SOME PROPERTIES OF DISTRIBUTIONS ON LIE GROUPS

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1. Introduction. Let $G$ be a separable Lie group and let $V$ be a complete, metrizable, topological vector space. The underlying space of $G$ is a separable real analytic manifold so that we can define, by the methods of L. Schwartz (see [7], [12], [13]), the spaces $\mathcal{E}(V)$ of indefinitely differentiable maps of $G$ into $V$, and $\mathcal{D}(V)$ which consists of those maps in $\mathcal{E}(V)$ which are of compact carrier. Their duals are $\mathcal{D}'(V)$, the space of distributions on $G$ with values in $V$ (the dual of $V$), and $\mathcal{E}'(V)$ which is the space of distributions of compact carrier with values in $V$.

By using the group structure in $G$, we can define the convolution $S * f \in \mathcal{E}(C)$ for any $S \in \mathcal{D}'(V)$, $f \in \mathcal{D}(V)$, where $C$ is the complex plane. The main result of this paper is: Let $S \in \mathcal{D}'(V)$ have the property that $S * f \in \mathcal{D}'(C)$ whenever $f \in \mathcal{D}(V)$; then $S \in \mathcal{E}'(V)$. Moreover, the topology of $\mathcal{E}'(V)$ is that obtained by considering each $S \in \mathcal{E}'(V)$ as defining the continuous linear transformation $f \rightarrow S * f$ of $\mathcal{D}(V) \rightarrow \mathcal{D}(C)$ and then giving this set of transformations the compact-open topology (see [6]). This generalizes the result of [6] in case $G$ is a vector group and $V = C$.

This result is generalized to double coset spaces $L \setminus G/K$ where $L$ and $K$ are compact subgroups of $G$. In this form, the result will be used by the author and F. I. Mautner to generalize the Paley-Wiener theorem and the theory of mean-periodic functions of Schwartz (see [8]).

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2. Distributions on $G$. Instead of using the usual method of defining distributions on $G$, as for example in de Rham and Kodaira [12], we shall follow another approach which is more akin to the author's thesis [5]. We shall show that the two methods are equivalent.

By "function" we shall mean "complex-valued function" unless the contrary is specifically stated. "Linear" will mean "linear over the complex numbers" always. By $1$ we denote the identity in $G$, and by $\mathfrak{g}$ we denote the Lie algebra of $G$. For any $Y \in \mathfrak{g}$, we denote by $t \rightarrow \exp(tY)$ the unique one parameter subgroup in $G$ whose direction

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at 1 is $Y$. Let $V$ be a complete metrizable locally convex topological vector space.

The map $f$ of $G$ into $V$ is said to be differentiable in the direction $Y \in \mathfrak{g}$ at $x \in G$ if $\left\{ \left( \frac{df}{dt} \right) \left[ (\exp tY)x \right] \right\}_{t=0}$ exists; if this is the case, we set

$$
(D_Y f)(x) = \left\{ \left( \frac{df}{dt} \right) \left[ (\exp tY)x \right] \right\}_{t=0}.
$$

If $f$ is a continuous map of $G$ into $V$, we say that $f$ is in the domain of $D_Y$ if, for any $x \in G$, $f$ is continuously differentiable in the direction $Y$ at $x$. $D_Y f$ is then defined as the (continuous) map $x \mapsto (D_Y f)(x)$.

By $\mathcal{E}^0$ we denote the space of continuous maps of $G$ into $V$ with the topology of uniform convergence in $V$ on the compact sets of $G$. By the carrier of an $f \in \mathcal{E}^0$ we mean the closure of the set of points where $f \not= 0$. An operator on $G$ is a linear mapping of a subspace of $\mathcal{E}^0$ into $\mathcal{E}^0$. The operator $D$ is said to be closed if the conditions: $\{f_a\}$ in the domain of $D$, $f_a \to f$ and $Df_a \to h$ in $\mathcal{E}^0$, imply $f$ is in the domain of $D$ and $Df = h$.

**Proposition 1.** For any $Y$ in $\mathfrak{g}$, $D_Y$ is a closed operator.

**Proof.** It is clear that $D_Y$ is an operator. It remains to show that $D_Y$ is closed. Let $\{f_i\}$ be a sequence of functions in the domain of $D_Y$ such that $\{f_i\}$ and $\{D_Y f_i\}$ are Cauchy sequences in $\mathcal{E}^0$; call $f = \lim f_i$, $h = \lim D_Y f_i$, the limits being taken in $\mathcal{E}^0$. Let $Y, X_2, X_3, \ldots, X_n$ be a basis for $\mathfrak{g}$ and $N$ an open neighborhood of 1 in $G$ in which $\exp (t_1 Y) \exp (t_2 X_2) \cdots \exp (t_n X_n)$ form a coordinate system. It is clearly sufficient to prove that $f$ is in the domain of $D_Y$ at 1 and that $(D_Y f)(x) = h(x)$ for any $x \in N$.

Now, $\theta : (t_1, t_2, \ldots, t_n) \mapsto (\exp (t_1 Y), \exp (t_2 X_2), \ldots, \exp (t_n X_n))$ maps a circular neighborhood $M$ of 0 in real Euclidean $n$-space homeomorphically onto $N$. It is immediate from the definitions that a continuous map $p$ of $G$ into $V$ is differentiable in the direction $Y$ at 1 if and only if $p \theta$ has a continuous partial derivative in the direction $t_1$ at 0, and then

$$(D_Y p)(x) = \left( \frac{\partial p \theta}{\partial t_1} \right)(x)$$

for all $x$ in a suitable neighborhood of 1. From this and the known closure of $\partial / \partial t_i$ on Euclidean space, our assertion follows.

