ON THE TWO-ADIC DENSITY OF REPRESENTATIONS BY QUADRATIC FORMS

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1. Introduction. The problem of determining $A_q(S, T)$, the number of solutions of $X'SX = T \pmod{q}$, where $S^{(m)}$ and $T^{(n)}$ are symmetric integral matrices, has been considered by C. L. Siegel [2, pp. 539–547]. He obtained explicit formulas for $A_q(S, T)$ when $q = p^n$, where $p$ is a prime not dividing $2|S||T|$. We wish to determine both $A_2(S, T)$ and $A_{2s}(S, T)$ when $|S||T|$ is odd. Siegel has shown that the calculation of $A_{2s}(S, T)$, for $|S||T|$ odd, is sufficient to give results when the modulus is replaced by a higher power of 2. Moreover, his work for composite moduli does not exclude a power of 2 as a factor.

We shall follow the pattern of Siegel’s work, modifying it by the use of canonical forms established by B. W. Jones [1, pp. 715–727] and Gordon Pall for symmetric matrices in $G_2$, the ring of 2-adic integers. (Clearly, $A_q(S, T)$ depends only on the classes of $S$ and $T$ in $G_q$, the ring of $q$-adic integers). We shall calculate $A_2(S, T)$ combinatorially and $A_{2s}(S, T)$ by the use of exponential sums.

2. Recursion formula. For convenience, we state here the following theorem of Jones:

Every quadratic form with matrix in $G_2$ and with unit determinant, $D$, is equivalent to one of the following:

(a) $x_1^2 + x_2^2 + \cdots + ax_r x_{r+1} + bx_{r+1}^2 + cx_{r+1}^2$,

where $a, b, c$ take one of the following sets of values:

- $(1, 1, 1)$ or $(1, 3, 3)$ for $D \equiv 1 \pmod{8}$,
- $(1, 1, 5)$ or $(1, 3, 7)$ for $D \equiv 5 \pmod{8}$,
- $(1, 1, 3)$ or $(3, 3, 3)$ for $D \equiv 3 \pmod{8}$,
- $(1, 1, 7)$ or $(3, 3, 7)$ for $D \equiv 7 \pmod{8}$,

while if $r=2$, $b$ and $c$ take one of the following sets of values:

- $(1, 1)$ or $(3, 3)$ for $D \equiv 1 \pmod{8}$,
- $(1, 5)$ or $(3, 7)$ for $D \equiv 5 \pmod{8}$,
- $(1, 3)$ for $D \equiv 3 \pmod{8}$,
- $(1, 7)$ for $D \equiv 7 \pmod{8}$.

(b) A sum of binary forms of the two types: $f = 2x_1^2 + 2x_1 x_2 + 2x_2^2$,

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Here, we may at will choose one of types \( f \) and \( g \) and require that all but at most one of the binary forms be of that type.

When (a) applies, we will call the matrix of the form *even*; when (b) applies, we will call the matrix *odd*.

We assume hereafter that \( |S||T| \) is odd. Then we remark immediately, as in Siegel's paper, that all representations of \( T \) by \( S \) modulo \( 2^a \), where \( a = 1 \) or \( 3 \), are primitive. Following the line of Siegel's proof, we now obtain the recursion formula.

Taking \( T = T_0^{(r)} + T_1^{(n-r)} \), from the canonical forms above, we let \( \chi \) designate the first \( r \) columns of \( X \), where \( X'SX \equiv T \pmod{2^a} \). Then

\[
(1) \quad \chi'S\chi \equiv T_0 \pmod{2^a}.
\]

As remarked above, any solution \( \alpha \) of (1) is primitive, and so can be completed to a unimodular matrix \( U_1 = (\alpha A) \) in \( G_2 \). We wish to alter \( U_1 \) so that

\[
(2) \quad U_1'SU_1 \equiv \begin{pmatrix} T_0 & N' \\ N & S_1 \end{pmatrix} \pmod{2^a},
\]

with \( N \) designating an \( m-r \) by \( r \) null matrix. To do this, we call \( E \) the matrix obtained from \( U_1'SU_1 \) by deleting the first \( r \) columns and the last \( m-r \) rows. Then, noting that the determinant of \( T_0 \) is a 2-adic unit, we multiply \( T_0 \) by \( \lambda \) to achieve the desired form (2).

