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1. Introduction. Let G be a separable Lie group and let V be a complete, metrizable, topological vector space. The underlying space of G is a separable real analytic manifold so that we can define, by the methods of L. Schwartz (see [7], [12], [13]), the spaces $\mathscr{E}(V)$ of indefinitely differentiable maps of G into V, and $\mathscr{D}(V)$ which consists of those maps in $\mathscr{E}(V)$ which are of compact carrier. Their duals are $\mathscr{D}'(V)$, the space of distributions on G with values in V' (the dual of V), and $\mathscr{E}'(V)$ which is the space of distributions of compact carrier with values in V'.

By using the group structure in G, we can define the convolution $S*f \in \mathscr{E}(C)$ for any $S \in \mathscr{D}'(V)$, $f \in \mathscr{D}(V)$, where C is the complex plane. The main result of this paper is: Let $S \in \mathscr{D}'(V)$ have the property that $S*f \in \mathscr{D}(C)$ whenever $f \in \mathscr{D}(V)$; then $S \in \mathscr{E}'(V)$. Moreover, the topology of $\mathscr{E}'(V)$ is that obtained by considering each $S \in \mathscr{E}'(V)$ as defining the continuous linear transformation $f \to S*f$ of $\mathscr{D}(V) \to \mathscr{D}(C)$ and then giving this set of transformations the compact-open topology (see [6]). This generalizes the result of [6] in case G is a vector group and $V=C$.

This result is generalized to double coset spaces $L\backslash G/K$ where L and K are compact subgroups of G. In this form, the result will be used by the author and F. I. Mautner to generalize the Paley-Wiener theorem and the theory of mean-periodic functions of Schwartz (see [8]).

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2. Distributions on G. Instead of using the usual method of defining distributions on G, as for example in de Rham and Kodaira [12], we shall follow another approach which is more akin to the author's thesis [5]. We shall show that the two methods are equivalent.

By " function " we shall mean " complex-valued function " unless the contrary is specifically stated. " Linear " will mean " linear over the complex numbers " always. By 1 we denote the identity in G, and by $\mathfrak{g}$ we denote the Lie algebra of G. For any $Y \in \mathfrak{g}$, we denote by $t \to \exp(tY)$ the unique one parameter subgroup in G whose direction

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at 1 is Y. Let V be a complete metrizable locally convex topological vector space.

The map f of G into V is said to be differentiable in the direction $Y \in \mathfrak{g}$ at $x \in G$ if $\left\{\left(\frac{df}{dt}\right)[(\exp tY)x]\right\}_{t=0}$ exists; if this is the case, we set

$$(1) \qquad (D_Y f)(x) = \left\{\left(\frac{df}{dt}\right)[(\exp tY)x]\right\}_{t=0}.$$

If f is a continuous map of G into V, we say that f is in the domain of D_Y if, for any $x \in G$, f is continuously differentiable in the direction Y at x. $D_Y f$ is then defined as the (continuous) map $x \to (D_Y f)(x)$.

By $\mathscr{E}^0$ we denote the space of continuous maps of G into V with the topology of uniform convergence in V on the compact sets of G. By the *carrier* of an $f \in \mathscr{E}^0$ we mean the closure of the set of points where $f \neq 0$. An *operator* on G is a linear mapping of a subspace of $\mathscr{E}^0$ into $\mathscr{E}^0$. The operator D is said to be *closed* if the conditions: $\{f_\alpha\}$ in the domain of D, $f_\alpha \to f$ and $Df_\alpha \to h$ in $\mathscr{E}^0$, imply f is in the domain of D and $Df = H$.

PROPOSITION 1. *For any Y in $\mathfrak{g}$, D_Y is a closed operator.*

Proof. It is clear that D_Y is an operator.

It remains to show that D_Y is closed. Let $\{f_i\}$ be a sequence of functions in the domain of D_Y such that $\{f_i\}$ and $\{D_Y f_i\}$ are Cauchy sequences in $\mathscr{E}^0$; call $f = \lim f_i$, $h = \lim D_Y f_i$, the limits being taken in $\mathscr{E}^0$. Let $Y, X_2, X_3, \cdots, X_n$ be a basis for $\mathfrak{g}$ and N an open neighborhood of 1 in G in which $\exp(t_1 Y)\exp(t_2 X_2) \cdots \exp(t_n X_n)$ form a coordinate system. It is clearly sufficient to prove that f is in the domain of D_Y at 1 and that $(D_Y f)(x) = h(x)$ for any $x \in N$.

Now, $\theta: (t_1, t_2, \cdots, t_n) \to (\exp(t_1 Y), \exp(t_2 X_2), \cdots, \exp(t_n X_n))$ maps a circular neighborhood M of 0 in real Euclidean n-space homeomorphically onto N. It is immediate from the definitions that a continuous map p of G into V is differentiable in the direction Y at 1 if and only if $p\theta$ has a continuous partial derivative in the direction t_1 at 0, and then

$$(D_Y p)(x) = \left(\frac{\partial p\theta}{\partial t_1}\right)(x)$$

for all x in a suitable neighborhood of 1. From this and the known closure of $\partial/\partial t_1$ on Euclidean space, our assertion follows.

