FUNCTIONALS ASSOCIATED WITH A CONTINUOUS TRANSFORMATION

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1. Let \( T: z = t(w), w \in R_0 \), be a continuous transformation from a simply connected polygonal region \( R_0 \) in the Euclidean plane \( \pi \), into Euclidean three-space. The transformation \( T \) is a representation for an \( F \)-surface of the type of the 2-cell in Euclidean three-space, which will be called, in brief, a surface \( S \). \([4, \text{II. 3.7, II. 3.44}]\).

In connection with transformation \( T \), T. Radó defines a non-negative (possibly infinite) functional \( \alpha(T) \), which he shows is independent of the representation \( T \) for the surface \( S \). \([4, \text{V. 1.6}]\). Radó calls \( \alpha(T) \) the lower area of the surface, and it plays an important role in the study of surface area.

P. V. Reichelderfer has also defined a non-negative (possibly infinite) functional \( eA(S) \), which he calls the essential area of the surface \( S \). \([5, \text{p. 274}]\). It too is an important concept in surface area theory.

The question arises as to what relationship exists between the lower area \( \alpha(T) \) and the essential area \( eA(S) \). In this paper, we show that \( eA(S) = \alpha(T) \). In addition, we introduce certain other functionals, which we show yield the same value as that of \( eA(S) \) and \( \alpha(T) \). These functionals, as well as \( eA(S) \) and \( \alpha(T) \), will be defined in § 3, after a discussion in § 2 of necessary topological concepts.

2. Let \( M \) be a metric space. If \( A \subset M \), then \( M - A, c(A), i(A), \) and \( fr(A) \) denote respectively, the complement, closure, interior, and frontier of \( A \). If \( A \subset M, B \subset M \), then \( A \cup B, A \cap B \), and \( A - B \) denote the union, intersection, and difference of \( A \) and \( B \). \( \phi \) denotes the empty set.

Let \( F: z = f(w), w \in M \), be a continuous transformation from a metric space \( M \) into a metric space \( N \). If \( P \subset M \), the symbol \( F|P \) denotes the transformation \( F \) with its domain restricted to \( P \).

If \( z \in N \), let \( (F|P)^{-1}z \) denote the set of points \( w \) such that \( w \in P, f(w) = z \). If \( (F|P)^{-1}z \neq \phi \), then the components of \( (F|P)^{-1}z \) are called maximal model components for \( z \) under \( F|P \). If a maximal model component for \( z \) under \( F|P \) is a continuum, then it is called a maximal model continuum (henceforth abbreviated m.m.c.) for \( z \) under \( F|P \).

Now let \( F: \tilde{z} = f(w), w \in R_0 \), be a continuous transformation from a
simply connected polygonal region $R_o$ in the Euclidean plane $\pi$ into the Euclidean plane $\mathbb{E}$.

If $R$ is a Jordan region, $R \subset R_o$, let $C_1, \cdots, C_{n-1}$ denote the interior boundary curves, if any, of $R$, oriented in the negative sense, and let $C_n$ denote the exterior boundary curve of $R$, oriented the positive sense. If $\bar{z} \in F(\bigcup_{i=1}^n C_i)$, let $\mu(\bar{z}, F, R) = 0$. If $\bar{z} \notin F(\bigcup_{i=1}^n C_i)$, let $\mu(\bar{z}, F, R) = \sum_{i=1}^n \lambda(\bar{z}, F, C_i)$, where $\lambda(\bar{z}, F, C_i)$ denotes the topological index of $\bar{z}$ with respect to the oriented closed curve $F(C_i)$, [4, II. 4.34, IV. 1.24]. If $\bar{z} \in \pi$, then $\mu(\bar{z}, F, R)$ is an integer.

If $P$ is a Jordan region or a domain, $P \subset R_o$, we shall call $P$ an admissible set.

Suppose $P$ is an admissible set, and consider $F \setminus P$: $\bar{z} = f(w)$, $w \in P$. Suppose $\gamma$ is a maximal model component for $\bar{z}$ under $F \setminus P$. If, for every open set $G$ containing $\gamma$, there is a Jordan region $R$ such that $\gamma \subset i(R)$, $R \subset G \cap i(P)$, (note that this implies that $\gamma$ is a continuum), and such that $\mu(\bar{z}, F, R) \neq 0$, then we say that $\gamma$ is an essential maximal model continuum, (henceforth abbreviated e.m.m.c.), for $\bar{z}$ under $F \setminus P$.

If $P$ and $Q$ are admissible sets, $Q \subset P$, and if $\bar{z} \in \pi$, then $\kappa(\bar{z}, F \setminus P, Q)$ will denote the number of e.m.m.c.'s for $\bar{z}$ under $F \setminus P$ which are contained in $i(Q)$. $\kappa(\bar{z}, F \setminus P, Q)$ is possibly infinite, while, if finite, it is a non-negative integer. It may be shown that

$$\kappa(\bar{z}, F \setminus P, Q) = \kappa(\bar{z}, F \setminus P, Q) = \kappa(\bar{z}, F, Q).$$

Further, it is clear that if $P_1, \cdots, P_n$ is a collection of admissible sets with disjoint interiors, and if $P_j \subset Q$ for $j = 1, \cdots, n$, then $\sum_{j=1}^n \kappa(\bar{z}, F, P_j) \leq \kappa(\bar{z}, F, Q)$.

