

Pacific Journal of Mathematics

NEIGHBOR RELATIONS ON THE CONVEX OF CYCLIC PERMUTATIONS

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NEIGHBOR RELATIONS ON THE CONVEX OF CYCLIC PERMUTATIONS

I. HELLER

1. Introduction and summary. Two vertices of a polyhedron are called neighbors of order k when they have a face of dimension k , and none of lower dimension, in common. $K(P)$ denotes the maximum value of k for a given polyhedron P . For the convex hull (polyhedron) P_n of all permutations of n elements (represented by square matrices of order n and interpreted as points in n^2 -space) it was shown [1 and 2] that $K(P) = \lfloor n/2 \rfloor$ (that is, the largest integer not exceeding $n/2$), which is rather small as compared with $\dim P_n = (n-1)^2$. For the convex hull Q_n of all cyclic permutations of n elements that leave no element fixed, $H. Kuhn$ performed computations showing that any two vertices of Q_5 but not any two vertices of Q_6 are neighbors of order 1, which means that $K(Q_5) = 1$ and $K(Q_6) > 1$. The present note, dealing with general n , proves, for $n \geq 8$:

$$(1) \quad K(Q_n) = K(P_n) - 1 = \frac{n}{2} - 1 \quad \text{if } n = 4m + 2$$

$$(2) \quad K(Q_n) = K(P_n) = \left\lfloor \frac{n}{2} \right\rfloor \quad \text{if } n \neq 4m + 2$$

For $n = 1, 2, \dots, 6, 7$, $K(Q_n) = 0, 0, 1, 1, 1, 2, 2$ respectively.

2. A permutation p of n numbered elements is customarily represented by a matrix (p_{ij}) , where

$$p_{ij} = \begin{cases} 1 & \text{when } p \text{ sends } i \text{ into } j \\ 0 & \text{otherwise.} \end{cases}$$

To the product of permutations then corresponds the product of the associated matrices under ordinary matrix multiplication, and therefore the same symbol will be used for a permutation and its matrix.

The following facts from [1] and [2] regarding neighbor relations on P_n will be used in the sequel:

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$$(2.1) \quad K(P_n) = \left[\frac{n}{2} \right]$$

(2.2) p_1 and p_2 are neighbors of order k on P_n if and only if $p_1^{-1}p_2$ is a product of k disjoint cycles (not counting cycles of length 1)

(2.3) If c_1, c_2, \dots, c_k are disjoint cycles and F is the face of lowest dimension that contains the two vertices

$$p \text{ and } \bar{p} = pc_1c_2 \cdots c_k,$$

then F has the 2^k vertices

$$pc_{i_1}c_{i_2} \cdots c_{i_s} \quad (0 \leq s \leq k).$$

3. If the vertices of a convex polyhedron Q are a subset of the vertices of a convex polyhedron P , let two vertices q_1, q_2 of Q be neighbors of order k on P and k^* on Q :

$$k = k(q_1, q_2; P), \quad k^* = k^*(q_1, q_2; Q).$$

Let

$$F = F(q_1, q_2; P), \quad F^* = F^*(q_1, q_2; Q)$$

be the face of lowest dimension of P respectively Q that contains q_1 and q_2 , so that

$$k = \dim A(F), \quad k^* = \dim A(F^*),$$

where $A(F)$ and $A(F^*)$ denote the "affine span" of F and F^* respectively, which is also obtained as the intersection of all hyperplanes that support P respectively Q and contain q_1 and q_2 (with the understanding that A is the entire space when such hyperplanes do not exist); then

$$(3.1) \quad F \supseteq F^*,$$

hence

$$(3.2) \quad A(F) \supseteq A(F^*),$$

and therefore

$$(3.3) \quad k \geq k^*.$$

Proof of (3.1). The line segment joining q_1 and q_2 goes through the interior of F^* (otherwise q_1 and q_2 would have a face of lower dimension in common). Therefore any hyperplane through q_1 and q_2 necessarily contains interior points of F^* .

Further, the vertices of Q , hence in particular those of F^* , are also vertices of P . Therefore any hyperplane that supports P supports F^* .