Now, let $Y_1, Y_2, \cdots, Y_n$ be a basis for $\mathfrak{g}$. We set

$$D_1 = D_{Y_1},\ D_2 = D_{Y_2},\ \cdots,\ D_n = D_{Y_n}$$

and we call $\mathfrak{D}$ the family $(D_1, D_2, \cdots, D_n)$ so $\mathfrak{D}$ is a family of closed operators. By means of D we can now define, by the methods of [5], the complete, locally convex, Hausdorff, topological vector spaces $\mathscr{D}$ (or $\mathscr{D}(V)$) of indefinitely differentiable maps of compact carrier of G into V, and $\mathscr{E}$ (or $\mathscr{E}(V)$) of all indefinitely differentiable maps of G into V. $\mathscr{E}$ is a metrizable space; a sequence $\{f_i\}$ converges to zero in $\mathscr{E}$ if and only if for any operator $D^* = D_{j_1} D_{j_2} \cdots D_{j_r}$, $D_{j_m} \in \mathfrak{D}$, $Df \to 0$ uniformly in V on every compact set of G. The topology of $\mathscr{D}$ may be described as follows: For each compact set K, let $\mathscr{D}_K$ be the subspace of $\mathscr{D}$ consisting of those maps of $\mathscr{D}$ which have their carriers in K; the topology of $\mathscr{D}_K$ is that induced by $\mathscr{E}$. Then of all possible locally convex topologies which induce on each $\mathscr{D}_K$ the topology of $\mathscr{D}_K$ that may be given to the set of functions of $\mathscr{D}$, $\mathscr{D}$ is given the strongest (see [4]).

PROPOSITION 2. *The spaces $\mathscr{D}$ and $\mathscr{E}$ are the same as those we would have obtained by considering G as an indefinitely differentiable manifold.*[1]

Proof. Let N be a neighborhood of 1 in G in which $(\exp t_1 Y_1 \exp t_2 Y_2 \cdots \exp t_n Y_n)$ form a coordinate system. Then it is clearly sufficient to prove the theorem for the restrictions of the functions of $\mathscr{E}$ and $\mathscr{D}$ to N. The result now follows by the method of the proof of Proposition 1.

PROPOSITION 3. *$\mathscr{D}$ and $\mathscr{E}$ are reflexive topological spaces.*[2]

Proof. We prove the theorem first for $\mathscr{E}$. Since $\mathscr{E}$ is metrizable, it is sufficient to prove that $\mathscr{E}$ is a Montel space, that is, that the bounded sets of $\mathscr{E}$ are relatively compact (of compact closure). Let then B be a bounded set in $\mathscr{E}$. Let N be a compact neighborhood of 1 in G in which $(\exp t_1 Y_1 \exp t_2 Y_2 \cdots \exp t_n Y_n)$ form a coordinate system. Since G is separable, we can find a sequence of points $a_i \in G$ such that $G = \bigcup (\textit{interior } Na_i)$.

It is easily seen that it is sufficient to show that, for any i, and for any integers $r_1, r_2, \cdots, r_m$, if we set $D^* = D_{r_1} D_{r_2} \cdots D_{r_m}$, then the set $\{D^* f\}_{f \in B}$ is equicontinuous on $a_i N$. It follows immediately as in the proof of Proposition 1 that the restrictions of the maps $D^* f$ have the property that (if we identify them with maps on a circular neighborhood of zero in Euclidean n-space) their partial derivatives in all direc-

[1] That is, by applying the method of de Rham and Kodaira [**12**].

[2] See [**3**].

tions are uniformly bounded for $f \in B$. As is well-known, this implies the equicontinuity of $\{D^* f\}_{f \in B}$ on $a_i N$; hence Proposition 3 is established as regards the space $\mathscr{E}$.

If L is a bounded set in $\mathscr{D}$, then all the maps of L have their carriers in a fixed compact set K of G, that is, $L \subset \mathscr{D}_K$. Since the topology induced by $\mathscr{D}$ on $\mathscr{D}_K$ is also the topology induced by $\mathscr{E}$ on $\mathscr{D}_K$, L is bounded in $\mathscr{E}$. Thus, L is relatively compact in $\mathscr{E}$, hence in $\mathscr{D}_K$, hence also in $\mathscr{D}$ which concludes the proof of Proposition 3.

A sequence of open, relatively compact (that is, of compact closure) sets $K_i \subset G$ will be called a *scattered resolution* of G (see [5]) if $\bigcup K_i = G$ and if, given any compact set $K \subset G$, only a finite number of the K_i meet K. Given any scattered resolution $\{K_i\}$ of G, there exists a *partition of unity* $\{h_i\}$ relative to it; by this is meant that the indefinitely differentiable functions h_i have the properties that:

1. For each i, *carrier* $h_i \subset K_i$.

2. For any $x \in G$, $\Sigma h_i(x) = 1$.

(This sum has meaning because all but a finite number of terms are zero.) To establish the existence of the partition of unity $\{h_i\}$, we have only to note that the scattered resolution $\{K_i\}$ can be "refined" to a scattered resolution $\{L_i\}$ by coordinate neighborhoods (that is, each K_i is contained in a union of a finite number of L_j). The existence of a partition of unity relative to $\{L_i\}$ is readily verified and, in turn, implies immediately the existence of a partition of unity relative to $\{K_i\}$.