If $P$ is an admissible set, then $\kappa(\bar{z}, F, P)$, $\bar{z} \in \pi$, is a lower semi-continuous function, and hence is a Lebesgue measurable function. $\int_{F(P)} \kappa(\bar{z}, F, P) d\bar{z}$ will denote the Lebesgue integral of $\kappa(\bar{z}, F, P)$ over the set $F(P)$.

3. Let $R_o$ be a simply connected polygonal region in the Euclidean plane $\pi$. We shall consider the following types of collections of sets (where it is to be understood that the collections consist of a finite number of sets, each of which is contained in $R_o$):

1. Collections of disjoint simply connected polygonal regions.
2. Collections of disjoint polygonal regions.
3. Collections of simply connected Jordan regions with disjoint interiors.
4. Collections of Jordan regions, with disjoint interiors.
5. Collections of disjoint simply connected domains.
(6) Collections of disjoint domains.

Collections of the type described in (j) will be called collections of class $j$, $j=1, \ldots, 6$. If $A \subset \mathbb{R}_0$, and if $\Phi$ is a collection of class $j$ such that $R \in \Phi$ implies $R \subset A$, then we shall say that $\Phi$ is a collection of class $j$ in $A$.

The transformation $T: z=t(w), \ w \in \mathbb{R}_0$, described in § 1, may be written $T: z=t(w)=(x_1(w), x_2(w), x_3(w)), \ w \in \mathbb{R}_0$, where $x_1(w), x_2(w),$ and $x_3(w)$ are the rectangular coordinates of $t(w)$. We now define three plane transformations.

$$T_1: \ z_1=t_1(w)=(x_2(w), x_3(w)), \ w \in \mathbb{R}_0$$

$$T_2: \ z_2=t_2(w)=(x_3(w), x_1(w)), \ w \in \mathbb{R}_0$$

$$T_3: \ z_3=t_3(w)=(x_1(w), x_2(w)), \ w \in \mathbb{R}_0.$$  

For $i=1, 2, 3$, $T_i: z_i=t_i(w), \ w \in \mathbb{R}_0$, is a continuous transformation from $\mathbb{R}_0$ into the Euclidean plane $\pi_i$.

If $P$ is an admissible set, (see § 2), let $g(T_i, P)=\int_{T_i(P)} \kappa(z_i, T_i, P)dz_i$, for $i=1, 2, 3$, and let $G(T, P)=[\sum_{i=1}^3 (g(T_i, P))^2]^{1/2}$. These quantities are non-negative and possibly infinite.

If $\Phi$ is a collection of admissible sets, let $g(T, \Phi)=\sum_{P \in \Phi} g(T, P)$, for $i=1, 2, 3$, and let $G(T, \Phi)=\sum_{P \in \Phi} G(T, P)$.

For $j=1, \ldots, 6$, let $a_j(T)=\text{l.u.b.} G(T, \Phi)$, where the least upper bound is taken with respect to all collections $\Phi$ of class $j$. These quantities are non-negative, possibly infinite. We note that $a_6(T)$ is precisely the lower area $\alpha(T)$, and $a_3(T)$ is the essential area $eA(S)$, discussed in § 1, [4, V. 1.3], [5, p. 274].

The purpose of this paper is to show that the functionals $a_j(T)$, $j=1, \ldots, 6$, all yield the same value.

4. It is quite obvious from the definitions set forth in § 3. that $a_3(T) \leq a_2(T) \leq a_1(T)$, $a_1(T) \leq a_5(T) \leq a_6(T)$, and $a_3(T) \leq a_6(T)$.

Further, if $R_1, \ldots, R_n$ is a collection of class 3, then $i(R_1), \ldots, i(R_n)$ is a collection of class 5, while, for $k=1, \ldots, n$, and $i=1, 2, 3$, we have, (see § 2), $\kappa(z_i, T_i, R_k)=\kappa(z_i, T_i, i(R_k))$. From this it follows that $a_3(T) \leq a_6(T)$. The same type of reasoning shows that $a_6(T) \leq a_4(T)$.

5. If $D$ is a domain, $D \subset \mathbb{R}_0$, then there exists a sequence $\{R_n\}$ of polygonal regions, such that $R_n \subset i(R_{n+1})$ for each $n$, and $\bigcup_{n=1}^\infty R_n = D$, [4, I. 2.48]. Then $\lim_{n \to \infty} \kappa(z_i, T_i, R_n) = \kappa(z_i, T_i, D)$, for $i=1, 2, 3$, [4; IV.
1.43], and this implies that \( a_0(T) \leq a_3(T) \).