Above establishes that any hyperplane H that supports P and contains q_1 and q_2 necessarily contains F^* , since it supports F^* and contains points interior to F^* . Therefore

$$A(F) \supseteq F^* ,$$

which, in conjunction with

$$P \supset Q \supset F^* ,$$

implies

$$F^* \subseteq P \cap A(F) .$$

This completes the proof of (3.1), since the right hand side of the last relation equals F .

A somewhat sharper form of (3.1) may be noted as

LEMMA 1. *The vertices of F^* are among the vertices of F .*

The proof is immediate from (3.1) and the fact that the vertices of F^* are vertices of P , and a vertex of P contained in F is vertex of F .

From (3.3) it follows that $\max k^* \leq \max k$, that is

$$(3.4) \quad K(Q) \leq K(P)$$

4. At this point it is convenient to first establish some auxiliary facts. p, q, c denote permutations of n elements, for fixed n .

LEMMA 2. *If*

$$c_1, c_2, \dots, c_r, c_{r+1}, \dots, c_s$$

is a set of s disjoint cycles, and

$$c' = c_1 c_2 \dots c_r, \quad c'' = c_{r+1} c_{r+2} \dots c_s$$

then

$$(4.1) \quad c' + c'' = I + c' c''$$

Proof. Obvious (note that a cycle of less than n elements is still represented as an n by n matrix, with 1's along the main diagonal for fixed elements).

LEMMA 3. Under the assumptions of Lemma 1, let

$$(4.2) \quad q, qc', qc'', qc'c' = \bar{q}$$

be vertices of a polyhedron R . Then

a hyperplane H through q and \bar{q} that supports R contains qc' and qc'' ,

and consequently

$F(q, \bar{q}; R)$ contains qc' and qc'' (obviously as vertices).

This lemma will be used in the particular case where $R=Q_n$ or P_n .

Proof of Lemma 3. Using parentheses to denote the inner product, let H , given by $(h, x) = \alpha$, contain q and \bar{q} but not contain qc' (say); that is

$$(h, q) = (h, \bar{q}) = \alpha, \quad (h, qc') = \alpha + \beta, \quad \beta \neq 0.$$

By (4.1) and (4.2)

$$qc' + qc'' = q + \bar{q},$$

hence

$$(h, qc'') = (h, q + \bar{q} - qc') = 2\alpha - (\alpha + \beta) = \alpha - \beta,$$

so that H separates qc' from qc'' and therefore does not support R .

LEMMA 4. If

$$k = \left[\begin{matrix} n \\ 2 \end{matrix} \right], \quad 2s \leq k$$

$$q = (12 \cdots n)$$

$$c_i = (i, i+k) \quad (i=1, 2, \cdots, k),$$

then the product of q with $2s$ distinct c_i ,

$$qc_{i_1}c_{i_2} \cdots c_{i_{2s}}$$

is an n -cycle.

Proof. Since the c_i are disjoint, they commute, and may be arranged in such manner that

$$i_1 < i_2 < \cdots < i_{2s};$$

then

$$\begin{aligned}
 & (1 \cdots n)(i_1, i_1+k)(i_2, i_2+k) \cdots (i_{2s-1}, i_{2s-1}+k)(i_{2s}, i_{2s}+k) \\
 = & (1 \cdots i_1, i_1+k+1, \cdots i_2+k, i_2+1, \cdots i_3, i_3+k+1, \cdots i_4+k, i_4+1 \cdots \\
 & \cdots i_{2s-1}, i_{2s-1}+k+1, \cdots i_{2s}+k, i_{2s}+1, \cdots \\
 & i_1+k, i_1+1, \cdots i_2, i_2+k+1, \cdots i_3+k, i_3+1, \cdots i_4, i_4+k+1, \cdots \\
 & \cdots i_{2s-1}+k, i_{2s-1}+1, \cdots i_{2s}, i_{2s}+k+1, \cdots n) .
 \end{aligned}$$

It is easily verified above relation also holds, with proper changes, for $i_1=1$ and for $2s=k, 2k=n$.