Now if there exists a \( C \), with its first \( r \) columns congruent to \( \alpha \) (mod \( 2^a \)), such that \( C'SC \equiv T \pmod{2^a} \), we complete \( C \) to a unimodular matrix in \( G_2 \), say \( U_2 = (CA_1) \). Since \( U_1 \) and \( U_2 \) are both completions of \( \alpha \), consideration of \( U_1'SU_2 \) shows us that

\[
(3) \quad C \equiv U_1' \begin{pmatrix} T^{(r)} & B \\ N & C_1 \end{pmatrix} \pmod{2^a},
\]

where \( C_1 \) and the \( r \)-rowed \( B \) are in \( G_2 \). Using (2) and (3) in \( C'SC \equiv T \pmod{2^a} \), we find that \( B \) is null and that \( C_1'S_1C_1 \equiv T_1 \pmod{2^a} \). Thus, we obtain each different solution \( X \pmod{2^a} \) exactly once by first determining all different solutions \( \chi \) (mod \( 2^a \)) of (1), then finding a \( U_1 \) as above for each such \( \chi \), and finally determining for the corresponding \( S_1 \) all different solutions of \( X'S_1X \equiv T_1 \pmod{2^a} \). Thus

\[
A_{2^a}(S, T) = \sum_\alpha A_{2^a}(S_1, T_1).
\]

3. **Combinatorial calculation of** \( A_2(S, T) \). We use canonical forms,
taken modulo 2, in the following cases:

Case 1. We assume $T$ even and $S$ odd. Here we clearly have no solution.

Case 2. We assume both $S$ and $T$ even.

2.1. For $n=1$, $A_2(S, T)=2^{m-1}$.

Proof. We seek solutions $\{x_i\}$ such that

$$\sum_{i=1}^{m} x_i = 1 \pmod{2}.$$

Since a parity change in one $x_i$ changes the parity of the sum, we see that $A_2(S, T)$ is half of $2^m$.

2.2. For $n=2$, $A_2(S, T)=2^{m-1} \cdot 2^{m-2}$, for even $m$.

$$A_2(S, T)=(2^{m-1}-1) \cdot 2^{m-2}, \text{ for odd } m.$$

Proof. We use Case 2.1 with the recursion formula. We wish to show that for every solution $\alpha$ of (4), except one where $m$ is odd and each component of $\alpha$ is 1, $A_2(S, T)>0$; that is, $S_1$ is even. Here we have the additional conditions:

$$\sum_{i=1}^{m} y_i = 1 \pmod{2},$$

$$\sum_{i=1}^{m} x_i y_i = 0 \pmod{2}.$$

But there is an obvious $\{y_i\}$ satisfying (5) and (6) with any solution $\{x_i\}$ of (4) which has a zero element; and clearly there is no such $\{y_i\}$ if all the elements of $\{x_i\}$ are 1. Hence, we have our result.

2.3. For general $m$ and $n$, ($n>1$),

$$A_2(S, T)=F(m) \cdot F(m-1) \cdots F(m-n+2) \cdot 2^{m-n},$$

where $F(m)=2^{m-1}$ for even $m$ and $F(m)=2^{m-1}-1$ for odd $m$.

Proof. Now $S_1$ depends only on $\alpha$ and not on $n$, so that Case 2.2 tells us that $S_1$ is even except when $m$ is odd and each element of $\alpha$ is 1. Then the above result follows easily from the recursion formula.

Case 3. We assume both $S$ and $T$ odd.

