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NEIGHBOR RELATIONS ON THE CONVEX OF CYCLIC PERMUTATIONS

ISIDORE HELLER

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I. HELLER

1. Introduction and summary. Two vertices of a polyhedron are called neighbors of order k when they have a face of dimension k, and none of lower dimension, in common. $K(P)$ denotes the maximum value of k for a given polyhedron P. For the convex hull (polyhedron) P_n of all permutations of n elements (represented by square matrices of order n and interpreted as points in n^2-space) it was shown [1 and 2] that $K(P)=[n/2]$ (that is, the largest integer not exceeding $n/2$), which is rather small as compared with $\dim P_n=(n-1)^2$. For the convex hull Q_n of all cyclic permutations of n elements that leave no element fixed, *H.* Kuhn performed computations showing that any two vertices of Q_5 but not any two vertices of Q_6 are neighbors of order 1, which means that $K(Q_5)=1$ and $K(Q_6)>1$. The present note, dealing with general n, proves, for $n \geqq 8$:

$$K(Q_n)=K(P_n)-1=\frac{n}{2}-1 \quad \text{if} \quad n=4m+2 \tag{1}$$

$$K(Q_n)=K(P_n)=\left[\frac{n}{2}\right] \quad \text{if} \quad n \neq 4m+2 \tag{2}$$

For $n=1, 2, \cdots 6, 7$, $K(Q_n)=0, 0, 1, 1, 1, 2, 2$ respectively.

2. A permutation p of n numbered elements is customarily represented by a matrix (p_{ij}), where

$$p_{ij}=\begin{cases} 1 & \text{when } p \text{ sends } i \text{ into } j \\ 0 & \text{otherwise.} \end{cases}$$

To the product of permutations then corresponds the product of the associated matrices under ordinary matrix multiplication, and therefore the same symbol will be used for a permutation and its matrix.

The following facts from [1] and [2] regarding neighbor relations on P_n will be used in the sequal:

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(2.1) $$K(P_n)=\left[\frac{n}{2}\right]$$

(2.2) p_1 and p_2 are neighbors of order k on P_n if and only if $p_1^{-1}p_2$ is a product of k disjoint cycles (not counting cycles of length 1)

(2.3) If $c_1, c_2, \cdots, c_k$ are disjoint cycles and F is the face of lowest dimension that contains the two vertices

$$p \text{ and } \bar{p}=pc_1c_2\cdots c_k ,$$

then F has the 2^k vertices

$$pc_{i_1}c_{i_2}\cdots c_{i_s} \qquad (0\leqq s\leqq k) .$$

3. **If the vertices of** a convex polyhedron Q are a subset of the vertices of a convex polyhedron P, let two vertices q_1, q_2 of Q be neighbors of order k on P and k^* on Q:

$$k=k(q_1, q_2; P) , \quad k^*=k^*(q_1, q_2; Q) .$$

Let

$$F=F(q_1, q_2; P) , \quad F^*=F^*(q_1, q_2; Q)$$

be the face of lowest dimension of P respectively Q that contains q_1 and q_2, so that

$$k=\dim A(F) , \quad k^*=\dim A(F^*) ,$$

where $A(F)$ and $A(F^*)$ denote the "affine span" of F and F^* respectively, which is also obtained as the intersection of all hyperplanes that support P respectively Q and contain q_1 and q_2 (with the understanding that A is the entire space when such hyperplanes do not exist) ; then

(3.1) $$F\supseteqq F^* ,$$

hence

(3.2) $$A(F)\supseteqq A(F^*) ,$$

and therefore

(3.3) $$k\geqq k^* .$$

Proof of (3.1). The line segment joining q_1 and q_2 goes through the interior of F^* (otherwise q_1 and q_2 would have a face of lower dimension in common). Therefore any hyperplane through q_1 and q_2 necessarily contains interior points of F^*.

Further, the vertices of Q, hence in particular those of F^*, are also vertices of P. Therefore any hyperplane that supports P supports F^*.

Above establishes that any hyperplane H that supports P and contains q_1 and q_2 necessarily contains F^*, since it supports F^* and contains points interior to F^*. Therefore

$$A(F) \supseteqq F^* ,$$

which, in conjunction with

$$P \supset Q \supset F^* ,$$

implies

$$F^* \subseteqq P \cap A(F) .$$

This completes the proof of (3.1), since the right hand side of the last relation equals F.