In addition, if \( D \) is simply connected, then the polygonal regions \( R_n, n=1, 2, \cdots \), may be chosen to be simply connected, and thus \( a_3(T) \leq a_4(T) \).

6. The inequalities in § 4 and in § 5 yield \( a_1(T)=a_2(T)=a_3(T) \) and \( a_4(T)=a_5(T)=a_6(T) \), while \( a_7(T) \geq a_8(T) \). To establish the equality of these six functionals, therefore, it is sufficient to show that \( a_1(T) \geq a_2(T) \).

Note that if \( G(T, R_0)=+\infty \), then \( a_1(T)=+\infty \), and so \( a_9(T) \geq a_4(T) \). Thus we shall assume henceforth, without loss of generality, that \( G(T, R_0) \leq +\infty \). This in turn implies that if \( \phi \) is any collection of class \( j, j=1, \cdots, 6 \), then \( G(T, \phi) \leq \sum_{i=1}^{3} g(T, \phi) \leq \sum_{i=1}^{3} g(T, R_0) \leq 3G(T, R_0) \). Consequently, \( a_j(T) \leq 3G(T, R_0) \leq +\infty \), that is, \( a_j(T) \) is finite, \( j=1, \cdots, 6 \).

7. In this section, we suppose that all sets considered are subsets of the Euclidean plane \( \pi \).

Suppose \( A \) and \( B \) are connected sets, \( C \) is a closed set and \( A \cup B \subset \pi - C \). We shall say that \( C \) separates \( A \) and \( B \) if \( A \) and \( B \) are contained in distinct components of \( \pi - C \).

Suppose that \( C \) is closed, \( C \subset R \), where \( R \) is a polygonal region. Let \( Q_1, \cdots, Q_{q-1} \) denote the bounded components of \( \pi - R \), if any, and let \( Q_q \) be the unbounded component of \( \pi - R \). We shall say that \( C \) separates in \( R \) if there exists \( k, 1 \leq k \leq q-1 \), such that \( C \) separates \( Q_k \) and \( Q_q \).

Let \( \mathcal{G} \) be an upper semi-continuous collection of continua \( \gamma \), such that \( \bigcup_{\gamma \in \mathcal{G}} \gamma = R \), [4; II. 1.10]. Let \( E \) be the set of points belonging to continuia of \( \mathcal{G} \) which separate in \( R \). Then \( E \) is closed. If \( R-E \neq \phi \), let \( M \) be a component of \( R-E \), and let \( N=M \cap i(R) \). Then there exist a finite number of sets, \( \gamma_1, \cdots, \gamma_q \), such that either \( \gamma_k = \phi \), or else \( \gamma_k \) is a continuum of \( \mathcal{G}, k=1, \cdots, q \), and such that \( \text{fr}(N) \cap i(R) \subset \bigcup_{k=1}^{q} \gamma_k \).

Suppose further that \( R' \) is a polygonal region, and \( R \subset N \). Let \( Q'_1, \cdots, Q'_{t-1} \) denote the bounded components of \( \pi - R' \), if any, and let \( Q'_t \) denote the unbounded component \( \pi - R' \). Suppose also that \( Q_k \not\subset N, k=1, \cdots, t \). Let \( \mathcal{H} \) be an upper semi-continuous collection of continua \( \gamma' \) such that \( \bigcup_{\gamma' \in \mathcal{H}} \gamma' = R' \), and such that if \( \gamma' \in \mathcal{H} \), then there exists \( \gamma \in \mathcal{G} \) for which \( \gamma' \subset \gamma \). Then no continuum of \( \mathcal{H} \) separates in \( R' \).

Next, suppose \( \mathcal{F} \) is an upper semi-continuous collection of continua \( \gamma \), for which \( \gamma \subset \gamma = R' \), and such that no continuum of \( \mathcal{F} \) separates in \( R \). Suppose \( \mathcal{S} \) is an upper semicontinuous collection of continua \( \gamma' \),
\[ \bigcup \gamma' = R', \text{ such that if } \gamma' \in \mathcal{L}, \text{ there exists } \gamma \in \mathcal{F} \text{ for which } \gamma' \subseteq \gamma. \]

Then no continuum of \( \mathcal{L} \) separates in \( R' \).

8. We now state several lemmas concerning the transformation \( T \) defined in §1 and §3. It is assumed that \( G(T, R_0) < +\infty \).

**Lemma 1.** If \( R \) is a polygonal region, \( R \subset R_0 \), then, for \( i=1, 2, 3 \), there exists a set \( K_i, K_i \subset T_i(R) \subset \pi_i \), for which \( m(K_i) = 0 \), (where \( m(K_i) \) denotes the Lebesgue measure of \( K_i \)), and such that if \( z_i \notin K_i \), then every m.m.c. \( \gamma \) for \( z_i \) under \( T_i \mid R \) is also an m.m.c. for \( z_i \) under \( T_i \mid R \). [1; vol. 10, p. 287].