Now, let $Y_1, Y_2, \ldots, Y_n$ be a basis for $\mathfrak{g}$. We set $D_1 = D_{Y_1}$, $D_2 = D_{Y_2}$, \ldots, $D_n = D_{Y_n}$.
and we call $\mathfrak{D}$ the family $(D_1, D_2, \ldots, D_n)$ so $\mathfrak{D}$ is a family of closed operators. By means of $D$ we can now define, by the methods of [5], the complete, locally convex, Hausdorff, topological vector spaces $\mathfrak{D}$ (or $\mathfrak{D}(V)$) of indefinitely differentiable maps of compact carrier of $G$ into $V$, and $\mathcal{E}$ (or $\mathcal{E}(V)$) of all indefinitely differentiable maps of $G$ into $V$. $\mathcal{E}$ is a metrizable space; a sequence $\{f_i\}$ converges to zero in $\mathcal{E}$ if and only if for any operator $D^* = D_{j_1}D_{j_2} \ldots D_{j_r}$, $D_{j_{m_1}} \in \mathfrak{D}$, $Df \to 0$ uniformly in $V$ on every compact set of $G$. The topology of $\mathfrak{D}$ may be described as follows: For each compact set $K$, let $\mathfrak{D}_K$ be the subspace of $\mathfrak{D}$ consisting of those maps of $\mathfrak{D}$ which have their carriers in $K$; the topology of $\mathfrak{D}_K$ is that induced by $\mathcal{E}$. Then of all possible locally convex topologies which induce on each $\mathfrak{D}_K$ the topology of $\mathfrak{D}_K$ that may be given to the set of functions of $\mathfrak{D}, \mathcal{E}$ is given the strongest (see [4]).

**Proposition 2.** The spaces $\mathfrak{D}$ and $\mathcal{E}$ are the same as those we would have obtained by considering $G$ as an indefinitely differentiable manifold.

*Proof.* Let $N$ be a neighborhood of 1 in $G$ in which $(\exp t_1Y_1 \exp t_2Y_2 \cdots \exp t_nY_n)$ form a coordinate system. Then it is clearly sufficient to prove the theorem for the restrictions of the functions of $\mathcal{E}$ and $\mathfrak{D}$ to $N$. The result now follows by the method of the proof of Proposition 1.

**Proposition 3.** $\mathfrak{D}$ and $\mathcal{E}$ are reflexive topological spaces.

*Proof.* We prove the theorem first for $\mathcal{E}$. Since $\mathcal{E}$ is metrizable, it is sufficient to prove that $\mathcal{E}$ is a Montel space, that is, that the bounded sets of $\mathcal{E}$ are relatively compact (of compact closure). Let then $B$ be a bounded set in $\mathcal{E}$. Let $N$ be a compact neighborhood of 1 in $G$ in which $(\exp t_1Y_1 \exp t_2Y_2 \cdots \exp t_nY_n)$ form a coordinate system. Since $G$ is separable, we can find a sequence of points $a_i \in G$ such that $G = \bigcup (\text{interior } Na_i)$.

It is easily seen that it is sufficient to show that, for any $i$, and for any integers $r_1, r_2, \ldots, r_m$, if we set $D^* = D_{r_1}D_{r_2} \cdots D_{r_m}$, then the set $\{D^*f\}_{f \in B}$ is equicontinuous on $a_iN$. It follows immediately as in the proof of Proposition 1 that the restrictions of the maps $D^*f$ have the property that (if we identify them with maps on a circular neighborhood of zero in Euclidean $n$-space) their partial derivatives in all direc-

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1 That is, by applying the method of de Rham and Kodaira [12].

2 See [3].
tions are uniformly bounded for $f \in B$. As is well-known, this implies the equicontinuity of $\{D^*f\}_{f \in B}$ on $a_i N$; hence Proposition 3 is established as regards the space $E$.

If $L$ is a bounded set in $\mathscr{D}$, then all the maps of $L$ have their carriers in a fixed compact set $K$ of $G$, that is, $L \subseteq \mathscr{D}_K$. Since the topology induced by $\mathscr{D}$ on $\mathscr{D}_K$ is also the topology induced by $E$ on $\mathscr{D}_K$, $L$ is bounded in $E$. Thus, $L$ is relatively compact in $E$, hence in $\mathscr{D}_K$, hence also in $\mathscr{D}$, which concludes the proof of Proposition 3.

A sequence of open, relatively compact (that is, of compact closure) sets $K_i \subseteq G$ will be called a scattered resolution of $G$ (see [5]) if $\bigcup K_i = G$ and if, given any compact set $K \subseteq G$, only a finite number of the $K_i$ meet $K$. Given any scattered resolution $\{K_i\}$ of $G$, there exists a partition of unity $\{h_i\}$ relative to it; by this is meant that the indefinitely differentiable functions $h_i$ have the properties that:

1. For each $i$, carrier $h_i \subseteq K_i$.
2. For any $x \in G$, $\sum h_i(x) = 1$.

(This sum has meaning because all but a finite number of terms are zero.) To establish the existence of the partition of unity $\{h_i\}$, we have only to note that the scattered resolution $\{K_i\}$ can be "refined" to a scattered resolution $\{L_i\}$ by coordinate neighborhoods (that is, each $K_i$ is contained in a union of a finite number of $L_i$). The existence of a partition of unity relative to $\{L_i\}$ is readily verified and, in turn, implies immediately the existence of a partition of unity relative to $\{K_i\}$.