A somewhat sharper form of (3.1) may be noted as

LEMMA 1. *The vertices of F^* are among the vertices of F.*

The proof is immediate from (3.1) and the fact that the vertices of F^* are vertices of P, and a vertex of P contained in F is vertex of F.

From (3.3) it follows that $\max k^* \leqq \max k$, that is

$$K(Q) \leqq K(P) \tag{3.4}$$

4. At this point it is convenient to first establish some auxiliary facts. p, q, c denote permutations of n elements, for fixed n.

LEMMA 2. *If*

$$c_1, c_2, \cdots, c_r, c_{r+1}, \cdots, c_s$$

is a set of s disjoint cycles, and

$$c' = c_1 c_2 \cdots c_r , \quad c'' = c_{r+1} c_{r+2} \cdots c_s$$

then

$$c' + c'' = I + c'c'' \tag{4.1}$$

Proof. Obvious (note that a cycle of less than n elements is still represented as an n by n matrix, with 1's along the main diagonal for fixed elements).

LEMMA 3. *Under the assumptions of Lemma* 1, *let*

$$q,\ qc',\ qc'',\ qc'c''=\bar{q} \tag{4.2}$$

be vertices of a polyhedron R. Then

> *a hyperplane H through q and $\bar{q}$ that supports R contains qc' and qc'',*

and consequently

> *$F(q, \bar{q}; R)$ contains qc' and qc'' (obviously as vertices).*

This lemma will be used in the particular case where $R=Q_n$ or P_n.

Proof of Lemma 3. Using parentheses to denote the inner product, let H, given by $(h, x)=\alpha$, contain q and $\bar{q}$ but not contain qc' (say); that is

$$(h, q)=(h, \bar{q})=\alpha\ , \quad (h, qc')=\alpha+\beta, \quad \beta\neq 0\ .$$

By (4.1) and (4.2)

$$qc'+qc''=q+\bar{q}\ ,$$

hence

$$(h, qc'')=(h, q+\bar{q}-qc')=2\alpha-(\alpha+\beta)=\alpha-\beta\ ,$$

so that H separates qc' from qc'' and therefore does not support R.

LEMMA 4. *If*

$$k=\left[\frac{n}{2}\right],\quad 2s\leqq k$$

$$q=(12\cdots n)$$

$$c_i=(i,\ i+k) \qquad (i=1,\ 2,\ \cdots k)\ ,$$

then the product of q with $2s$ distinct c_i,

$$qc_{i_1}c_{i_2}\cdots c_{i_2}$$

is an n-cycle.

Proof. Since the c_i are disjoint, they commute, and may be arranged in such manner that

$$i_1<i_2<\cdots<i_{2s}\ ;$$

then

$$(1\cdots n)(i_1, i_1+k)(i_2, i_2+k)\cdots(i_{2s-1}, i_{2s-1}+k)(i_{2s}, i_{2s}+k)$$
$$=(1\cdots i_1, i_1+k+1, \cdots i_2+k, i_2+1, \cdots i_3, i_3+k+1, \cdots i_4+k, i_4+1\cdots$$
$$\cdots i_{2s-1}, i_{2s-1}+k+1, \cdots i_{2s}+k, i_{2s}+1, \cdots$$
$$i_1+k, i_1+1, \cdots i_2, i_2+k+1, \cdots i_3+k, i_3+1, \cdots i_4, i_4+k+1, \cdots$$
$$\cdots i_{2s-1}+k, i_{2s-1}+1, \cdots i_{2s}, i_{2s}+k+1, \cdots n) .$$

It is easily verified above relation also holds, with proper changes, for $i_1=1$ and for $2s=k$, $2k=n$.