**Lemma 2.** If \( R \) is a polygonal region, \( R \subset R_0 \), then, for \( i=1, 2, 3 \), there exists a set \( B_i, B_i \subset T_i(R) \subset \pi_i \), for which \( m(B_i) = 0 \), and such that \( \bigcup T_i(\gamma) \subset B_i \), where the union is extended over every e.m.m.c. \( \gamma \) under \( T_i \mid R \) such that \( \pi - \gamma \) has more than one component. [3; pp. 593-6].

**Lemma 3.** Suppose \( R \) is a polygonal region, \( R \subset R_0 \). Suppose that, for \( i=1, 2, 3 \), \( F_i \) is a bounded Lebesgue measurable set, \( F_i \subset \pi_i \). Then, given \( \varepsilon > 0 \), there exists a closed, totally disconnected set \( E_i \), such that \( E_i \subset F_i \) and

\[ \int_{E_i} \kappa(z_i, T_i, R)dz_i > \int_{F_i} \kappa(z_i, T_i, R)dz_i - \varepsilon. \]

9. As stated previously, we wish to show that \( a_j(T) = a_k(T) \), \( j, k = 1, \cdots, 6 \), and it was noted in §6 that to do this, it is sufficient to show that \( a_i(T) \geq a_j(T) \) under the assumption that \( G(T, R_0) < +\infty \). The proof that \( a_i(T) \geq a_j(T) \) when \( G(T, R_0) < +\infty \) will be a consequence of Theorem 1 and Theorem 2, which we now consider.

**Theorem 1.** If \( R \) is a polygonal region, \( R \subset R_0 \), then, given \( \varepsilon > 0 \), there is a collection \( \Phi_i \) of class 2 in \( R \), and a subcollection \( \Psi_i \) of \( \Phi_i \) such that

(a) \( g(T_i, \Phi_i) > g(T_i, R) - \varepsilon \), \quad \text{ } \quad i=1, 2, 3.

(b) \( g(T_i, \Psi_i) > g(T_i, R) - \varepsilon \).

(c) If \( \bar{R} \in \Psi_i \), then no m.m.c. under \( T_i \mid \bar{R} \) separates in \( \bar{R} \).

(d) If \( \bar{R} \in \Psi_i \), and if, for some \( i, 1 \leq i \leq 3 \), no m.m.c. under \( T_i \mid \bar{R} \) separates in \( R \), then no m.m.c. under \( T_i \mid \bar{R} \) separates in \( \bar{R} \).

(There exist similar collections \( \Phi_i, \Psi_i \), and \( \Phi_j, \Psi_j \), having similar properties relative to the transformations \( T_2 \) and \( T_3 \) respectively.)
Proof. (1) If $R$ is simply connected, then $\Phi_i$ and $\Psi_i$ may both be chosen to consist of $R$ alone.

(2) If $R$ is not simply connected, let $Q_1, \ldots, Q_{n-1}$ denote the bounded components of $\pi - R$ and let $Q_n$ be the unbounded component of $\pi - R$. Let $r_1, \ldots, r_q$ denote the disjoint simple closed polygons which constitute the frontier of $R$, in such a way that $r_k = \text{fr}(Q_k)$, $k=1, \ldots, q$. Consider $T_i| R$: $z_i = t_i(w)$, $w \in R$. Let $\mathcal{C}$ denote the collection of all m.m.c.'s under $T_i| R$. Then $\mathcal{C}$ is an upper semi-continuous collection of continua $\gamma$, such that $\bigcup \gamma = R$, and the statements of § 7 apply. Let $E$ be the set of points which belong to m.m.c.'s under $T_i| R$ which separate in $R$. $E$ is closed.

(3) If $E$ is empty, then $\Phi_i$ and $\Psi_i$ may both be chosen to consist of $R$ alone.

If $i(R) \subset E$, then $E=R$. In this case, every m.m.c. $\gamma$ under $T_i| R$ is such that $\pi - \gamma$ has more than one component. Consequently, by Lemma 2, there is a set $B_i, B_i \subset T_i(R) \subset \pi_i$, $m(B_i)=0$, such that $\bigcup T_i(\gamma) \subset B_i$, where the union is extended over every e.m.m.c. $\gamma$ under $T_i| R$. If $z_i \notin B_i$, we have $\kappa(z_i, T_i, R)=0$, so $g(T_i, R)=0$. Thus in this case we may let $\Phi_i$ consist of $R$ alone, and we may let $\Psi_i$ be the empty collection.

(4) From (3), we may assume $E \neq \Phi_i, E \neq R$. Then $R-E \neq \Phi_i$. $R-E$ is open relative to $R$, and the components of $R-E$ are open relative to $R$, and form at most a countably infinite collection. These components will be denoted by $C_1, C_2, \ldots$. Let $D_j = C_j \cap i(R)$ for each $j$. $D_j$ is non-empty, open, and connected for each $j$.