By $\mathscr{D}'$ (or $\mathscr{D}'(V)$ we denote the dual of $\mathscr{D}$ with the topology of uniform convergence on the bounded (compact) sets of $\mathscr{D}$. It can be shown (see [7]) that, $\mathscr{D}'$ can also be described as the space of continuous linear maps of $\mathscr{D}(C) \to V'$, this space of maps being given the compact-open topology. For this reason, $\mathscr{D}'$ is usually called the space of *distributions on G with values in V'*. In this paper, we shall call the elements of $\mathscr{D}'$ *distributions*.

For any distribution S, and any open set O in G, we say that S vanishes on O if $S \cdot f = 0$ for any $f \in \mathscr{D}$ whose carrier is contained in O. Because of the existence of partitions of unity, we can easily show that if S vanishes on O_α where O_α are open sets, then S vanishes also on $\bigcup O_\alpha$. Thus there is a largest open set on which S vanishes. The *carrier* of S is defined as the complement of this set.

$\mathscr{E}'$ (or $\mathscr{E}'(V)$) is the dual of $\mathscr{E}$. It can be shown, as in [13], that $\mathscr{E}'$ consists of all distributions of compact carrier.

For any $S \in \mathscr{D}'$, by $\bar{S}$ is meant the distribution $f \to \overline{S \cdot \bar{f}}$ for $f \in \mathscr{D}$, where $\bar{f}(x)=\overline{f(x)}$ for any $x \in G$.

By $G\times G$ we denote the direct product of G with itself; $G\times G$ is again a Lie group whose underlying manifold is the Cartesian product of the underlying manifold of G with itself. By ${}_2\mathscr{D}$, ${}_2\mathscr{E}$, ${}_2\mathscr{D}'$, ${}_2\mathscr{E}'$ we denote the spaces on $G\times G$ corresponding to $\mathscr{D}$, $\mathscr{E}$, $\mathscr{D}'$ $\mathscr{E}'$ respectively.

Let k be a continuous map on $G\times G$ and $x \in G$. Then by $k_{X_1=x}$ we mean the map on G: $y \to k(x, y)$. Suppose that, for all $x \in G$, $k_{X_1=x}$ is in a space U of mappings on G. Then by k_1 we mean the mapping $x \to k_{X_1=x}$ of $G \to U$. Let L be a map defined on U; then we say that k is in the domain of L, and we denote by $L_2 k$ the map

$$x \to Lk_{X_1=x}$$

for $x \in G$. If the range of L is again a space of mappings on G, then we say also that k is in the domain of L_{2l} and we shall denote by $L_{2l}k$ the mapping on $G\times G$:

$$(x,\ y) \to Lk_{X_1=x}(y)\ .$$

L_{2l} is called the *lift* of L to $G\times G$. We define $k_{X_2=x}$, k_2, L_1, L_{1l} similarly.

We can now define, as in [5], two products involving distributions and functions:

For any $S \in \mathscr{D}'$, $k \in {}_2\mathscr{D}$, then we have two *inner products*: $S_1 k$ and $S_2 k$ which are both in $\mathscr{D}$.

For any S, $U \in \mathscr{D}'$ we define the *direct products* $S_1 \times U_2$ and $S_2 \times U_1 \in {}_2\mathscr{D}'$ by

$$S_1\times U_2\cdot k = S\cdot U_2 k, \qquad S_2\times U_1\cdot k = S\cdot U_1 k$$

for any $k \in {}_2\mathscr{D}$.

The direct products define continuous bilinear maps which are commutative, while the inner products are only separately continuous bilinear maps. (If V, W, X are topological vector spaces and $t: V\times W \to X$ is a bilinear map, then t is called separately continuous (see [4], [5]) if, for B, B' any bounded sets in V, W respectively, the maps

$$w \to t(b,\ w),\ v \to t(v,\ b')$$

are, for $b \in B$, $b' \in B'$, equicontinuous linear maps of $W \to X$ and $V \to X$ respectively.)

By $\{Q_i\}$ we shall denote an enumeration of the operators $D_{r_1} D_{r_2} \cdots D_{r_m}$ with Q_1=identity.

For f a continuous map defined on G, $\check{f}$ is the map $x \to f(x^{-1})$.

We shall denote by η the function on G defined by $dxg=\eta(g)dx$, where dx is a left invariant Haar measure. It is known that $\eta \in \mathscr{E}(C)$

and, moreover, η is a homomorphism on G. By ω we denote the function on G defined by $dx^{-1}=\omega(x)dx$. Again, $\omega\in\mathscr{E}(C)$ and ω is a homomorphism on G. It is readily verified that $\omega(y)=\eta(y^{-1})$ for any $y\in G$. For any $S\in\mathscr{D}'$, we write $\check{S}\cdot f=S\cdot\omega\check{f}$ for any $f\in\mathscr{D}$.