In similar straightforward fashion one easily proves:

LEMMA 5. *If q is an n-cycle and d is a 3-cycle, then qd is an n-cycle if and only if the elements of d occur in q in the same cyclic order as in d.*

LEMMA 6. *If q is an n-cycle and the 2-cycle $(ij)\neq(km)$, then $q(ij)(km)$ is an n-cycle if and only if the pair i, j separates the pair k, m in q.*

5. The case $n=4m$, $n=4m+1$; $m\geqq 2$.

$$K(Q_n)=K(P_n) \qquad (n=4m,\ 4m+1;\ m\geqq 2) \tag{5.1}$$

Proof. Because of (3.4), it is sufficient to show that $K(Q_n)\geqq K(P_n)$; this will be achieved by showing that for a particular pair of vertices q, $\bar{q}$

$$k(q, \bar{q};\ Q_n)\geqq\left[\frac{n}{2}\right]=K(P_n) . \tag{5.2}$$

Now let $2m=k$, so that $n\geqq 2k$, choose

$$\begin{cases} q=(12\cdots n) \\ c_s=(i,\ i+k) \qquad (i=1, 2\cdots k) \\ \bar{q}=qc_1c_2\cdots c_k=qc , \end{cases} \tag{5.3}$$

and denote by c' the product of an even number (including 0 and k) of the c_i, by c'' the product of the remaining c_i (whose number is also even, since k is even):

$$\begin{cases} c'=c_{i_1}c_{i_2}\cdots c_{i_{2s}} \qquad (0\leqq 2s\leqq k) \\ c'c''=c_1c_2\cdots c_k=c . \end{cases} \tag{5.4}$$

(It should be noted that the now following proof of $k^*(q, \bar{q}\,;\, Q_n) \geqq k$ does not depend on the special assumption $n=4m$, $4m+1$ and $k=2m$, but rather holds in general for any pair n, k, where k is even and $n \geqq 2k$; this fact will be used in § 9).

The qc' are vertices of Q_n (by Lemma 4) and therefore (by Lemma 3) they are also vertices of $F^* = F(q, \bar{q}\,;\, Q_n)$.

To verify (5.2), that is

$$\dim A(F^*) \geqq k\,,$$

consider the following subset of $k+1$ vertices of F^*:

$$q_1 = qc_1c_1 = q,\ q_2 = qc_1c_2,\ \cdots\ q_k = qc_1c_k,\ q_{k+1} = qc = \bar{q}\,. \tag{5.5}$$

The q_i of (5.5) are linearly independent.

Proof. Assume

$$\lambda qc + \sum_{i=1}^{k} \lambda_i q_i = 0\,. \tag{5.6}$$

Successive application of (4.1) to

$$c = c_1c_2 \cdots c_k$$

yields

$$c = c_1[c_2 + \cdots + c_k - (k-2)I]\,, \tag{5.7}$$

and (5.6) becomes

$$\lambda qc_1[c_2 + \cdots + c_k - (k-2)I] + \sum_{i=1}^{k} \lambda_i qc_1c_i = 0$$

that is

$$qc_1[\lambda_1 c_1 - \lambda(k-2)I + \sum_{i=2}^{k} (\lambda_i + \lambda)c_i] = 0$$

or, equivalently, since q and c_1 are nonsingular matrices

$$\lambda_1 c_1 - \lambda(k-2)I + \sum_{i=2}^{k} (\lambda_i + \lambda)c_i = 0 \tag{5.8}$$

Since the c_i are disjoint cycles (5.8) implies

$$\lambda_1 = 0;\ \lambda_i + \lambda = 0\ (i = 2, \cdots k);\ \lambda(k-2) = 0$$

which, in conjunction with $k \neq 2$ (following from $m \geqq 2$), further implies

$$\lambda = 0\,, \quad \lambda_i = 0\,.$$

This verifies that the $k+1$ q_i of (5.5) are linearly independent, so that the dimension of their linear span is $k+1$, and therefore the dimension of their affine span equal to k. This completes the proof of (5.2) and hence of (5.1)

6. The case $n=4m$, $n=4m+1$; $m=1$. Removing the restriction $m \geq 2$ in (5.1) leaves the cases $n=4$ and $n=5$ still to be considered

$$K(Q_n)=1 \qquad (n=4,\ 5) \tag{6.1}$$

Proof. Since, by (3.4) and (2.1), $K(Q_n) \leq 2$, one only has to show that $K(Q_n) \neq 2$.

Assume there were two vertices q and $\bar{q}$ of Q_n such that

$$k^*(q,\ \bar{q}\ ;\ Q_n)=2\ .$$

Then, by (3.4), (3.3) and (2.1)

$$k(q,\ \bar{q};\ P_n)=2\ ,$$

which by (2.2) implies that $q^{-1}\bar{q}$ is a product of two disjoint cycles, say c_1, c_2, so that $\bar{q}=qc_1c_2$.