(5) Suppose $\gamma$ is an e.m.m.c. under $T_i| R$. Then $\gamma \subset i(R)$. Hence either $\gamma \subset E$ or else $\gamma \subset i(R) \cap (R-E) = \bigcup D_j$.

In the first case, $\gamma$ separates in $R$, and so $\pi - \gamma$ has more than one component. By Lemma 2, there is a set $B_i, B_i \subset T_i(R) \subset \pi_i$, $m(B_i)=0$, and $\bigcup T_i(\gamma) \subset B_i$, where the union is extended over every e.m.m.c. $\gamma$ under $T_i| R$ for which $\pi - \gamma$ has more than one component.

In the second case, since $D_j$ is a component of $\bigcup D_j$, there exists $j$ such that $\gamma \subset D_j$. Hence $\gamma$ is an e.m.m.c. under $T_i| D_j$. This implies that if $z_i \notin B_i$, then $\sum_{j=1}^m \kappa(z_i, T_i, D_j) = \kappa(z_i, T_i, R)$. Since $m(B_i)=0$, we have $\sum_{j=1}^m g(T_i, D_j) = g(T_i, R)$. There is an integer $n$ such that $\sum_{j=1}^n g(T_i, D_j) > g(T_i, R) - \varepsilon/2$.

(6) For each $j, j=1, \ldots, n$, and for each $k, k=1, \ldots, q$, we have, from § 7, a set $\gamma_{jk}$ such that either $\gamma_{jk} = \Phi_i$, or else $\gamma_{jk}$ is an e.m.m.c. under $T_i| R$, such that $\text{fr}(D_j) \cap i(R) \subset \bigcup_{j,k} \gamma_{jk}$, for each $j, j=1, \ldots, n$.

Therefore, $\bigcup_{j=1}^n \text{fr}(D_j) \cap i(R) \subset \bigcup_{j=1}^n \bigcup_{k=1}^q \gamma_{jk}$, and $T_i(\bigcup_{j=1}^n \text{fr}(D_j) \cap i(R))$ is
a finite set. Also, \( \bigcup_{j=1}^{n} \text{fr}(D_j) \subset \bigcup_{j=1}^{n} \gamma_j \cup \text{fr}(R) = \bigcup_{j=1}^{n} \gamma_j \cup \bigcup_{k=1}^{q} \gamma_k \).

(7) Let \( F = \bigcup_{j=1}^{n} c(D_j) \). \( F \) is closed, \( F \subset R \), and \( R - F \) is open relative to \( R \). Let \( C_{n+1}, C_{n+2}, \ldots \) denote the components of \( R - F \). These components are open relative to \( R \), and form at most a countably infinite collection. For each \( j \), let \( \text{D'} \mathcal{n+j} = \bigcap_{j=1}^{n+j} \bigcup_{k=1}^{q} \gamma_k \). \( \text{D'} \mathcal{n+j} \) is open and connected. (We are assuming \( R - F \neq \emptyset \). If \( R - F = \emptyset \), the proof is essentially the same and somewhat simpler.)

Also, it is easily seen that \( \bigcup_{j=1}^{n} \text{fr}(\text{D'} \mathcal{n+j}) \subset \bigcup_{j=1}^{n} \bigcup_{k=1}^{q} \gamma_k \),

\[
\left( \bigcup_{j=1}^{n} \text{fr}(D_j) \right) \cup \left( \bigcup_{j=1}^{n} \text{fr}(D'_{n+j}) \right) \cup \left( \bigcup_{j=1}^{n} \text{fr}(\text{D'} \mathcal{n+j}) \right) = R,
\]

and

\[
\left( \bigcup_{j=1}^{n} D_j \right) \cup \left( \bigcup_{j=1}^{n} D'_{n+j} \right) \cup \left( \bigcup_{j=1}^{n} \text{fr}(D_j) \right) \cup \left( \bigcup_{k=1}^{q} \gamma_k \right) = R.
\]

(8) Consider the transformation \( T_z \mid R : z_2 = t_z(w) \), \( w \in R \). Let \( \gamma \) be an e.m.m.c. under \( T_z \mid R \). Then either \( \gamma \) intersects \( \left( \bigcup_{j=1}^{n} \text{fr}(D_j) \right) \cap i(R) \), or not.

In the first case, from (7), \( \gamma \) intersects \( \bigcup_{j=1}^{n} \text{fr}(D_j) \cap i(R) \). In (6), we have seen that \( T_i \left( \bigcup_{j=1}^{n} \text{fr}(D_j) \cap i(R) \right) \) is a finite set, so \( T_i \left( \bigcup_{j=1}^{n} \text{fr}(D_j) \cap i(R) \right) \) is a set of measure zero. Then \( \bigcup_{j=1}^{n} T_i(\gamma) \subset T_i \left( \bigcup_{j=1}^{n} \text{fr}(D_j) \cap i(R) \right) \), where the union is extended over every e.m.m.c. \( \gamma \) under \( T_z \mid R \) such that \( \gamma \cap \left( \bigcup_{j=1}^{n} \text{fr}(D_j) \right) \cap i(R) \neq \emptyset \).