In similar straightforward fashion one easily proves :

LEMMA 5. *If q is an n -cycle and d is a 3-cycle, then qd is an n -cycle if and only if the elements of d occur in q in the same cyclic order as in d .*

LEMMA 6. *If q is an n -cycle and the 2-cycle $(ij) \neq (km)$, then $q(ij)(km)$ is an n -cycle if and only if the pair i, j separates the pair k, m in q .*

5. The case $n=4m, n=4m+1; m \geq 2$.

$$(5.1) \quad K(Q_n) = K(P_n) \quad (n=4m, 4m+1; m \geq 2)$$

Proof. Because of (3.4), it is sufficient to show that $K(Q_n) \geq K(P_n)$; this will be achieved by showing that for a particular pair of vertices q, \bar{q}

$$(5.2) \quad k(q, \bar{q}; Q_n) \geq \left[\frac{n}{2} \right] = K(P_n) .$$

Now let $2m=k$, so that $n \geq 2k$, choose

$$(5.3) \quad \begin{cases} q = (12 \cdots n) \\ c_s = (i, i+k) & (i=1, 2 \cdots k) \\ \bar{q} = qc_1c_2 \cdots c_k = qc \end{cases}$$

and denote by c' the product of an even number (including 0 and k) of the c_i , by c'' the product of the remaining c_i (whose number is also even, since k is even):

$$(5.4) \quad \begin{cases} c' = c_{i_1}c_{i_2} \cdots c_{i_{2s}} & (0 \leq 2s \leq k) \\ c'c'' = c_1c_2 \cdots c_k = c \end{cases}$$

(It should be noted that the now following proof of $k^*(q, \bar{q}; Q_n) \geq k$ does not depend on the special assumption $n=4m, 4m+1$ and $k=2m$, but rather holds in general for any pair n, k , where k is even and $n \geq 2k$; this fact will be used in § 9).

The qc' are vertices of Q_n (by Lemma 4) and therefore (by Lemma 3) they are also vertices of $F^{*k} = F(q, \bar{q}; Q_n)$.

To verify (5.2), that is

$$\dim A(F^{*k}) \geq k,$$

consider the following subset of $k+1$ vertices of F^{*k} :

$$(5.5) \quad q_1 = qc_1c_1 = q, \quad q_2 = qc_1c_2, \dots, \quad q_k = qc_1c_k, \quad q_{k+1} = qc = \bar{q}.$$

The q_i of (5.5) are linearly independent.

Proof. Assume

$$(5.6) \quad \lambda qc + \sum_{i=1}^k \lambda_i q_i = 0.$$

Successive application of (4.1) to

$$c = c_1c_2 \cdots c_k$$

yields

$$(5.7) \quad c = c_1[c_2 + \cdots + c_k - (k-2)I],$$

and (5.6) becomes

$$\lambda qc_1[c_2 + \cdots + c_k - (k-2)I] + \sum_{i=1}^k \lambda_i qc_1c_i = 0$$

that is

$$qc_1[\lambda_1c_1 - \lambda(k-2)I + \sum_{i=2}^k (\lambda_i + \lambda)c_i] = 0$$

or, equivalently, since q and c_1 are nonsingular matrices

$$(5.8) \quad \lambda_1c_1 - \lambda(k-2)I + \sum_{i=2}^k (\lambda_i + \lambda)c_i = 0$$

Since the c_i are disjoint cycles (5.8) implies

$$\lambda_1 = 0; \quad \lambda_i + \lambda = 0 \quad (i=2, \dots, k); \quad \lambda(k-2) = 0$$

which, in conjunction with $k \neq 2$ (following from $m \geq 2$), further implies

$$\lambda = 0, \quad \lambda_i = 0.$$

This verifies that the $k+1$ q_i of (5.5) are linearly independent, so that the dimension of their linear span is $k+1$, and therefore the dimension of their affine span equal to k . This completes the proof of (5.2) and hence of (5.1)

6. The case $n=4m, n=4m+1; m=1$. Removing the restriction $m \geq 2$ in (5.1) leaves the cases $n=4$ and $n=5$ still to be considered

$$(6.1) \quad K(Q_n)=1 \quad (n=4, 5)$$

Proof. Since, by (3.4) and (2.1), $K(Q_n) \leq 2$, one only has to show that $K(Q_n) \neq 2$.