By $\mathscr{D}'$ (or $\mathscr{D}'(V)$) we denote the dual of $\mathscr{D}$ with the topology of uniform convergence on the bounded (compact) sets of $\mathscr{D}$. It can be shown (see [7]) that, $\mathscr{D}'$ can also be described as the space of continuous linear maps of $\mathscr{D}(C) \rightarrow V$, this space of maps being given the compact-open topology. For this reason, $\mathscr{D}'$ is usually called the space of distributions on $G$ with values in $V$. In this paper, we shall call the elements of $\mathscr{D}'$ distributions.

For any distribution $S$, and any open set $O$ in $G$, we say that $S$ vanishes on $O$ if $S \cdot f = 0$ for any $f \in \mathscr{D}$ whose carrier is contained in $O$. Because of the existence of partitions of unity, we can easily show that if $S$ vanishes on $O_\alpha$ where $O_\alpha$ are open sets, then $S$ vanishes also on $\bigcup O_\alpha$. Thus there is a largest open set on which $S$ vanishes. The carrier of $S$ is defined as the complement of this set.

$\mathscr{E}'$ (or $\mathscr{E}'(V)$) is the dual of $\mathscr{E}$. It can be shown, as in [13], that $\mathscr{E}'$ consists of all distributions of compact carrier.

For any $S \in \mathscr{D}'$, by $\overline{S}$ is meant the distribution $f \rightarrow S \cdot f$ for $f \in \mathscr{D}$,
where \( \tilde{f}(x) = \overline{f(x)} \) for any \( x \in G \).

By \( G \times G \) we denote the direct product of \( G \) with itself; \( G \times G \) is again a Lie group whose underlying manifold is the Cartesian product of the underlying manifold of \( G \) with itself. By \( \mathcal{D}, \mathcal{C}, \mathcal{D}', \mathcal{C}' \) we denote the spaces on \( G \times G \) corresponding to \( \mathcal{D}, \mathcal{C}, \mathcal{D}', \mathcal{C}' \) respectively.

Let \( k \) be a continuous map on \( G \times G \) and \( x \in G \). Then by \( k_{x_1 \times x_2} \) we mean the map on \( G \) : \( y \rightarrow k(x, y) \). Suppose that, for all \( x \in G \), \( k_{x_1 \times x_2} \) is in a space \( U \) of mappings on \( G \). Then by \( k \) we mean the mapping \( x \rightarrow k_{x_1 \times x_2} \) of \( G \rightarrow U \). Let \( L \) be a map defined on \( U \); then we say that \( k \) is in the domain of \( L \), and we denote by \( L_k \) the map

\[
x \rightarrow Lk_{x_1 \times x_2}
\]

for \( x \in G \). If the range of \( L \) is again a space of mappings on \( G \), then we say also that \( k \) is in the domain of \( L_{d1} \) and we shall denote by \( L_{d1}k \) the mapping on \( G \times G \):

\[
(x, y) \rightarrow Lk_{x_1 \times x_2}(y).
\]

\( L_{d1} \) is called the lift of \( L \) to \( G \times G \). We define \( k_{x_1 \times x_2}, k_2, L_1, L_{d1} \) similarly.

We can now define, as in \([5]\), two products involving distributions and functions:

For any \( S \in \mathcal{D}' \), \( k \in \mathcal{D} \), then we have two inner products: \( S_1k \) and \( S_2k \) which are both in \( \mathcal{D} \).

For any \( S, U \in \mathcal{D}' \) we define the direct products \( S_1 \times U_2 \) and \( S_2 \times U_1 \) by

\[
S_1 \times U_2 \cdot k = S \cdot U_2k, \quad S_2 \times U_1 \cdot k = S \cdot U_1k
\]

for any \( k \in \mathcal{D} \).

The direct products define continuous bilinear maps which are commutative, while the inner products are only separately continuous bilinear maps. (If \( V, W, X \) are topological vector spaces and \( t : V \times W \rightarrow X \) is a bilinear map, then \( t \) is called separately continuous (see \([4]\), \([5]\)) if, for \( B, B' \) any bounded sets in \( V, W \) respectively, the maps

\[
w \rightarrow t(b, w), \quad v \rightarrow t(v, b')
\]

are, for \( b \in B, b' \in B' \), equicontinuous linear maps of \( W \rightarrow X \) and \( V \rightarrow X \) respectively.)

By \( \{Q_i\} \) we shall denote an enumeration of the operators \( D_{r_1}D_{r_2} \cdots D_{r_m} \) with \( Q_1 = \text{identity} \).

For \( f \) a continuous map defined on \( G \), \( \tilde{f} \) is the map \( x \rightarrow f(x^{-1}) \).

We shall denote by \( \gamma \) the function on \( G \) defined by \( dxg = \gamma(g)dx \), where \( dx \) is a left invariant Haar measure. It is known that \( \gamma \in \mathcal{C}(C) \)
and, moreover, \( \gamma \) is a homomorphism on \( G \). By \( \omega \) we denote the function on \( G \) defined by \( dx^{-1} = \omega(x)dx \). Again, \( \omega \in C(C) \) and \( \omega \) is a homomorphism on \( G \). It is readily verified that \( \omega(y) = \gamma(y^{-1}) \) for any \( y \in G \). For any \( S \in D \), we write \( S \cdot f = S \cdot \omega \) for any \( f \in D \).