3.1. For $n=2$, $A_2(S, T)=(2^m-1) \cdot 2^{m-1}$.

Proof. We want solutions, $\{x_i\}$ and $\{y_i\}$ of

$$x_1 y_2 + x_2 y_3 + \cdots + x_{m-1} y_m + x_m y_{m-1} = 1 \pmod{2}.$$
Now \{x_i\} cannot be null if (7) is to hold; also there is an obvious \{y_i\} satisfying (7) for each non-null \{x_i\}. Let us fix a non-null \{x_i\} and call any \{y_i\} satisfying (7) with our fixed \{x_i\} a "solution", otherwise a "non-solution". Then, since, modulo 2, the sum of two "solutions" is a "non-solution" and the sum of a "solution" with a "non-solution" is a "solution", we have our result.

3.2. For general \(m\) and \(n\),

\[A_2(S, T) = (2^m - 1) \cdot 2^{m-1} \cdot (2^{m-2} - 1) \cdot 2^{m-3} \cdots (2^{m-n+1} - 1) \cdot 2^{m-n+1} .\]

Proof. Equivalent matrices in \(G_2\) have the same parity, which is clearly unchanged when the matrices are taken modulo 2. Thus, from (2), since \(S\) is odd, so is

\[S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .\]

Hence \(S\) is odd, and our result follows.

Case 4 We assume that \(S\) is even and \(T\) odd.

4.1. For \(n=2\), \(A_2(S, T) = (2^{m-1} - 1) 2^{m-2}, \) if \(m\) is odd.
\[A_2(S, T) = (2^{m-1} - 2) 2^{m-2}, \] if \(m\) is even.

Proof. We want solutions \(\{x_i\}\) and \(\{y_i\}\), of

\[\sum_{i=1}^{m} x_i = 0, \quad \sum_{i=1}^{m} y_i = 0, \quad \sum_{i=1}^{m} x_i y_i = 1 ,\]

all taken modulo 2. Let us fix \(\{x_i\}\) satisfying the first of these and consider the \(2^{m-1}\) incongruent \(\{y_i\}\) which satisfy the second. Of these \(\{y_i\}\), we call those satisfying the final congruence with our fixed \(\{x_i\}\) "solutions" and those not doing so "non-solutions". By an argument similar to that used in Case 3.1, we see that exactly half the \(2^{m-1}\) choices of \(\{y_i\}\) are "solutions", except when \(\{x_i\}\) is the null vector or, with \(m\) even, \((1, 1, \ldots, 1)\). There is no "solution" \(\{y_i\}\) corresponding to either of these exceptional \(\{x_i\}\).

4.2. For general \(m\) and \(n\),

\[A_2(S, T) = (2^{m-1} - p) 2^{m-2} (2^{m-3} - p) 2^{m-4} \cdots (2^{m-n+1} - p) 2^{m-n} ,\]

where \(p=1\) for odd \(m\) and \(p=2\) for even \(m\).

Proof. Using (2) again, we observe that \(S\) is even. (See Case 3.2.) Then the recursion formula implies our result.

4. Determination of \(A_3(S, T)\). We will assume throughout the fol-
following cases that \( S \) and \( T \) are in appropriate canonical forms as given in §2.

**Case 1.** We assume \( T \) is even.

Clearly, \( A_S(S, T) = 0 \) for \( S \) odd and \( T \) even; so we will also assume \( S \) is even.

1.1. Let \( n=1 \). Here \( T=(t) \). For \( \omega \) a primitive 8th root of unity, we have

\[
8A_S(S, T) = \sum_{h, a \mod 8} \omega^r, \quad Y = h(a, s_1^2 + \cdots + a_m s_m^2 - t),
\]

where \( h \) and the elements \( a_1, a_2, \ldots, a_m \) of the vector \( a \) run through a complete residue system modulo 8, and where the diagonal elements of \( S \) are the odd \( s_1, s_2, \ldots, s_m \). Calling

\[\sum_{\alpha \mod 8} a h^a s^2 = \left[ h s \right],\]

we get

\[
8A_S(S, T) = \sum_{i=1}^{m} [h s_i] [h s_j] \cdots [h s_m] \omega^{-h^2} + s^m.
\]

We observe that \( [h s_i] = 4\omega^{h s_i} \) for odd \( h \); \( [h s_i] = 0 \) for \( h \equiv 4 \mod 8 \); \( [h s_i] = 4\sqrt{2} \omega \) for \( h s_i \equiv 2 \mod 8 \); and \( [h s_i] = 4\sqrt{2} \omega^2 \) for \( h s_i \equiv 6 \mod 8 \).