Assume there were two vertices q and \bar{q} of Q_n such that

$$k^*(q, \bar{q}; Q_n)=2.$$

Then, by (3.4), (3.3) and (2.1)

$$k(q, \bar{q}; P_n)=2,$$

which by (2.2) implies that $q^{-1}\bar{q}$ is a product of two disjoint cycles, say c_1, c_2 , so that $\bar{q}=qc_1c_2$.

Since q and \bar{q} are cycles of the same length (namely n), c_1c_2 is necessarily an even permutation, so that c_1 and c_2 are both of length 2.

Now let F be the lowest dimensional face of P_n containing q and \bar{q} . Then, by (2.3), F has the 4 vertices

$$q, \bar{q}, qc_1, qc_2.$$

of which the last two are not n -cycles and therefore not vertices of F^* . Hence, by Lemma 1, F^* has only the two vertices q and \bar{q} , which implies $k^*=1$ in contradiction to the assumption that $k^*=2$. This completes the proof of (6.1).

7. The case $n=4m+3; m \neq 1$.

$$(7.1) \quad K(Q_n)=K(P_n) \quad (n=4m+3, m \neq 1),$$

including $m=0$.

Proof. Because of (3.4) it is again sufficient to point out two vertices, q, \bar{q} , of Q_n , such that

$$(7.2) \quad k^*(q, \bar{q}; Q_n) \geq K(P_n)=2m+1.$$

For $k=2m$, let q, c_1, c, c', c'' be defined as in (5.3) and (5.4), let $d=(2k+1, 2k+2, 2k+3)$, and $\bar{q}=qcd$.

By Lemmas 4 and 5 the qc' and $qc'd$ are vertices of Q_n for all c' of (5.4), and by Lemma 3 they are also vertices of $F^{*k}(q, \bar{q}; Q_n)$. To prove that

$$\dim A(F^{*k}) \geq 2m + 1,$$

it is shown that the dimension of the linear span of F^{*k} is $\geq 2m + 2 = k + 2$, in verifying that the $k + 2$ vertices of F^{*k}

$$(7.3) \quad q_1 = q = qc_1c_1, q_2 = qc_1c_2, \dots, q_k = qc_1c_k, q_{k+1} = qd, \quad q_{k+2} = \bar{q} = qcd$$

are linearly independent.

Assume

$$(7.4) \quad \sum_{i=1}^{k+2} \lambda_i q_i = 0$$

or, equivalently, substituting for q_i their expressions from (7.3), omitting the non singular common factor qc_1 , and writing μ_i for λ_{k+i} ,

$$(7.5) \quad \sum_{i=1}^k \lambda_i c_i + \mu_1 c_1 d + \mu_2 c_2 c_3 \dots c_k d = 0.$$

Application of (4.1) yields for the left hand side of (7.5)

$$\sum_{i=1}^k \lambda_i c_i + \mu_1 (c_1 + d - I) + \mu_2 [c_2 + \dots + c_k + d - (k-1)I],$$

so that (7.4) is equivalent to

$$(7.6) \quad (\lambda_1 + \mu_1)c_1 + \sum_{i=2}^k (\lambda_i + \mu_2)c_i + (\mu_1 + \mu_2)d - [\mu_1 + (k-1)\mu_2]I = 0$$

Since the c_i and d are disjoint cycles, (7.6) implies

$$(7.7) \quad \begin{cases} \lambda_1 + \mu_1 = 0 \\ \lambda_i + \mu_2 = 0 & (i=2, 3, \dots, k) \\ \mu_1 + \mu_2 = 0 \\ \mu_1 + (k-1)\mu_2 = 0 \end{cases}$$

The last two relations of (7.7) imply (because of the assumption $m \neq 1$, hence $k \neq 2, k-1 \neq 1$)

$$\mu_1 = \mu_2 = 0,$$

which in conjunction with the first two relations of (7.7) implies

$$\lambda_i = 0 \quad (i=1, 2, \dots, k),$$

so that all coefficients of (7.4) vanish; this proves that the q_i of (7.4)

are linearly independent, and completes the proof of (7.2) and hence (7.1).

8. The case $n=7$ (excepted in § 7).

$$(8.1) \quad K(Q_7) = K(P_7) - 1 = 2$$

Proof. By (3.4) and (2.1)

$$K(Q_7) \leq 3.$$

To see that equality cannot hold, let $q = (12 \cdots 7)$.