Since q and $\bar{q}$ are cycles of the same length (namely n), c_1c_2 is necessarily an even permutation, so that c_1 and c_2 are both of length 2.

Now let F be the lowest dimensional face of P_n containing q and $\bar{q}$. Then, by (2.3), F has the 4 vertices

$$q,\ \bar{q},\ qc_1,\ qc_2\ .$$

of which the last two are not n-cycles and therefore not vertices of F^*. Hence, by Lemma 1, F^* has only the two vertices q and $\bar{q}$, which implies $k^*=1$ in contradiction to the assumption that $k^*=2$. This completes the proof of (6.1).

7. The case $n=4m+3$; $m \neq 1$.

$$K(Q_n)=K(P_n) \qquad (n=4m+3,\ m \neq 1)\ , \tag{7.1}$$

including $m=0$.

Proof. Because of (3.4) it is again sufficient to point out two vertices, q, $\bar{q}$, of Q_n, such that

$$k^*(q,\ \bar{q};\ Q_n) \geq K(P_n)=2m+1\ . \tag{7.2}$$

For $k=2m$, let q, c_i, c, c', c'' be defined as in (5.3) and (5.4), let $d=(2k+1,\ 2k+2,\ 2k+3)$, and $\bar{q}=qcd$, By Lemmas 4 and 5 the qc' and $qc'd$ are vertices of Q_n for all c' of (5.4), and by Lemma 3 they are also vertices of $F^*(q, \bar{q}; Q_n)$. To prove that

$$\dim A(F^*) \geqq 2m+1 ,$$

it is shown that the dimension of the linear span of F^* is $\geqq 2m+2=k+2$, in verifying that the $k+2$ vertices of F^*

$$(7.3) \qquad q_1=q=qc_1c_1,\ q_2=qc_1c_2,\ \cdots,\ q_k=qc_1c_k,\ q_{k+1}=qd, \quad q_{k+2}=\bar{q}=qcd$$

are linearly independent.

Assume

$$(7.4) \qquad \sum_{i=1}^{k+2} \lambda_i q_i=0$$

or, equivalently, substituting for q_i their expressions from (7.3), omitting the non singular common factor qc_1, and writing μ_i for λ_{k+i} ,

$$(7.5) \qquad \sum_{i=1}^{k} \lambda_i c_i+\mu_1 c_1 d+\mu_2 c_2 c_3 \cdots c_k d=0 .$$

Application of (4.1) yields for the left hand side of (7.5)

$$\sum_{i=1}^{k} \lambda_i c_i+\mu_1(c_1+d-I)+\mu_2[c_2+\cdots+c_k+d-(k-1)I] ,$$

so that (7.4) is equivalent to

$$(7.6) \qquad (\lambda_1+\mu_1)c_1+\sum_{i=2}^{k}(\lambda_i+\mu_2)c_i+(\mu_1+\mu_2)d-[\mu_1+(k-1)\mu_2]I=0$$

Since the c_i and d are disjoint cycles, (7.6) implies

$$(7.7) \qquad \begin{cases} \lambda_1+\mu_1=0 & \\ \lambda_i+\mu_2=0 & (i=2, 3, \cdots k) \\ \mu_1+\mu_2=0 & \\ \mu_1+(k-1)\mu_2=0 & \end{cases}$$

The last two relations of (7.7) imply (because of the assumption $m \neq 1$, hence $k \neq 2$, $k-1 \neq 1$)

$$\mu_1=\mu_2=0 ,$$

which in conjunction with the first two relations of (7.7) implies

$$\lambda_i=0 \qquad (i=1, 2, \cdots k) ,$$

so that all coefficients of (7.4) vanish; this proves that the q_i of (7.4) are linearly independent, and completes the proof of (7.2) and hence (7.1).

8. The case $n=7$ (excepted in § 7).

(8.1) $$K(Q_7)=K(P_7)-1=2$$

Proof. By (3.4) and (2.1)

$$K(Q_7)\leqq 3 .$$

To see that equality cannot hold, let $q=(12\cdots 7)$.