3. Convolution on G. For any continuous map f of G into V and any $x\in G$ we define the translations

$$(\mathfrak{L}(x)f)(y)=f(x^{-1}y) \qquad (\mathfrak{R}(x)f)(y)=f(yx)$$

for any $y\in G$.

PROPOSITION 4. *$(x, f)\to\mathfrak{L}(x)f$ and $(x, f)\to\mathfrak{R}(x)f$ are continuous maps of $G\times\mathscr{D}\to\mathscr{D}$ and also of $G\times\mathscr{E}\to\mathscr{E}$.*

Proof. We shall establish the theorem for the map $(x, f)\to\mathfrak{L}(x)f$ of $G\times\mathscr{D}\to\mathscr{D}$; the other parts of the proposition may be established by similar methods. By the results of Dieudonné and Schwartz (see [4], [5]) it is sufficient to prove that this is a continuous map of $G\times\mathscr{D}_K\to\mathscr{D}$ for any compact set K of G. Since the map is linear in f and a homomorphism in x, it is sufficient to prove continuity at $f=0$ and $x=1$. Let K be a given compact set in G and choose K' a compact set in G so large that K' contains the carriers of all $\mathfrak{L}(x)f$ for $x\in\mathscr{D}_K$. Let M be a neighborhood of zero in $\mathscr{D}_{K'}$. Then we can find operators $Q_1, Q_2, \cdots, Q_r$, and continuous semi-norms $\rho_1. \rho_2, \cdots, \rho_n$ on V, and a positive number a so that M contains the set of $h\in\mathscr{D}_{K'}$ which satisfy

$$\max_{y\in G,\, i}\rho_i[(Q_jh)(y)]\leq a$$

for $j=1, 2, \cdots, r$.

For any $p\in\mathscr{D}$, any k, and $x, z\in G$,

$$\begin{aligned}(D_k\mathfrak{L}(z)p)(x)&=\left\{\left[\left(\frac{d}{dt}\right)\mathfrak{L}(z)p\right][(\exp tY_k)\,x]\right\}_{t=0}\\&=\left\{\left[\left(\frac{d}{dt}\right)p\right][z^{-1}(\exp tY_k)x]\right\}_{t=0}\\&=\left\{\left(\frac{dp}{dt}\right)[z^{-1}(\exp tY_k)zz^{-1}x]\right\}_{t=0}\\&=\left\{\left(\frac{dp}{dt}\right)[(\exp tz^{-1}Y_kz)z^{-1}x]\right\}_{t=0}.\end{aligned}$$

Now, write $z^{-1}Y_kz=\Sigma\, c_{kl}(z)Y_l$ where (c_{kl}) is the matrix of the adjoint representation of G on g. Then we have

$$(D_k \mathfrak{L}(z)p)(x) = (D_{z^{-1}Y_k z} f)(z^{-1}x) .$$

We also have

$$D_{z^{-1}Y_z} p = \Sigma\, c_{kj}(z) D_j p .$$

The functions c_{kj} are continuous and even indefinitely differentiable on G. Hence, we can find an $A > 0$ so that

$$\max_{z \in K} |c_{kj}(z)| \leqq A$$

for all k, j.

It follows immediately from this that we can be assured that, for $q \in \mathscr{D}_K$, $z \in K$,

$$\max_{x \in G,\, i} \rho_i[(D_K \mathfrak{L}(z)q)(x)]$$

will be small by making

$$\max_{x \in G,\, j,\, i} \rho_i[(D_j q)(x)]$$

sufficiently small. Proposition 4 now follows by iteration, since each Q_l is of the form $D_{r_1} D_{r_2} \cdots D_{r_m}$.

For any continuous map f on G, $\mathfrak{L}f$ is the map on $G \times G$: $(x, y) \to f(x^{-1}y)$; $\mathfrak{L}^* f$ is the map on $G \times G$: $(x, y) \to f(xy)$. By the method of proof of Proposition 1, we can establish

PROPOSITION 5. *$f \to \mathfrak{L}f$ and $f \to \mathfrak{L}^* f$ are continuous linear maps of $\mathscr{E} \to_2 \mathscr{E}$.*

We are now in a position to define the convolution product involving distributions and functions. The definition differs slightly from that of Schwartz [13]: For any $S \in \mathscr{D}'$, $f \in \mathscr{D}$, $x \in G$, we set

$$(1) \qquad (S * f)(x) = \overline{S} \cdot \mathfrak{L}(x) \check{f}$$

This formula can also be considered valid if $S \in \mathscr{E}'$ and $f \in \mathscr{E}$.

PROPOSITION 6. *$(S, f) \to S * f$ is a separately continuous map of*

(a) $\mathscr{E}' \times \mathscr{E} \to \mathscr{E}(C)$

(b) $\mathscr{E}' \times \mathscr{D} \to \mathscr{D}(C)$

(c) $\mathscr{D}' \times \mathscr{D} \to \mathscr{E}(C)$.

which is antilinear in S and linear in f.