Because of (2.1) and (3.3), only such \bar{q} must be considered where

$$k(q, \bar{q}; P_7) = 3.$$

By (2.2) the last relation is only possible for

$$\bar{q} = qc_1c_2d,$$

where c_1, c_2, d are disjoint cycles.

For \bar{q} to be a 7-cycle it is necessary (not sufficient) that c_1c_2d be even, that is, that two of them, say c_1 and c_2 , be transpositions and d a 3 cycle.

For the same reason, among the 8 vertices of $F(q, \bar{q}; P_7)$ determined by (2.3), at most 4 are 7-cycles, namely

$$(8.2) \quad q_1 = q, q_2 = qc_1c_2, q_3 = qd, q_4 = \bar{q} = qc_1c_2d,$$

so that, by Lemma 1, $F^*(q, \bar{q}; Q_7)$ has at most the 4 vertices (8.2).

However, application of (4.1) yields

$$q_1 + q_4 = q(I + c_1c_2d) = q(I + c_1c_2 + d - I) = q_2 + q_3$$

which is a relation

$$\sum \lambda_i c_i = 0 \quad \text{with} \quad \sum \lambda_i = 0,$$

therefore

$$\dim A(F^*) \leq 2.$$

It has thus been established that

$$K(Q_7) \leq 2.$$

To complete the proof of (8.1), choose

$$(8.3) \quad q = (12 \cdots 7), c_1 = (13), c_2 = (24), d = (567).$$

Then each q_i of (8.2) is a 7-cycle (by Lemmas 4 and 5) and a

vertex of $F^*(q, \bar{q}; Q_7)$ (by Lemma 3.) The last 3 of these q_i are linearly independent. This establishes, for this particular face F^* ,

$$\dim A(F^*)=2,$$

and completes the proof of (8.1).

9. The case $n=4m+2$.

$$(9.1) \quad K(Q_n)=K(P_n)-1=2m \quad (n=4m+2).$$

The proof is achieved in showing

$$(9.2) \quad K(Q_n) \leq K(P_n)-1=2m$$

$$(9.3) \quad K(Q_n) \geq K(P_n)-1=2m.$$

To verify (9.2), assume $K(Q_n) > K(P_n)-1$, which, by (3.4) and (2.1), implies $K(Q_n)=K(P_n)=2m+1$.

Then there must be a pair of vertices q and \bar{q} on Q_n such that

$$k^*(q, \bar{q}; Q_n)=2m+1,$$

and hence, by (3.3) and (2.1),

$$k(q, \bar{q}; P_n)=2m+1,$$

which, by (2.2) implies

$$\bar{q}=qc_1c_2 \cdots c_{2m+1},$$

where the c_i are disjoint cycles, and therefore necessarily transpositions, because of $n=2(2m+1)$. Then however, the product of the c_i is an odd permutation, and \bar{q} cannot be an n -cycle if q is one. This proves (9.2).

To verify (9.3), consider first the case $m \geq 2$. Setting $2m=k$, the construction from (5.3) through the end of § 5 proves the existence of q, \bar{q} with $k^*(q, \bar{q}; Q_n)=k$, which implies $K(Q_n) \geq k$.

For $m=1$, that is, $n=6$, choose

$$q=(12 \cdots 6), \quad d_1=(123), \quad d_2=(456), \quad \bar{q}=qd_1d_2.$$

Then, by Lemma 5, the 4 points

$$q, \quad qd_1, \quad qd_2, \quad \bar{q}=qd_1d_2$$

are 6-cycles, and therefore, by Lemma 3, vertices of

$$F^*(q, \bar{q}; Q_6).$$

This implies $\dim A(F^*) \geq 2$ (since not more than two vertices can be on

a line), that is,

$$k^*(q, \bar{q}; Q_6) \geq 2.$$

Finally (if one wants to split hairs) for $m=0$, that is, $n=2$, (9.3) amounts to asserting the existence of at least one 2-cycle; for $q=\bar{q}=(12)$, $F^*(q, \bar{q}; Q_2)=q$, $k^*=0$, hence $K(Q_2) \geq 0$. This completes the proof of (9.1).

The relations (5.1), (6.1), (7.1), (8.1), and (9.1) constitute the statement at the end of § 1.

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