and $(m_2, k_2)=1$, then the total number of solutions of the system (1) in $GF(q)$ is $q^{n-2}$.

A similar application of Theorem 1 is the following. First let $S$ be
where \((k_1, k_2) = 1\). Also if \(M\) is the least common multiple of \(m_3, \ldots, m_n\), assume \((eM, k_i) = 1\) and \((dM, k_i) = 1\). In place of (5) we employ the following transformation in (2), following Carlitz [3]:

\[
y = \lambda^a \\
y_i = \lambda^{-b} M^m z_i \quad (i = 3, \ldots, n),
\]

where \(ak_j k_i + bM(sk_i e + rk_i d) + c(q-1) = 1\), \((a, q-1) = 1\). Exactly as above follows the next theorem.

**Theorem 3.** The total number of solutions of (8) subject to the conditions stated above is \(q^{n-2}\).

Also [3] suggests the following generalization of Theorem 2. Let \(f_3(x_3), f_4(x_4), \ldots, f_n(x_n)\) and \(g_3(x_3), g_4(x_4), \ldots, g_n(x_n)\) be homogeneous polynomials of degrees \(em_3, em_4, \ldots, em_n\) and \(dm_3, dm_4, \ldots, dm_n\) respectively, where now \((x_i) = (x_{i1}, x_{i2}, \ldots, x_{is_i}) \quad (i = 3, \ldots, n)\). Thus by the same argument follows the next theorem.

**Theorem 4.** Replacing in (8) \(x_i^{m_i}\) by \(f_i(x_i)\) and \(x_i^{m_i}\) by \(g_i(x_i)\), \((i = 3, \ldots, n)\), then the total number of solutions of the resulting system is \(q^{s_3 + \cdots + s_n}\).

3. Now let \(F\) be the rational field and let \(f\) and \(g\) in (1) be polynomials with integral coefficients. If \(x_3, \ldots, x_n\) are restricted to be integers, then \(x_1\) and \(x_2\) in any solution must be integers.

In the equation \(rk_1 + sk_2 = 1\) we may assume that \(r > 0, s < 0\). In place of system (1) write

\[
x_1^{k_1} = 1 = \frac{1}{f(x_3, \ldots, x_n)} = f(x_3, \ldots, x_n) \\
x_2^{k_2} = g(x_3, \ldots, x_n).
\]

we assume as in Theorem 2 that \(f\) and \(g\) are homogeneous of degrees \(m_1\) and \(m_2\) respectively, \((m_1, k_1) = 1\) and \((m_2, k_2) = 1\). Let \(a, b\) and \(c\) satisfy \(ak_1 k_2 + b(rk_1 m_2 - sk_2 m_1) + c(q-1) = 1\), \((a, q-1) = 1\); then (5) determines a family of solutions in integers of

\[
y^{k_1 k_2} = f^{k_1 k_2}(y_3, \ldots, y_n) g^{k_2}(y_3, \ldots, y_n),
\]

\(y \neq 0\). By Theorem 1, (3) determines a family of solutions of (10) with
Theorem 5. If $f$ and $g$ are homogeneous polynomials with integral coefficients of degrees $m_1$ and $m_2$ respectively, $(m_1, k_1)=1$ and $(m_2, k_2)=1$ then a family of solutions in integers may be found for equations (1) by the method above.

See [2] for remarks on the solution of equation (11) under the above hypotheses. Note especially the above method does not in general give all solutions.

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References

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