Proof. (a) Let j be fixed and write $A=D_j$. We find from the definitions that, for $S\in\mathscr{E}'$, $f\in\mathscr{E}$, $S*f\in\mathscr{E}(C)$ and, moreover,

$$(2)\qquad [A(S*f)](x)=\overline{S}\cdot(A_{1/}\mathfrak{L}\check{f})_{X_1=x}\ .$$

From this it follows by iteration that, for any $Q=Q_s$, we have

$$(3)\qquad [Q(S*f)](x)=\overline{S}\cdot(Q_{1/}\mathfrak{L}\check{f})_{X_1=x}\ .$$

Part (a) results immediately from (3) together with Proposition 5.

(b) By a result of Dieudonné and Schwartz (see [4]) it is sufficient to prove that, for K a compact set in G, $(S, f)\rightarrow S*f$ is a separately continuous map of $\mathscr{E}'\times\mathscr{D}_K\rightarrow\mathscr{D}$. Now, it is obvious that

$$(4)\qquad (carrier\ S*f)\subset(carrier\ S)(carrier\ f)\ .$$

Our assertion now follows from (a) above and the fact that $\mathscr{D}_K$ has the topology induced by $\mathscr{E}$.

(c) This is proven by essentially the same reasoning as that employed in the proof of (a) above.

4. $\mathscr{E}$ as a space of linear transformations. In this section we shall prove our main result.

THEOREM 1. *Let $S\in\mathscr{D}'$ have the property that $S*f\in\mathscr{D}(C)$ whenever $f\in\mathscr{D}$; then $S\in\mathscr{E}'$.*

Proof. Let us suppose that S satisfies the hypotheses of Theorem 1, and let K be a fixed compact set in G. We shall show first that there exists a compact set $K'\subset G$ such that $S*\mathscr{D}_K\subset\mathscr{D}_{K'}$. Assume this is not the case, and let $\{K_i\}$ be a compact exhaustion of G. (That is, each K_i is a compact set which is the closure of a nonempty open set. Moreover, $K_i\subset K_{i+1}$ and $\bigcup K_i=G$.) We shall produce a sequence $\{g_i\}$ with the following properties:

1. Each $g_i\in\mathscr{D}_K$.
2. Σg_i converges in $\mathscr{D}_K$.
3. There is a sequence of positive numbers m_i with $m_{i+1}-m_i\geqq 1$ for all i such that

$$carrier\ (S*g_i)\subset K_{m_i}$$
$$carrier\ (S*g_{i+1})\not\subset K_{m_i}\ .$$

4. There is a sequence of points $a_i\in G$ such that a_i is a point of $K_{m_i}-K_{m_{i-1}}$ (where K_{m_0} is the empty set) for which $(S*g_i)(a_i)\neq 0$ and

$$|(S * g_{i+k})(a_i)| \leqq \frac{1}{3^k} |(S * g_i)(a_i)|$$

for all $k > 0$.

Suppose that the sequences $\{g_i\}$, $\{m_i\}$, $\{a_i\}$ can be found. Then for any $i > 1$,

$$\begin{aligned} |(S * \Sigma\, g_j)(a_i)| &= |\Sigma(S * g_j)(a_i)| \\ &= |\Sigma_{j \geqq i}(S * g_j)(a_i)| \\ &\geqq |(S * g_j)(a_i)| - \sum_{j>i} |(S * g_j)(a_i)| \\ &\geqq |(S * g_i)(a_i)| \left[1 - \sum_{j \geqq 1} \frac{1}{3^j}\right] \\ &= \frac{1}{2} |(S * g_i)(a_i)| \\ &> 0 . \end{aligned}$$

Since the set $\{a_i\}$ is clearly not contained in any compact set of G, we conclude that $S * \Sigma\, g_j$ is not of compact carrier, which contradicts our hypothesis.

It remains to define the sequences $\{g_i\}$, $\{m_i\}$, and $\{a_i\}$. Let $g_1 \in \mathscr{D}_K$ be chosen so that $S * g_1 \neq 0$. Let a_1 be any point in G for which $(S * g_1)(a_1) \neq 0$, and choose $m_1 > 0$ so that

$$\textit{carrier } (S * g_1) \subset K_{m_1} .$$

Assume that $g_1, \cdots, g_k$, $a_1, \cdots, a_k$, $m_1, \cdots, m_k$ have been defined with the required properties; we shall now define g_{k+1}, a_{k+1}, m_{k+1}. Now, by our assumption, there is an $f \in \mathscr{D}_K$ such that

$$\textit{carrier } (S * f) \not\subset K_{m_k+1} .$$

Let m_{k+1} be chosen so that *carrier* $(S * f) \subset K_{m_{k+1}}$, and let a_{k+1} be some point in $K_{m_{k+1}} - K_{m_k}$ such that $(S * f)(a_{k+1}) \neq 0$. Define

$$g_{k+1} = \frac{f}{\max(1, \max\limits_{x \in G} |f(x)| 3^{k+1}) \max\limits_{j,\, i \leqq k} (1, \max\limits_{x \in G}{}_j[(Q_i f)(x)], [|(S * g_i)(a_i)|]^{-1})}$$

The sequences $\{g_i\}$, $\{m_i\}$, $\{a_i\}$ are thus defined. It is clear that conditions 1, 3, 4 are satisfied. Further, each $g_i \in \mathscr{D}_K$ and, for R any semi-norm on D_K of the kind used to define the topology of that space, it is clear that

$$\Sigma\, R(g_1) > \infty .$$

Thus $\Sigma\, g_i$ converges in $\mathscr{D}_K$.