Because of (2.1) and (3.3), only such $\bar{q}$ must be considered where

$$k(q, \bar{q}\,;\, P_7)=3 .$$

By (2.2) the last relation is only possible for

$$\bar{q}=qc_1c_2d ,$$

where c_1, c_2, d are disjoing cycles.

For $\bar{q}$ to be a 7-cycle it is necessary (not sufficient) that c_1c_2d be even, that is, that two of them, say c_1 and c_2, be transpositions and d a 3 cycle.

For the same reason, among the 8 vertices of $F(q, \bar{q}\,;\, P_7)$ determined by (2.3), at most 4 are 7-cycles, namely

(8.2) $$q_1=q,\ q_2=qc_1c_2,\ q_3=qd,\ q_4=\bar{q}=qc_1c_2d ,$$

so that, by Lemma 1, $F^*(q, \bar{q}\,;\, Q_7)$ has at most the 4 vertices (8.2).

However, application of (4.1) yields

$$q_1+q_4=q(I+c_1c_2d)=q(I+c_1c_2+d-I)=q_2+q_3$$

which is a relation

$$\Sigma\, \lambda_i c_i=0 \quad \text{with} \quad \Sigma\, \lambda_i=0 ,$$

therefore

$$\dim A(F^*)\leqq 2 .$$

It has thus been established that

$$K(Q_7)\leqq 2 .$$

To complete the proof of (8.1), choose

(8.3) $$q=(12\cdots 7),\ c_1=(13),\ c_2=(24),\ d=(567) .$$

Then each q_i of (8,2) is a 7-cycle (by Lemmas 4 and 5) and a vertex of $F^*(q, \bar{q}\,;\, Q_7)$ (by Lemma 3.) The last 3 of these q_i are linearly independent. This establishes, for this particular face F^*,

$$\dim A(F^*)=2\,,$$

and completes the proof of (8.1).

9. The case $n=4m+2$.

$$K(Q_n)=K(P_n)-1=2m \qquad (n=4m+2). \tag{9.1}$$

The proof is achieved in showing

$$K(Q_n)\leqq K(P_n)-1=2m \tag{9.2}$$

$$K(Q_n)\geqq K(P_n)-1=2m\,. \tag{9.3}$$

To verify (9.2), assume $K(Q_n)>K(P_n)-1$, which, by (3.4) and (2.1), implies $K(Q_n)=K(P_n)=2m+1$.

Then there must be a pair of vertices q and $\bar{q}$ on Q_n such that

$$k^*(q, \bar{q}\,;\, Q_n)=2m+1\,,$$

and hence, by (3.3) and (2.1),

$$k(q, \bar{q}; P_n)=2m+1\,,$$

which, by (2.2) implies

$$\bar{q}=qc_1c_2\cdots c_{2m+1}\,,$$

where the c_i are disjoint cycles, and therefore necessarily transpositions, because of $n=2(2m+1)$. Then however, the product of the c_i is an odd permutation, and $\bar{q}$ cannot be an n-cycle if q is one. This proves (9.2).

To verify (9.3), consider first the case $m\geqq 2$. Setting $2m=k$, the construction from (5.3) through the end of § 5 proves the existence of $q, \bar{q}$ with $k^*(q, \bar{q}; Q_n)=k$, which implies $K(Q_n)\geqq k$.

For $m=1$, that is, $n=6$, choose

$$q=(12\cdots 6),\ d_1=(123),\ d_2=(456),\ \bar{q}=qd_1d_2\,.$$

Then, by Lemma 5, the 4 points

$$q,\ qd_1,\ qd_2,\ \bar{q}=qd_1d_2$$

are 6-cycles, and therefore, by Lemma 3, vertices of

$$F^*(q, \bar{q}\,;\, Q_6)\,.$$

This implies $\dim A(F^*)\geqq 2$ (since not more than two vertices can be on a line), that is,

$$k^*(q, \bar{q};\ Q_6) \geqq 2\,.$$

Finally (if one wants to split hairs) for $m=0$, that is, $n=2$, (9.3) amounts to asserting the existence of at least one 2-cycle; for $q=\bar{q}=(12)$, $F^*(q, \bar{q}; Q_2)=q$, $k^*=0$, hence $K(Q_2)\geqq 0$. This completes the proof of (9.1).

The relations (5.1), (6.1), (7.1), (8.1), and (9.1) constitute the statement at the end of § 1.

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