Then, let us call \( u = \sum_{i=1}^{m} s_i - t \mod 8 \), and define \( f(u) = 1 \) for \( u \equiv 0 \mod 8 \), \( f(u) = -1 \) for \( u \equiv 4 \mod 8 \), and \( f(u) = 0 \) for \( u \equiv 0 \mod 4 \). Also define

\[
K = (-1)^{(s_1-1)/2} + (-1)^{(s_2-1)/2} + \cdots + (-1)^{(s_m-1)/2} - 2t \quad \mod 8.
\]

Then direct calculation gives from (9),

\[
8A_S(S, T) = 8^m + 4^{m+1} f(u) + 2(4\sqrt{2})^m \cos \frac{K\pi}{4}.
\]

1.2. Let \( n=2 \). We will (a) ascertain when \( S \) is even and (b) show that two even \( S \)'s corresponding to different solutions \( a \) are equivalent in \( G_2 \). Then the result follows from the recursion formula.

(a) Let \( T=t_1 + t_2 \). Since parity is the same modulo 2 or modulo 8, we see from §3, Case 2.2, that of all solutions, \( a \), of \( rS \equiv t_1 \mod 8 \), those and only those which reduce, modulo 2, to the vector \((1, 1, \ldots, 1)\) will yield odd \( S \)'s. For such an \( a \), \( \sum_{i=1}^{m} a_i s_i \equiv t_1 \mod 8 \) implies \( \sum_{i=1}^{m} s_i \equiv t_1 \mod 8 \). But, equally well, if \( S \) and \( t_i \) are such that \( \sum_{i=1}^{m} s_i \equiv t_i \mod 8 \),


then $\sum_{i=1}^{m} a_i s_i \equiv t_i \pmod{8}$ holds for arbitrary odd $a_i$. Thus, if $\sum_{i=1}^{m} s_i \equiv t_i \pmod{8}$, we get $4^m$ number of $\alpha$'s, solutions of $x^t S x \equiv t_i \pmod{8}$, which yield odd $S_i$'s; otherwise, none.

(b) Now let $a$ be such that $S_i$ is even. From [1], we see that two even matrices of odd determinant, which are congruent modulo 8, are in the same class in $G_2$. Thus, using (2), we obtain:

$$t_1 |S| = |S| \pmod{8} \quad \text{and} \quad \lambda(t_1 + S) = \lambda(S),$$

where $\lambda(S)$ is the class invariant defined as 1 if $4j$ or $4j+1$ of the diagonal elements of a diagonalized form of $S$ are congruent to 3 modulo 4 and $-1$ if $4j+2$ or $4j+3$ are congruent to 3 modulo 4. These two conditions determine uniquely, independently of $a$, the class of $S_i$ in $G_i$.

**Example.** Let $S$ be of type $(1, 3, 3)$ as given in §2, $m > 3$, and $t_i = 5$. Then the determinantal relation gives an even $S_i$ of type $(1, 1, 5)$ or $(1, 3, 7)$. But the $\lambda$-condition admits only the second of the two, so any even $S_i$ is of type $(1, 3, 7)$.

Thus we have

$$8^m A(S, T) = (8^m + 4^{m+1} \cos \left(\pi K_0 / 4\right)) - 8 \cdot 4^m h(u_0))$$

$$\times (8^{m-1} + 4^m \cos \left(\pi K_1 / 4\right)),$$

where $u_0$ and $K_0$ are arguments obtained from $S$ and $t_1$ as above; $u_i$ and $K_i$ are arguments similarly obtained from $S_i$ and $t_i$; and $h(u_0)$ is defined as 1 if $u_0 \equiv 0 \pmod{8}$ and as 0 otherwise.