3. Convolution on \( G \). For any continuous map \( f \) of \( G \) into \( V \) and any \( x \in G \) we define the translations

\[
(\mathcal{L}(x)f)(y) = f(x^{-1}y) \quad (\mathcal{R}(x)f)(y) = f(yx)
\]

for any \( y \in G \).

**Proposition 4.** \((x, f) \rightarrow \mathcal{L}(x)f \) and \((x, f) \rightarrow \mathcal{R}(x)f \) are continuous maps of \( G \times D \rightarrow D \) and also of \( G \times \mathcal{E} \rightarrow \mathcal{E} \).

**Proof.** We shall establish the theorem for the map \((x, f) \rightarrow \mathcal{L}(x)f \) of \( G \times D \rightarrow D \); the other parts of the proposition may be established by similar methods. By the results of Dieudonné and Schwartz (see [4], [5]) it is sufficient to prove that this is a continuous map of \( G \times \mathcal{D}_K \rightarrow D \) for any compact set \( K \) of \( G \). Since the map is linear in \( f \) and a homomorphism in \( x \), it is sufficient to prove continuity at \( f = 0 \) and \( x = 1 \). Let \( K \) be a given compact set in \( G \) and choose \( K' \) a compact set in \( G \) so large that \( K' \) contains the carriers of all \( \mathcal{L}(x)f \) for \( x \in \mathcal{D}_K \). Let \( M \) be a neighborhood of zero in \( \mathcal{D}_K \). Then we can find operators \( Q_1, Q_2, \ldots, Q_r \), and continuous semi-norms \( \rho_1, \rho_2, \ldots, \rho_n \) on \( V \), and a positive number \( a \) so that \( M \) contains the set of \( h \in \mathcal{D}_K \) which satisfy

\[
\max_{y \in O, t} \rho_j[(Q_jh)(y)] \leq a
\]

for \( j = 1, 2, \ldots, r \).

For any \( p \in D \), any \( k \), and \( x, z \in G \),

\[
(D_kQ(z)p)(x) = \left[ \left( \frac{d}{dt} \right) Q(z)p \right] \left( \exp tY_k x \right) \bigg|_{t=0}
\]

\[
= \left[ \left( \frac{d}{dt} \right) p \right] \left( z^{-1} \exp tY_k x \right) \bigg|_{t=0}
\]

\[
= \left( \frac{dp}{dt} \right) \left( z^{-1} \exp tY_k x \right) \bigg|_{t=0}
\]

\[
= \left( \frac{dp}{dt} \right) \left( (\exp tY_k z) z^{-1} x \right) \bigg|_{t=0}
\]

Now, write \( z^{-1} Y_k z = \sum c_{kl}(z)Y_l \) where \( (c_{kl}) \) is the matrix of the adjoint representation of \( G \) on \( g \). Then we have
We also have

\[ (D_z \mathcal{L}(z)p)(x) = (D_{z^{-1}y_z}f)(z^{-1}x) \, . \]

We also have

\[ D_{z^{-1}y_z}p = \sum c_{kj}(z)D_j p \, . \]

The functions \(c_{kj}\) are continuous and even indefinitely differentiable on \(G\). Hence, we can find an \(A > 0\) so that

\[ \max_{z \in K} |c_{kj}(z)| \leq A \]

for all \(k, j\).

It follows immediately from this that we can be assured that, for \(q \in \mathcal{D}_K, z \in K\),

\[ \max_{x \in G} \rho_i[(D_k \mathcal{L}(z)q)(x)] \]

will be small by making

\[ \max_{x \in G, j, i} \rho_i[(D_j q)(x)] \]

sufficiently small. Proposition 4 now follows by iteration, since each \(Q_t\) is of the form \(D_{r_1}D_{r_2} \cdots D_{r_m}\).

For any continuous map \(f\) on \(G\), \(\mathcal{L}f\) is the map on \(G \times G: (x, y) \rightarrow f(x^{-1}y)\); \(\mathcal{L}^*f\) is the map on \(G \times G: (x, y) \rightarrow f(xy)\). By the method of proof of Proposition 1, we can establish

**Proposition 5.** \(f \rightarrow \mathcal{L}f\) and \(f \rightarrow \mathcal{L}^*f\) are continuous linear maps of \(\mathcal{E} \rightarrow \mathcal{E}'\).

We are now in a position to define the convolution product involving distributions and functions. The definition differs slightly from that of Schwartz [13]: For any \(S \in \mathcal{D}', f \in \mathcal{D}\), \(x \in G\), we set

\[ (S * f)(x) = \mathcal{S} \cdot \mathcal{L}(x)f \]

This formula can also be considered valid if \(S \in \mathcal{E}'\) and \(f \in \mathcal{E}\).