In the second case, \( \gamma \subset \left( \bigcup_{j=1}^{n} D_j \right) \cup \left( \bigcup_{j=1}^{n} D'_{n+j} \right) \), from (7). If there exists \( j, 1 \leq j \leq n \), such that \( \gamma \cap D_j \neq \emptyset \), then, since \( \gamma \) is connected, and \( \gamma \cap \text{fr}(D_j) = \emptyset \), it follows that \( \gamma \subset D_j \), \( \gamma \) is an e.m.m.c. under \( T_z \mid D_j \).

If there is a \( j \) such that \( \gamma \cap D_{n+j} \neq \emptyset \), then the same reasoning shows that \( \gamma \) is an e.m.m.c. under \( T_z \mid D'_{n+j} \).

Hence, if \( z_2 \notin T_z \left( \bigcup_{j=1}^{n} \text{fr}(D_j) \cap i(R) \right) \), we have

\[
\sum_{j=1}^{n} \kappa(z_2, T_z, D_j) + \sum_{j=1}^{n} \kappa(z_2, T_z, D'_{n+j}) = \kappa(z_2, T_z, R).
\]

Since

\[
T_z \left( \bigcup_{j=1}^{n} \text{fr}(D_j) \cap i(R) \right)
\]

is a set of measure zero, we have

\[
\sum_{j=1}^{n} g(T_z, D_j) + \sum_{j=1}^{n} g(T_z, D'_{n+j}) = g(T_z, R).
\]
(In similar fashion, $\sum_{j=1}^{n} g(T, D_j) + \sum_{j=1}^{n'} g(T, D_{n+j}) = g(T, R).$)

(9) Choose $n'$ so that

$$\sum_{j=1}^{n} g(T, D_j) + \sum_{j=1}^{n'} g(T, D_{n+j}) > g(T, R) - \varepsilon/2, \quad \text{for } i=1, 2, 3.$$  

We can determine polygonal regions $R_j, j=1, \ldots, n+n'$, so that $R_j \subset D_j$ and no component of $\pi - R_j$ is contained in $D_j$ for $j=1, \ldots, n$, and so that $R_{n+j} \subset D_{n+j}$, and no component of $\pi - R_{n+j}$ is contained in $D_{n+j}$, for $j=1, \ldots, n'$, and such that

$$\sum_{j=1}^{n} g(T, R_j) > \sum_{j=1}^{n'} g(T, D_j) + \sum_{j=1}^{n'} g(T, D_{n+j}) - \varepsilon/2,$$

for $i=1, 2, 3$.

Let $m=n+n'$. Then

$$\sum_{j=1}^{m} g(T, R_j) > g(T, R) - \varepsilon, \quad \text{for } i=1, 2, 3,$$

and

$$\sum_{j=1}^{n} g(T, R_j) > g(T, R) - \varepsilon.$$

For each $j, j=1, \ldots, n$, consider the transformation $T_{j|R_j} : z = t_{j}(w), w \in R_j$. Let $\mathcal{H}_j$ denote the collection of m.m.c.'s under $T_{j|R_j}$. Then $\mathcal{H}_j$ is an upper semi-continuous collection of continua $\gamma'$, with $\bigcup_{\gamma' \in \mathcal{H}_j} \gamma' = R_j$.

Further, if $\gamma' \in \mathcal{H}_j$, there exists $\gamma \in \mathcal{H}_j$, such that $\gamma' \subset \gamma$. In addition, no component of $\pi - R_j$ is contained in $D_j$. From §7, no continuum of $\mathcal{H}_j$ separates in $R_j$, that is no m.m.c. under $T_{j|R_j}$ separates in $R_j$, $j=1, \ldots, n$.

In a similar fashion, we find from §7 that if, for some $i, 1 \leq i \leq 3$, no m.m.c. under $T_{i|R}$ separates in $R$, then no m.m.c. under $T_{i|R_j}$ separates in $R_j, j=1, \ldots, n$.

(10) Let $\varphi_i$ be the collection consisting of the disjoint polygonal regions $R_i, \ldots, R_m$, and let $\psi_i$ be the collection consisting of $R_i, \ldots, R_n$. These collections satisfy the requirements of the theorem. Assertions (a), (b), (c), and (d) of the theorem have been verified in (9).

10. We now prove the following.

**Theorem 2.** Let $R$ be a polygonal region, $R \subset R_0$, and give $\varepsilon > 0$. Let $i_1, \ldots, i_h, 1 \leq h \leq 3$, denote those subscripts, if any, such that no
m.m.c. under $T_j|R$ separates in $R$, $j=1, \ldots, h$. Then there exists a collection $\Phi$ of class 1 in $R$ such that $g(T_{ij}, \Phi) > g(T_{ij}, R) - \varepsilon$, $j=1, \ldots, h$.