1.3. Let $n \geq 2$. Since the process of obtaining an $S_i$ from a given pair, $S$ and $t_i$, is the same for $n = 2$ and for $n > 2$, we may use 1.2 above to obtain

$$8^n A_n(S, T) = (8^{n-1} + 4^n f(u_{n-1}) + 2(4^n \sqrt{2}) \cos (\pi K_{n-1} / 4))$$

$$\times \prod_{j=m-n+2}^{m} (8^{j+1} f(u_{m-j}) + 2(4^{j} \sqrt{2}) \cos (\pi K_{m-j} / 4)) - 8 \cdot 4^j h(u_{m-j})),$$

where, for each $i$, $u_i$, and $K_i$ come from $S_i$ and $t_{i+1}$, as above.

(The process of finding successive $S_i$ and $t_i$, and hence of successive $K_i$, $f(u_i)$, and $h(u_i)$, is easy in practice, as evidenced by the example above. Explicit but complicated formulas could be given.)

**Case 2.** We assume $S$ and $T$ are both odd. We will first take $n = 2$.

2.1. We suppose that

$$T = \begin{pmatrix} b & 1 \\ 1 & b \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
where \( b = 0 \) or \( 2 \). Then we seek solutions of:

\[
F(x) = 2(x_1x_2 + x_3x_4 + \cdots + x_{m-1}x_m) \equiv b \pmod{8}
\]

\[
G(y) = 2(y_1y_2 + y_3y_4 + \cdots + y_{m-1}y_m) \equiv b \pmod{8}
\]

\[
H(x, y) = x_1y_2 + x_2y_3 + x_3y_4 + \cdots + x_{m-1}y_{m-1} + x_my_m \equiv 1 \pmod{8}.
\]

Thus

\[
8^2A_d(S, T) = \sum_{h,k, \xi, \eta} \omega^{(F-b)h + (G-b)k + (H-1)t},
\]

where \( \omega = e^{\pi i \xi}; \) and \( h, k, l \), and the components of the vectors \( \xi \) and \( \eta \) all run through complete residue systems modulo 8. Then, letting

\[
(10) \quad R = \sum_{x_1, x_2, y_1, y_2} \omega^{xp}, \quad EXP = 2x_1x_2h + 2y_1y_2k + (x_3y_4 + x_4y_3),
\]

we get

\[
(11) \quad 8^2A_d(S, T) = \sum_{h,k,t} R^{m/2} \omega^{-1 - bh - bk}.
\]

We note that, for \( l \) odd, replacement of \( h \) by \( lh \), of \( k \) by \( lk \), of \( x_i \) by \( lx_i \), and of \( y_j \) by \( ly_j \) in \( EXP \), the displayed exponent of \( (10) \), shows that \( \sum_{h,k} R^{m/2} \) is independent of \( l \). A similar argument works for \( l \equiv 2 \pmod{4} \).

For \( l \equiv 0 \pmod{8} \), we have

\[
R = 2^{1 + r(h)} \cdot 2^{1 + r(k)},
\]

where \( r(t) = 0 \) if \( t \equiv 1 \pmod{2} \), \( r(t) = 1 \) if \( t \equiv 2 \pmod{4} \), and \( r(t) = 2 \) if \( t \equiv 0 \pmod{4} \).