To complete the proof of Theorem 1, let us assume that S is not of compact carrier, and let K be a given compact symmetric neighborhood of 1 in G. It is clear that we can choose an open set U in G such that S does not vanish on U and such that $U \cap K'$ is empty, where K' is a compact symmetric set such that $\overline{S} * \mathscr{D}_K \subset \mathscr{D}_{K'}$. It follows easily that we can find a $g \in G$, and an $f \in D$ such that *carrier* $f \subset Kg \subset U$, $S \cdot f \neq 0$.

On the other hand, by definition,

$$S \cdot \check{f} = (\overline{S} * \mathfrak{L}(g) f)(g^{-1}) .$$

But, *carrier* $f \subset Kg$ implies *carrier* $\mathfrak{L}(g) f \subset K$ because K is symmetric. Also, $g \notin K'$ because $1 \in K$ and $K' \cap U$ is empty. Since K' is symmetric, also $g^{-1} \notin K'$. Thus, $S \cdot \check{f} = (\overline{S} * \mathfrak{L}(g) f)(g^{-1}) = 0$; this contradiction completes the proof of Theorem 1.

The set of distributions of $\mathscr{E}'$ forms a vector space of continuous linear mappings of $\mathscr{D} \to \mathscr{D}$ under convolution; we give this space the compact-open topology (see [6]) and obtain a topological vector space J. A fundamental system of neighborhoods of zero in J consists of all sets N for which we can find a compact set K in $\mathscr{D}$ and a neighborhood of zero M in $\mathscr{D}$ so that N consists of those $S \in \mathscr{E}'$ with $S * h \in M$ for all $h \in K$. By Proposition 1 of § 5 of [6], we would have obtained the same topologies if we had considered the distributions of $\mathscr{E}'$ as defining, under convolution, continuous linear maps of $\mathscr{D}' \to \mathscr{D}'$.

THEOREM 2. *The natural map* $u: \mathscr{E}' \to J$ *is a topological isomorphism onto.*

Proof. u is clearly one-to-one, linear, and onto. Moreover, J is given the weakest topology to make the maps

$$S \to (u^{-1} S) * f$$

of $J \to \mathscr{D}$ equicontinuous for f in any compact set of $\mathscr{D}$; by Proposition 6 this implies that u is continuous.

Since u^{-1} is linear, we need verify continuity only at zero. Let T be a neighborhood of zero in $\mathscr{E}'$; there is a bounded set $\beta \subset \mathscr{E}$ so that T contains the set of $S \in \mathscr{E}'$ which satisfy $|S \cdot b| \leqq 1$ for all of $b \in \beta$.

Let K be an open symmetric neighborhood of 1 in G whose closure is compact. Then it is clear that we can find a sequence of points $a_i \in G$ such that $\{a_i K\}$ is a scattered resolution of G (see § 2). We can also insure that, if a is one of the a_i, so is a^{-1}. Let $\{h\}$ be a partition of unity relative to this scattered resolution (see § 2). It is readily verified by the method of proof of Proposition 1 of § 3 that, for each i, the set B_i of functions $\mathfrak{L}(a_i)(h_i f)$ for $f \in \beta$ is bounded in $\mathscr{D}$. For each j there is a double sequence $s_j = M_{jik}$ of positive numbers so that B_j is contained in the bounded (in $\mathscr{D}$) set L_j of all $g \in D$ whose carriers are contained in K and which satisfy

$$\max_{x \in G} \rho_k[(Q_i g)(x)] \leqq M_{jik}$$

for all i. From the denumerable number of double sequences s_j we construct a double sequence $s = \{M_{ik}\}$ of positive numbers such that, for each j, $M_{jik} \leqq M_{ik}$ for all but a finite number of i, k. Hence, for each j, we can find an $e_j > 0$ so that $e_j M_{jik} \leqq M_{ik}$ for all i, k; we can clearly make $e_j = e_l$ if $a_j = a_l^{-1}$.

Let A be the set of $f \in \mathscr{D}$ for which

1. *carrier* $f \subset K$

2. $\max\limits_{x \in G} \rho_k[(Q_i f)(x)] \leqq M_{ik}$ for all i, k,

so A is bounded in $\mathscr{D}$. Let M be the neighborhood of zero in $\mathscr{D}$ consisting of those $h \in \mathscr{D}$ with

$$\max_{x \in a_j K} \rho_k h(x) \leqq e_j d_j$$

for all j, where d_j are positive numbers which satisfy $\Sigma\, d_j = 1$. Call N the set of $S \in J$ with $S * f \in M$ for all $f \in A$, so N is a neighborhood of zero in J; we claim that $u^{-1}(N) \subset T$.