**Proposition 6.** \((S, f) \rightarrow S * f\) is a separately continuous map of

(a) \(\mathcal{E}' \times \mathcal{E} \rightarrow \mathcal{E}(C)\)

(b) \(\mathcal{E}' \times \mathcal{D} \rightarrow \mathcal{D}(C)\)

(c) \(\mathcal{D}' \times \mathcal{D} \rightarrow \mathcal{E}(C)\).

which is antilinear in \(S\) and linear in \(f\).
Proof. (a) Let \( j \) be fixed and write \( A = D_j \). We find from the definitions that, for \( S \in \mathcal{E}' \), \( f \in \mathcal{E} \), \( S \ast f \in \mathcal{E}(C) \) and, moreover,

\[
(A(S \ast f))(x) = \overline{S} \cdot (A_{|\sqrt{\mu}} \hat{f})_{x_1 = x}.
\]

From this it follows by iteration that, for any \( Q = Q_s \), we have

\[
(Q(S \ast f))(x) = \overline{S} \cdot (Q_{|\sqrt{\mu}} \hat{f})_{x_1 = x}.
\]

Part (a) results immediately from (3) together with Proposition 5.

(b) By a result of Dieudonné and Schwartz (see [4]) it is sufficient to prove that, for \( K \) a compact set in \( G \), \( (S, f) \mapsto S \ast f \) is a separately continuous map of \( \mathcal{E}' \times \mathcal{D}_K \to \mathcal{D} \). Now, it is obvious that

\[
\text{carrier } S \ast f \subset (\text{carrier } S)(\text{carrier } f).
\]

Our assertion now follows from (a) above and the fact that \( \mathcal{D}_K \) has the topology induced by \( \mathcal{E} \).

(c) This is proven by essentially the same reasoning as that employed in the proof of (a) above.

4. \( \mathcal{E} \) as a space of linear transformations. In this section we shall prove our main result.

**Theorem 1.** Let \( S \in \mathcal{D}' \) have the property that \( S \ast f \in \mathcal{D}(C) \) whenever \( f \in \mathcal{D} \); then \( S \in \mathcal{E}' \).

Proof. Let us suppose that \( S \) satisfies the hypotheses of Theorem 1, and let \( K \) be a fixed compact set in \( G \). We shall show first that there exists a compact set \( K' \subset G \) such that \( S \ast \mathcal{D}_K \subset \mathcal{D}_{K'} \). Assume this is not the case, and let \( \{K_i\} \) be a compact exhaustion of \( G \). (That is, each \( K_i \) is a compact set which is the closure of a nonempty open set. Moreover, \( K_i \subset K_{i+1} \) and \( \bigcup K_i = G \).) We shall produce a sequence \( \{g_i\} \) with the following properties:

1. Each \( g_i \in \mathcal{D}_K \).
2. \( \Sigma g_i \) converges in \( \mathcal{D}_K \).
3. There is a sequence of positive numbers \( m_i \) with \( m_{i+1} - m_i \geq 1 \) for all \( i \) such that

   \[
   \text{carrier } (S \ast g_i) \subset K_{m_i},
   \]

   \[
   \text{carrier } (S \ast g_{i+1}) \subset K_{m_i}.
   \]

4. There is a sequence of points \( a_i \in G \) such that \( a_i \) is a point of \( K_{m_i} \setminus K_{m_{i-1}} \) (where \( K_{m_0} \) is the empty set) for which \( (S \ast g_i)(a_i) \neq 0 \) and
\[(S \ast g_{i+k})(a_i) \leq \frac{1}{3^k} |(S \ast g_i)(a_i)|\]

for all \(k > 0\).

Suppose that the sequences \(\{g_i\}, \{m_i\}, \{a_i\}\) can be found. Then for any \(i > 1\),

\[
|\sum_{j=1}^{i} (S \ast g_j)(a_i)| = |\sum_j (S \ast g_j)(a_i)|
\]

\[
\geq |(S \ast g_j)(a_i) - \sum_{j<i} |(S \ast g_j)(a_i)|
\]

\[
\geq |(S \ast g_i)(a_i)| \left(1 - \sum_{j=1}^{i-1} \frac{1}{3^j}\right)
\]

\[
= \frac{1}{2} |(S \ast g_i)(a_i)|
\]

\[
> 0.
\]

Since the set \(\{a_i\}\) is clearly not contained in any compact set of \(G\), we conclude that \(S \ast \sum g_j\) is not of compact carrier, which contradicts our hypothesis.

It remains to define the sequences \(\{g_i\}, \{m_i\}, \{a_i\}\). Let \(g_i \in \mathcal{D}_K\) be chosen so that \(S \ast g_i \neq 0\). Let \(a_i\) be any point in \(G\) for which \((S \ast g_i)(a_i) \neq 0\), and choose \(m_i > 0\) so that

\[
\text{carrier } (S \ast g_i) \subset K_{m_i}.
\]

Assume that \(g_1, \ldots, g_k, a_1, \ldots, a_k, m_1, \ldots, m_k\) have been defined with the required properties; we shall now define \(g_{k+1}, a_{k+1}, m_{k+1}\). Now, by our assumption, there is an \(f \in \mathcal{D}_K\) such that

\[
\text{carrier } (S \ast f) \subset K_{m_{k+1}}.
\]

Let \(m_{k+1}\) be chosen so that \(\text{carrier } (S \ast f) \subset K_{m_{k+1}}\), and let \(a_{k+1}\) be some point in \(K_{m_{k+1}} - K_{m_k}\) such that \((S \ast f)(a_{k+1}) \neq 0\). Define

\[
g_{k+1} = \frac{f}{\max_{x \in G} |f(x)| \frac{1}{3^{k+1}}} \max_{j, i \in \mathbb{N}} \left(\max_{x \in G} |(Q_i, f)(x)|, \frac{1}{[|(S \ast g_j)(a_i)|]^{-i}}\right)
\]