Proof. We shall prove the theorem in the case where no m.m.c. under $T_i|R$ separates in $R$, for $i=1, 2, 3$. Then proofs in the remaining case are similar, and simpler.

(1) If $R$ is simply connected, then $\Phi$ may be chosen to consist of $R$ alone.

(2) If $R$ is not simply connected, then let $Q_1, \ldots, Q_{q-1}$ denote the bounded components of $\pi - R$, and let $Q_q$ denote the unbounded component of $\pi - R$. Let $r_1, \ldots, r_q$ denote the disjoint simple closed polygons which constitute the frontier of $R$ in such a way that $r_k = \text{fr}(Q_k), k=1, \ldots, q$.

By Lemma 1, there is for $i=1, 2, 3$, a set $K_i, K_i \subset T_i(R) \subset \pi_i$, such that $m(K_i) = 0$, and such that if $\gamma$ is an m.m.c. under $T_i|R$, and if $T_i(\gamma) \not\in K_i$, then $\gamma$ is an m.m.c. under $T_i|R$. By Lemma 3, there is for $i=1, 2, 3$, a closed and totally disconnected set $E_i$, such that $E_i \subset (\pi - K_i) \cap T_i(R)$, and such that

$$\int_{E_i} k(z, T_i, R)dz_i > \int_{(\pi - K_i) \cap T_i(R)} k(z, T_i, R)dz_i - \frac{\varepsilon}{2}. $$

Since

$$\int_{K_i} k(z, T_i, R)dz_i = 0,$$

we have

$$\int_{E_i} k(z, T_i, R)dz_i > g(T_i, R) - \frac{\varepsilon}{2}. $$

Let $\overline{E}_i = (T_i|R)^{-1}E_i$, for $i=1, 2, 3$. Then $\overline{E}_i$ is closed, and also, the components of $\overline{E}_i$ are m.m.c.'s under $T_i|R$. No component of $\overline{E}_i$ separates in $R$, and $\overline{E}_i$ does not separate in $R$, for $i=1, 2, 3$, [2; p. 117].

(3) Let $\gamma_i$ be a component of $\overline{E}_i$. Suppose $\gamma_i \cap \overline{E}_2 \neq \phi$. Then there is a component $\gamma_2$ of $\overline{E}_2$ such that $\gamma_i \cap \gamma_2 \neq \phi$. $\gamma_i$ and $\gamma_2$ are, respectively, m.m.c.'s under $T_i|R$ and $T_2|R$, while $T_i(\gamma_i) \not\in K_i$ and $T_2(\gamma_2) \not\in K_2$. Consequently, $\gamma_i$ and $\gamma_2$ are both m.m.c.'s under $T_i|R$, so $\gamma_i = \gamma_2$.

Therefore, if $\gamma_i$ is a component of $\overline{E}_i$, then $\gamma_i \cap \overline{E}_2$ is connected. Thus $\overline{E}_i \cup \overline{E}_2$ does not separate in $R$, [2; p. 120].

Let $\gamma_3$ be a component of $\overline{E}_3$. As above, either $\gamma_3 \cap \overline{E}_1 = \phi$ or else $\gamma_3 \cap \overline{E}_1 = \gamma_1$, and either $\gamma_3 \cap \overline{E}_2 = \phi$ or else $\gamma_3 \cap \overline{E}_2 = \gamma_2$. Hence, $\gamma_3 \cap (\overline{E}_1 \cup \overline{E}_2)$ is connected, and so $\overline{E}_1 \cup \overline{E}_2 \cup \overline{E}_3$ does not separate in $R$, [2; p. 120].
(4) Let \( \overline{E} = E_1 \cup E_2 \cup E_3 \). \( \overline{E} \) is closed, so \( \pi - \overline{E} \) is open. Also, since \( \overline{E} \) does not separate in \( R \), the components \( Q_k, k = 1, \ldots, q \), of \( \pi - R \) are contained in the same component of \( \pi - E \). Denote this component by \( D \). Since \( D \) is open and connected, there exist polygonal arcs \( p_h, k = 1, \ldots, q - 1 \), so that for each \( k \), \( p_h \cap \overline{E} = \emptyset \), and \( p_h \cup S_h \cup S_q \) is connected, where \( S_h = Q_h \cup r_h, k = 1, \ldots, q \).

Let \( G = i(R) - \bigcup_{k=1}^{q-1} p_h \). Then \( G \) is open, \( G \subset R \). Let \( D_1, \ldots, D_j, \ldots \) be the components of \( G \). For each \( j \), \( D_j \subset R \), and \( fr(D_j) \subset fr(G) \subset \pi - G = \bigcup_{k=1}^{q-1} (p_h \cup S_h \cup S_q) \cup S_q \). Then \( \pi - G \) is connected, so \( \pi - D_j \) is contained in a single component of \( \pi - D_j \). But each component of \( \pi - D_j \) contains just one component of \( fr(D_j) \), so \( \pi - D_j \) has only one component, that is, \( D_j \) is a simply connected domain, [2; p. 118].