Let us assume this is not the case; then we can find an $S \in N$ with $u^{-1}S \notin T$, that is, $S \in N$ but

$$|u^{-1}S \cdot f| > 1$$

for some $f \in \beta$. Now, $u^{-1}S$ is of compact carrier; thus we can find an r such that

$$\sum_{k=1}^{r} h_k(x) = 1$$

for any $x \in$ *carrier* $(u^{-1}S)$. Hence

$$|u^{-1}S \cdot f| \leqq |u^{-1}S \cdot h_1 f| + |u^{-1}S \cdot h_2 f| + \cdots + |u^{-1}S \cdot h_r f| \,. \tag{5}$$

It is clear from the definitions that, for each i,

$$e_i \mathfrak{L}(a_i)(h_i f) \in A \,.$$

Thus,

$$h_i f = \frac{1}{e_i} \mathfrak{L}(a_i^{-1}) g$$

for some $g \in A$, which gives, for $i=1, 2, \cdots, r$,

$$|u^{-1}S \cdot h_i f| = \frac{1}{e_i} |u^{-1}S \cdot \mathfrak{L}(a_i^{-1}) g|$$

$$= \frac{1}{e_i} |(\overline{u^{-1}S} * g)(a_i^{-1})|$$

$$\leqq \frac{1}{e_i} e_j d_j ,$$

where $a_j = a_i^{-1}$, because $g \in A$ and $u^{-1}S \in N$. Now, since $e_i = e_j$, we have

$$|u^{-1}S \cdot h_i f| \leqq d_j .$$

Applying this to equation (5) we obtain

$$|u^{-1}S \cdot f| \leqq d_{1'} + d_{2'} + \cdots + d_{r'} \leqq 1$$

(where we set $a_{j'} = a_j^{-1}$). This contradiction proves the theorem.

5. Extension of the main result. We assumed in §§ 2, 3, 4 that V is metrizable. In case V is not metrizable, then the spaces $\mathscr{E}$ and $\mathscr{D}$ can be defined as before, but E is no longer metrizable, and $\mathscr{D}$ is not an $\mathscr{L}\mathscr{F}$ space in the sence of Dieueonné and Schwartz [4]. However, there is no difficulty in extending the definition and continuity properties of the convolution product to this case. Theorem 1 can be extended to this case, but the proof of Theorem 2 does not extend to the case of V not metrizable. All that can be proven (and the proof is much simpler than the proof of Theorem 2 above) is that u is continuous and that u^{-1} is sequentially continuous and takes bounded sets into bounded sets. The continuity of u^{-1} is an open question.

We assume in the following that V is a complete, locally convex, Hausdorff, topological vector space. By V^* we denote the space of continuous linear maps of V into V with the compact-open topology, so V^* is again a complete, locally convex, Hausdorff, topological vector space.

Let K and L denote compact subgroups of G. By a *representation of K on V* we mean a continuous homomorphism U of K into V^*. Let U and W be representations of V of K and L respectively. By ${}_{UW}\mathscr{D}$ we denote the space of those $f \in \mathscr{D}(V^*)$ for which

$$\mathfrak{L}(k^{-1})\mathfrak{R}(l) f = U(k) f W(l) \tag{6}$$

for any $k \in K$, $l \in L$. We give ${}_{UW}\mathscr{D}$ the topology induced by $\mathscr{D}$. ${}_{UW}\mathscr{E}$

is defined similarly.

For any $T \in \mathscr{D}'(V^*)$, $g \in G$, we define $\mathfrak{L}(g)T$ and $\mathfrak{R}(g)T$ as the distributions

$$(7) \qquad \mathfrak{L}(g)T \cdot f = T \cdot \mathfrak{L}(g^{-1})f, \quad \mathfrak{R}(g)T \cdot f = \eta(g)T \cdot \mathfrak{R}(g^{-1})f$$

for any $f \in \mathscr{D}(V^*)$. (η was defined in § 2.) Let us denote by ${}_{UW}\mathscr{D}'$ the space of all $S \in \mathscr{D}'(V^*)$ which satisfy

$$(8) \qquad \mathfrak{L}(k)\mathfrak{R}(l)S \cdot f = S \cdot U(k^{-1})f\,W(l)$$

for any $f \in \mathscr{D}(V^*)$, $k \in K$, $l \in L$.[3] We shall write $U(k)SW(l^{-1}) \cdot f$ for the right side of (8). We give ${}_{UW}\mathscr{D}'$ the topology induced by $\mathscr{D}'(V^*)$. ${}_{UW}\mathscr{E}'$ is defined similarly.

We can easily show

PROPOSITION 7.

$$f \to P_{UW}f = \int_{K \times L} U(k)\mathfrak{L}(h)\mathfrak{R}(l)f\,V^{-1}(l)dkdl$$

(where dk and dl are the respective Haar measures on K and L so normalized that $\int_K dk = \int_L dl = 1$) defines continuous open projections of $\mathscr{D}(V^*)$ onto ${}_{UW}\mathscr{D}$ and $\mathscr{E}(V^*)$ onto ${}_{UW}\mathscr{E}$. Also

$$S \to P_{UW}S = \int_{K \times L} \mathfrak{L}(k^{-1})\mathfrak{R}(l)[U^{-1}(k)SV(l)]dkdl$$

defines continuous open projections of $\mathscr{D}'(V^*)$ onto ${}_{UW}\mathscr{D}'$ and $\mathscr{E}'(V^*)$ onto ${}_{UW}\mathscr{E}'$.