The sequences \(\{g_i\}, \{m_i\}, \{a_i\}\) are thus defined. It is clear that conditions 1, 3, 4 are satisfied. Further, each \(g_i \in \mathcal{D}_K\) and, for \(R\) any semi-norm on \(\mathcal{D}_K\) of the kind used to define the topology of that space, it is clear that

\[
\sum R(g_i) > \infty.
\]

Thus \(\sum g_i\) converges in \(\mathcal{D}_K\).
To complete the proof of Theorem 1, let us assume that $S$ is not of compact carrier, and let $K$ be a given compact symmetric neighborhood of 1 in $G$. It is clear that we can choose an open set $U$ in $G$ such that $S$ does not vanish on $U$ and such that $U \cap K'$ is empty, where $K'$ is a compact symmetric set such that $\bar{S} * D_K \subset D_{K'}$. It follows easily that we can find a $g \in G$, and an $f \in D$ such that carrier $f \subset Kg \subset U$, $S \cdot f \neq 0$.

On the other hand, by definition,

$$S \cdot \hat{f} = (\bar{S} * \mathcal{L}(g)f)(g^{-1}).$$

But, carrier $f \subset Kg$ implies carrier $\mathcal{L}(g)f \subset K$ because $K$ is symmetric. Also, $g \notin K'$ because $1 \in K$ and $K' \cap U$ is empty. Since $K'$ is symmetric, also $g^{-1} \notin K'$. Thus, $S \cdot \hat{f} = (\bar{S} * \mathcal{L}(g)f)(g^{-1}) = 0$; this contradiction completes the proof of Theorem 1.

The set of distributions of $\mathcal{E}'$ forms a vector space of continuous linear mappings of $D \rightarrow D$ under convolution; we give this space the compact-open topology (see [6]) and obtain a topological vector space $J$. A fundamental system of neighborhoods of zero in $J$ consists of all sets $N$ for which we can find a compact set $K$ in $D$ and a neighborhood of zero $M$ in $D$ so that $N$ consists of those $S \in \mathcal{E}'$ with $S * h \in M$ for all $h \in K$. By Proposition 1 of § 5 of [6], we would have obtained the same topologies if we had considered the distributions of $\mathcal{E}'$ as defining, under convolution, continuous linear maps of $D' \rightarrow D'$.

**Theorem 2.** The natural map $u : \mathcal{E}' \rightarrow J$ is a topological isomorphism onto.

**Proof.** $u$ is clearly one-to-one, linear, and onto. Moreover, $J$ is given the weakest topology to make the maps

$$S \rightarrow (u^{-1}S) * f$$

of $J \rightarrow D$ equicontinuous for $f$ in any compact set of $D$; by Proposition 6 this implies that $u$ is continuous.

Since $u^{-1}$ is linear, we need verify continuity only at zero. Let $T$ be a neighborhood of zero in $\mathcal{E}'$; there is a bounded set $\beta \subset \mathcal{E}$ so that $T$ contains the set of $S \in \mathcal{E}'$ which satisfy $|S \cdot b| \leq 1$ for all of $b \in \beta$.

Let $K$ be an open symmetric neighborhood of 1 in $G$ whose closure is compact. Then it is clear that we can find a sequence of points $a_i \in G$ such that $\{a_iK\}$ is a scattered resolution of $G$ (see § 2). We can also insure that, if $a$ is one of the $a_i$, so is $a^{-1}$. Let $\{h\}$ be a partition of unity relative to this scattered resolution (see § 2). It is readi-
ly verified by the method of proof of Proposition 1 of § 3 that, for
each $i$, the set $B_i$ of functions $\mathcal{L}(a_i)(h_i f)$ for $f \in \beta$ is bounded in $\mathcal{D}$. For each $j$ there is a double sequence $s_j = M_{jik}$ of positive numbers so that $B_j$ is contained in the bounded (in $\mathcal{D}$) set $L_j$ of all $g \in D$ whose carriers are contained in $K$ and which satisfy

$$\max_{x \in \mathcal{G}} \rho_{\mathcal{G}}((Q_i g)(x)) \leq M_{jik}$$

for all $i$. From the denumerable number of double sequences $s_j$, we construct a double sequence $s = \{M_{ik}\}$ of positive numbers such that, for each $j$, $M_{jik} \leq M_{ik}$ for all but a finite number of $i$, $k$. Hence, for each $j$, we can find an $e_j > 0$ so that $e_j M_{jik} \leq M_{ik}$ for all $i$, $k$; we can clearly make $e_j = e_i$ if $a_j = a_i^{-1}$.

Let $A$ be the set of $f \in \mathcal{D}$ for which

1. carrier $f \subset K$
2. $\max_{x \in \mathcal{G}} \rho_{\mathcal{G}}((Q_i f)(x)) \leq M_{ik}$ for all $i$, $k$,

so $A$ is bounded in $\mathcal{D}$. Let $M$ be the neighborhood of zero in $\mathcal{D}$ consisting of those $h \in \mathcal{D}$ with

$$\max_{x \in \mathcal{G}} \rho_{\mathcal{G}}(h(x)) \leq e_j d_j$$

for all $j$, where $d_j$ are positive numbers which satisfy $\Sigma d_j = 1$. Call $N$ the set of $S \in J$ with $S \ast f \in M$ for all $f \in A$, so $N$ is a neighborhood of zero in $J$; we claim that $u^{-1}(N) \subset T$.