(5) If, for some \( i, 1 \leq i \leq 3 \), \( \gamma \) is an e.m.m.c. under \( T_i|\overline{R} \), then \( \gamma \subset i(R) \). Either \( \gamma \cap \left( \bigcup_{h=1}^{q-1} p_h \right) \neq \emptyset \), or else \( \gamma \subset G \).

In the first case, \( T_i(\gamma) \notin E_i \), for otherwise

\[
\gamma \subset (T_i|R)^{-1}T_i(\gamma) \subset (T_i|R)^{-1}E_i = \overline{E_i},
\]

while

\[
\overline{E_i} \cap \left( \bigcup_{k=1}^{q-1} p_h \right) = \emptyset.
\]

Hence \( \gamma \subset G \) implies \( T_i(\gamma) \notin E_i \).

If \( \gamma \subset G \), then since \( \gamma \) is connected, it follows that \( \gamma \) is contained in a component \( D_j \) of \( G \), and \( \gamma \) is an e.m.m.c. under \( T_i|D_j \).

Therefore, if \( z_i \in E_i \), then each e.m.m.c. \( \gamma \) under \( T_i|R \), for which \( T_i(\gamma) = z_i \), is also an e.m.m.c. under \( T_i|D_j \), for some \( j \). Then

\[
\iint_{R_i} \kappa(z_i, T_i, R)dz_i = \sum_{j=1}^{\infty} \iint_{R_i} \kappa(z_i, T_i, D_j)dz_i,
\]

and

\[
\sum_{j=1}^{\infty} g(T_i, D_j) \geq \sum_{j=1}^{\infty} \iint_{R_j} \kappa(z_i, T_i, D_j)dz_i = \iint_{R_i} \kappa(z_i, T_i, R)dz_i \geq g(T_i, R) - \frac{\varepsilon}{2},
\]

for \( i = 1, 2, 3 \).

(6) There is an integer \( n \) for which

\[
\sum_{j=1}^{n} g(T_i, D_j) \geq g(T_i, R) - \frac{\varepsilon}{2},
\]
for \( i = 1, 2, 3 \). Each domain \( D_i \) is simply connected, so there is a collection \( R_i, \cdots, R_n \) of class 1, such that

\[
R_j \subset D_j \subset R, \text{ and } g(T, R_j) > g(T, D_j) - \frac{\varepsilon}{2n},
\]

for \( j = 1, \cdots, n, i = 1, 2, 3 \). Then

\[
\sum_{j=1}^{n} g(T, R_j) > g(T, R) - \varepsilon,
\]

and the collection \( R_i, \cdots, R_n \) serves as the collection \( \varphi \) in the statement of Theorem 2.

11. From Theorem 1 and Theorem 2, the following theorems are readily proved.

**Theorem 3.** If \( R \) is a polygonal region, \( R \subset R_0 \) and if \( \varepsilon > 0 \), then there is a collection \( \Phi_i \) of class 1 in \( R \) such that \( g(T, \Phi_i) > g(T, R) - \varepsilon \).

(Similar collections \( \Phi_2 \) and \( \Phi_3 \) exist relative to the transformations \( T_2 \) and \( T_3 \).)

**Theorem 4.** If \( R \) is a polygonal region, \( R \subset R_0 \), and if \( \varepsilon > 0 \), then there is a collection \( \Phi_3 \) of class 1 in \( R \) such that \( g(T, \Phi_3) > g(T_3, R) - \varepsilon \), and \( g(T_2, \Phi_3) > g(T_2, R) - \varepsilon \).

(Similar collections \( \Phi_2 \) and \( \Phi_3 \) exist relative to the transformations \( T_3 \) and \( T_3 \) and to the transformations \( T_2 \) and \( T_3 \).)

**Theorem 5.** If \( R \) is a polygonal region \( R \subset R_0 \), and if \( \varepsilon > 0 \), then there is a collection \( \Phi \) of class 1 in \( R \), such that \( g(T_1, \Phi) > g(T_1, R) - \varepsilon \), for \( i = 1, 2, 3 \).

12. From Theorem 5, it follows that if \( R \) is a polygonal region, \( R \subset R_0 \), and if \( \varepsilon > 0 \), then there is a collection of class 1 in \( R \), such that \( G(T, \Phi) > G(T, R) - \varepsilon \).

This in turn implies, of course, that if \( \Psi \) is a collection of class 2 in \( R_0 \), and if \( \varepsilon > 0 \), then there is a collection \( \Phi \) of class 1 in \( R_0 \), such that \( G(T, \Phi) > G(T, \Psi) - \varepsilon \). Hence \( a_i(T) \geq a_i(T) \), and so each of the functionals \( a_i(T) \), \( j = 1, \cdots, 6 \), defined in §3, yields the same value. We have shown in particular that the essential area of Reichelderfer, \( a_e(T) \) is equal to the lower area of Radó, \( a_d(T) \).

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REFERENCES


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