COROLLARY. *${}_{UW}\mathscr{D}'$ is the dual of ${}_{UW}\mathscr{D}$ and ${}_{UW}\mathscr{E}'$ is the dual of ${}_{UW}\mathscr{E}$.*

Proof. This is an immediate consequence of Proposition 6 and the fact that, for $S \in \mathscr{D}'$, $f \in \mathscr{D}$ (or for $S \in \mathscr{E}'$, $f \in \mathscr{E}$), we have $P_{UW}S \cdot f = S \cdot P_{UW}f$.

Suppose that $K = L$; then we see easily that the convolution defined in § 3 defines a separately continuous bilinear map of ${}_{UW}\mathscr{D}' \times {}_{WZ}\mathscr{D} \to {}_{UZ}\mathscr{E}(C)$ (where U, W, Z are representations of K on V). The method of proof of Theorems 1 and 2 can be used to show.

THEOREM 3. *${}_{UW}\mathscr{E}'$ consists of all $S \in {}_{UW}\mathscr{D}'$ such that $S * f \in {}_{UW}\mathscr{D}$ for any $f \in {}_{WW}\mathscr{D}$. The topology of ${}_{UW}\mathscr{E}'$ is sequentially the same as that obtained by considering the elements of ${}_{UW}\mathscr{E}'$ as defining (by convolution) continuous linear maps of ${}_{WW}\mathscr{D} \to {}_{UW}\mathscr{D}$ and giving this set the*

[3] Note that since L is compact, the restriction of η to L is 1.

compact-open topology τ. *Moreover, the bounded sets of* ${}_{UW}\mathscr{E}'$ *are the same as those of* τ.

REMARK 1. We do not know whether the topologies τ and that of ${}_{UW}\mathscr{E}'$ are the same. The difficulty is that, for $f \in {}_{WW}\mathscr{D}$, $g \in G$, $\mathfrak{L}(g)f$ is no longer in ${}_{WW}\mathscr{D}$.

REMARK 2. In case that $K=L$, V is finite dimensional, and U, W, X, Z are irreducible unitary representations of K on V, then it follows easily from the Schur orthogonality relations that $S*f=0$ for any $S \in {}_{UW}\mathscr{E}'$ $f \in {}_{XZ}\mathscr{D}$ if W is not equivalent to X.

REMARK 3. The conclusion of Theorem 3 does not necessarily hold if the space ${}_{WW}\mathscr{D}$ in the hypothesis of the theorem is replaced by ${}_{WZ}\mathscr{D}$ where Z is different from W, even if V is finite dimensional and U, W, Z are irreducible unitary representations of K. An example will be given in a forthcoming paper of the author and F. I. Mautner. (G can be taken as the complex unimodular group.)

6. General remarks. We have assumed that G is a separable Lie group. In the general case, the spaces $\mathscr{E}$ and $\mathscr{D}$ can be defined as before, but $\mathscr{E}$ will not be metrizable and $\mathscr{D}$ will not be an $\mathscr{L}\mathscr{F}$ space in the sense of Dieudonné and Schwartz [4] because $\mathscr{D}$ will be the inductive limit of a non-denumerable number of spaces, For this reason, the topology of $\mathscr{D}$ is best defined as follows: Let $\{f_{ij}\}=\sigma$ be a family of continuous functions on G such that

(a) For each i, only a finite number of j appear.

(b) Only a finite number of f_{ij} are different from zero on any compact set of G.

Then we define N_σ as the set of $h \in \mathscr{D}$ for which

$$\max_{x \in G} \rho_j[(f_{ij}(x)Q_jh)(x)] \leqq 1$$

for all i, j, where the Q_j are as in § 2, and $\{\rho_j\}$ denotes an enumeration of semi-norms which are sufficient to define the topology of V. The sets N_σ are seen to form a fundamental system of neighborhoods of zero of a locally convex topological vector space which we shall call $\mathscr{D}$. In case $\mathscr{D}$ is separable it is easily verified that the two definitions agree.

The advantage of the above definition is that it implies immediately the completeness of $\mathscr{D}$. For, the completion of $\mathscr{D}$ obviously consists of indefinitely differentiable maps. Moreover, if h is any map in the completion of $\mathscr{D}$, then, for any continuous function f on G, and any

k, it is easily seen that $\rho_k(fh)$ is a bounded function. This implies immediately that h is of compact carrier, hence $h \in \mathscr{D}$.

The properties of convolution can be extended to the nonseparable case and there is no difficulty in extending part of our main results. We can, as in § 5, prove only that the topology of $\mathscr{E}'$ is sequentially, and in regard to bounded sets, the same as the compact-open topology of the space of linear transformations of $\mathscr{D} \to \mathscr{D}$ (under convolution).

The results of § 5 on double coset spaces $K\backslash G/L$ can also be extended to functions invariant under a compact group of automorphisms of G (the group of automorphisms of G is given the compact-open topology).

In addition, the main results of this paper can be extended to locally compact groups. There $\mathscr{E}$ is replaced by the space of continuous functions, $\mathscr{D}$ the space of continuous functions of compact carrier, $\mathscr{D}'$ the space of measures and $\mathscr{E}'$ the space of measures of compact carrier.

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