Let us assume this is not the case; then we can find an $S \in N$ with $u^{-1}S \notin T$, that is, $S \in N$ but

$$|u^{-1}S \cdot f| > 1$$

for some $f \in \beta$. Now, $u^{-1}S$ is of compact carrier; thus we can find an $r$ such that

$$\sum_{k=1}^{r} h_k(x) = 1$$

for any $x \in \text{carrier } (u^{-1}S)$. Hence

$$|u^{-1}S \cdot f| \leq |u^{-1}S \cdot h_1 f| + |u^{-1}S \cdot h_2 f| + \cdots + |u^{-1}S \cdot h_r f| .$$

It is clear from the definitions that, for each $i$,

$$e_i \mathcal{L}(a_i)(h_i f) \in A .$$

Thus,
for some \( g \in A \), which gives, for \( i = 1, 2, \ldots, r \),

\[
|u^{-1}S \cdot h_i f| = \frac{1}{e_i} |u^{-1}S \cdot \mathcal{Q}(a_i^{-1}) g| \\
\quad = \frac{1}{e_i} |(u^{-1}S \ast g)(a_i^{-1})| \\
\quad \leq \frac{1}{e_i} e_j d_j ,
\]

where \( a_j = a_i^{-1} \), because \( g \in A \) and \( u^{-1}S \in N \). Now, since \( e_i = e_j \), we have

\[
|u^{-1}S \cdot h_i f| \leq d_j .
\]

Applying this to equation (5) we obtain

\[
|u^{-1}S \cdot f| \leq d_1 + d_2 + \cdots + d_r \leq 1
\]

(where we set \( a_j = a_j^{-1} \)). This contradiction proves the theorem.

5. Extension of the main result. We assumed in §§ 2, 3, 4 that \( V \) is metrizable. In case \( V \) is not metrizable, then the spaces \( \mathcal{E} \) and \( \mathcal{D} \) can be defined as before, but \( E \) is no longer metrizable, and \( \mathcal{D} \) is not an \( LF \) space in the sense of Dieuonné and Schwartz [4]. However, there is no difficulty in extending the definition and continuity properties of the convolution product to this case. Theorem 1 can be extended to this case, but the proof of Theorem 2 does not extend to the case of \( V \) not metrizable. All that can be proven (and the proof is much simpler than the proof of Theorem 2 above) is that \( u \) is continuous and that \( u^{-1} \) is sequentially continuous and takes bounded sets into bounded sets. The continuity of \( u^{-1} \) is an open question.

We assume in the following that \( V \) is a complete, locally convex, Hausdorff, topological vector space. By \( V^* \) we denote the space of continuous linear maps of \( V \) into \( V \) with the compact-open topology, so \( V^* \) is again a complete, locally convex, Hausdorff, topological vector space.

Let \( K \) and \( L \) denote compact subgroups of \( G \). By a representation of \( K \) on \( V \) we mean a continuous homomorphism \( U \) of \( K \) into \( V^* \). Let \( U \) and \( W \) be representations of \( V \) of \( K \) and \( L \) respectively. By \( \mathcal{D} \) we denote the space of those \( f \in \mathcal{D}(V^*) \) for which

\[
\mathcal{Q}(k^{-1})\mathcal{R}(l)f = U(k)f W(l)
\]

for any \( k \in K, \ l \in L \). We give \( \mathcal{D} \) the topology induced by \( \mathcal{D} \).
is defined similarly.

For any $T \in \mathcal{D}'(V^*)$, $g \in G$, we define $\mathcal{L}(g)T$ and $\mathcal{R}(g)T$ as the distributions
\[
(7) \quad \mathcal{L}(g)T \cdot f = T \cdot \mathcal{L}(g^{-1})f, \quad \mathcal{R}(g)T \cdot f = \eta(g)T \cdot \mathcal{R}(g^{-1})f
\]
for any $f \in \mathcal{D}(V^*)$. ($\eta$ was defined in §2.) Let us denote by $\mathcal{UW} \mathcal{D}'$ the space of all $S \in \mathcal{D}'(V^*)$ which satisfy
\[
(8) \quad \mathcal{L}(k)\mathcal{R}(l)S \cdot f = S \cdot U(k^{-1})f W(l)
\]
for any $f \in \mathcal{D}(V^*)$, $k \in K$, $l \in L$. We shall write $U(k)SW(l^{-1})f$ for the right side of (8). We give $\mathcal{UW} \mathcal{D}'$ the topology induced by $\mathcal{D}'(V^*)$. $\mathcal{UW} \mathcal{E}'$ is defined similarly.

We can easily show

**Proposition 7.**

\[
f \mapsto P_{\mathcal{UW}}f = \int_{K \times L} U(k)\mathcal{L}(h)\mathcal{R}(l)f V^{-1}(l)dkdl
\]

(whose $dk$ and $dl$ are the respective Haar measures on $K$ and $L$ so normalized that $\int_{K} dk = \int_{L} dl = 1$) defines continuous open projections of $\mathcal{D}(V^*)$ onto $\mathcal{UW} \mathcal{D}$ and $\mathcal{E}(V^*)$ onto $\mathcal{UW} \mathcal{E}$. Also
\[
S \mapsto P_{\mathcal{UW}}S = \int_{K \times L} \mathcal{L}(k^{-1})\mathcal{R}(l)[U^{-1}(k)SV(l)]dkdl
\]
defines continuous open projections of $\mathcal{D}'(V^*)$ onto $\mathcal{UW} \mathcal{D}'$ and $\mathcal{E}'(V^*)$ onto $\mathcal{UW} \mathcal{E}'$.

**Corollary.** $\mathcal{UW} \mathcal{D}'$ is the dual of $\mathcal{UW} \mathcal{D}$ and $\mathcal{UW} \mathcal{E}'$ is the dual of $\mathcal{UW} \mathcal{E}$.

**Proof.** This is an immediate consequence of Proposition 6 and the fact that, for $S \in \mathcal{D}'$, $f \in \mathcal{D}$ (or for $S \in \mathcal{E}'$, $f \in \mathcal{E}$), we have $P_{\mathcal{UW}}S \cdot f = S \cdot P_{\mathcal{UW}}f$.

Suppose that $K=L$; then we see easily that the convolution defined in §3 defines a separately continuous bilinear map of $\mathcal{UW} \mathcal{D}' \times_{wZ} \mathcal{D} \rightarrow \mathcal{UW} \mathcal{E}(C)$ (where $U$, $W$, $Z$ are representations of $K$ on $V$). The method of proof of Theorems 1 and 2 can be used to show.

**Theorem 3.** $\mathcal{UW} \mathcal{E}'$ consists of all $S \in \mathcal{UW} \mathcal{D}'$ such that $S \ast f \in \mathcal{UW} \mathcal{D}$ for any $f \in \mathcal{UW} \mathcal{D}$. The topology of $\mathcal{UW} \mathcal{E}'$ is sequentially the same as that obtained by considering the elements of $\mathcal{UW} \mathcal{E}'$ as defining (by convolution) continuous linear maps of $\mathcal{UW} \mathcal{D} \rightarrow \mathcal{UW} \mathcal{D}$ and giving this set the

---

3 Note that since $L$ is compact, the restriction of $\eta$ to $L$ is 1.
compact-open topology $\tau$. Moreover, the bounded sets of $uw^r$ are the same as those of $\tau$.

REMARK 1. We do not know whether the topologies $\tau$ and that of $uw^r$ are the same. The difficulty is that, for $f \in uw\mathcal{D}$, $g \in G$, $\mathcal{L}(g)f$ is no longer in $uw\mathcal{D}$.

REMARK 2. In case that $K=L$, $V$ is finite dimensional, and $U, W, X, Z$ are irreducible unitary representations of $K$ on $V$, then it follows easily from the Schur orthogonality relations that $S*f=0$ for any $S \in uw\mathcal{E}'f \in \mathcal{D}_x$ if $W$ is not equivalent to $X$.

REMARK 3. The conclusion of Theorem 3 does not necessarily hold if the space $uw\mathcal{D}$ in the hypothesis of the theorem is replaced by $wz\mathcal{D}$ where $Z$ is different from $W$, even if $V$ is finite dimensional and $U, W, Z$ are irreducible unitary representations of $K$. An example will be given in a forthcoming paper of the author and F. I. Mautner. ($G$ can be taken as the complex unimodular group.)

6. General remarks. We have assumed that $G$ is a separable Lie group. In the general case, the spaces $\mathcal{E}$ and $\mathcal{D}$ can be defined as before, but $\mathcal{E}$ will not be metrizable and $\mathcal{D}$ will not be an $\mathcal{LF}$ space in the sense of Dieudonné and Schwartz [4] because $\mathcal{D}$ will be the inductive limit of a non-denumerable number of spaces. For this reason, the topology of $\mathcal{D}$ is best defined as follows: Let $\{f_{ij}\}=\sigma$ be a family of continuous functions on $G$ such that

(a) For each $i$, only a finite number of $j$ appear.
(b) Only a finite number of $f_{ij}$ are different from zero on any compact set of $G$.

Then we define $N_\sigma$ as the set of $h \in \mathcal{D}$ for which

$$\max_{x \in G} \rho_j[(f_{ij}(x)Q_jh)(x)] \leq 1$$

for all $i, j$, where the $Q_j$ are as in § 2, and $\{\rho_j\}$ denotes an enumeration of semi-norms which are sufficient to define the topology of $V$. The sets $N_\sigma$ are seen to form a fundamental system of neighborhoods of zero of a locally convex topological vector space which we shall call $\mathcal{D}$. In case $\mathcal{D}$ is separable it is easily verified that the two definitions agree.

The advantage of the above definition is that it implies immediately the completnessness of $\mathcal{D}$. For, the completion of $\mathcal{D}$ obviously consists of indefinitely differentiable maps. Moreover, if $h$ is any map in the completion of $\mathcal{D}$, then, for any continuous function $f$ on $G$, and any
It is easily seen that $p_k(fh)$ is a bounded function. This implies immediately that $h$ is of compact carrier, hence $h \in \mathcal{D}$.

The properties of convolution can be extended to the nonseparable case and there is no difficulty in extending part of our main results. We can, as in § 5, prove only that the topology of $\mathcal{E}'$ is sequentially, and in regard to bounded sets, the same as the compact-open topology of the space of linear transformations of $\mathcal{D} \to \mathcal{D}$ (under convolution).

The results of § 5 on double coset spaces $K\backslash G/L$ can also be extended to functions invariant under a compact group of automorphisms of $G$ (the group of automorphisms of $G$ is given the compact-open topology).

In addition, the main results of this paper can be extended to locally compact groups. There $\mathcal{E}$ is replaced by the space of continuous functions, $\mathcal{D}$ the space of continuous functions of compact carrier, $\mathcal{D}'$ the space of measures and $\mathcal{E}'$ the space of measures of compact carrier.

REFERENCES

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