For \( l \equiv 4 \pmod{8} \) and \( h \) odd, we let \( z \equiv x_1h + 2y_2 \pmod{8} \), and replace \( y_2 \) by \( z \) as a variable in \( EXP \). Then, summing first on \( x_1 \), we get

\[
R = 2^8 + r(k).
\]

For \( l \equiv 4 \pmod{8} \) and \( h = 2h_1 \), we let \( z \equiv x_1h_1 + y_2 \pmod{8} \) and again replace \( y_2 \) by \( z \) as a variable in \( EXP \). Summing first on \( x_1 \) and \( z \), we readily get

\[
R = 2^9, \quad \text{for } h,k \equiv 1 \pmod{2}
\]

\[
R = 2^{10}, \quad \text{for } h,k \equiv 0 \pmod{4} \text{ or for } h,k \equiv 2 \pmod{4} \text{ and } k \equiv 1 \pmod{2}
\]

\[
R = 2^{11}, \quad \text{for } h,k \equiv 2 \pmod{4} \text{ and } k \equiv 0 \pmod{2}.
\]

Summing first on \( l \) in \( (11) \), we get by straightforward calculation:

\[
A_d(S, T) = 2^{5m/2} - (2^m + 2^{m/2} - 2), \quad \text{for } b = 0.
\]

\[
A_d(S, T) = 2^{5m/2} - (2^m - 3 \cdot 2^{m/2} + 2), \quad \text{for } b = 2.
\]

2.2. We suppose that
Then, using the same $R$ as before and letting
\[ v = \sum_{x,y,u,v} (8) \]
where
\[ P = 2(xy + x^2 + y)h + 2(uv + u^2 + v)k + (uy + vx + 2ua + 2vy)l, \]
we get
\[ (12) \quad A_8(S, T) = \sum_{h,k,l} R^{(m-2)/2} V^\omega - l - hh - bb - c. \]

To evaluate $V$, we use repeatedly:
\[ \sum_{a} \omega_{a}^2 \omega_{a}^2 a = 0, \text{ if } d \equiv 2 \pmod{4} \text{ or if } d \equiv 1 \pmod{2} \]
\[ = -4\omega^2 + 4, \text{ if } d \equiv 4 \pmod{8} \]
\[ = 4\omega^2 + 4, \text{ if } d \equiv 0 \pmod{8}. \]

We obtain:
(i) For $l$ odd, $V=64$.
(ii) $V$ is the same for $l=2$ and $l=6 \pmod{8}$.
(iii) For $l=0 \pmod{8}$, $V=g(h)g(k)$, where we define $g(t)=64$ for $t=0 \pmod{4}$, $g(t)=16$ for $t=1 \pmod{2}$, and $g(t)=-32$ for $t=2 \pmod{4}$.
(iv) For $l=4 \pmod{8}$, we have:
(a) When $h$ is odd, $V=16g(k)$.
(b) When $h$ or $k=0 \pmod{4}$, $V=2^9$.
(c) When $h=2 \pmod{4}$, $V=-2^4$, when $k$ is odd, and $V=-2^u$, when $k=2 \pmod{4}$.

We sum first on $l$ in (12), using our results for $R$ and considering only $l=0 \pmod{4}$. We get
\[ A_8(S, T) = 2^l \left( 2 \cdot 2^{(m-2)/2} - 2^{11(m-2)/2} - 2^{(m-2)} \right), \quad \text{for } b=0. \]
\[ A_8(S, T) = 2^l \left( 2 \cdot 2^{(m-2)/2} + 3 \cdot 2^{11(m-2)/2} + 2^{(m-2)} \right), \quad \text{for } b=2. \]

For $n>2$, when $S$ and $T$ are odd, we will use our results for $n=2$, along with the recursion formula. The successive canonical forms of $T, T_1, \cdots$ are clear; that is, $T_1$ is obtained from $T$ by removing the initial binary block, etc. $T_1$ is thus odd and known. From
\[ S_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = U_1 S U_1 \pmod{8}, \]
we deduce $-|S| = |S| \pmod{8}$ and the oddness of $S$. Thus $S_1$ is easily determined classwise uniquely. The same holds true, of course, for successive $S_i$.

**Case 3.** We assume $S$ is even and $T$ is odd. Considering first
Let $n=2$, we let $s_1, s_2, \cdots, s_m$ be the diagonal elements in the canonical form of $S$, and let $T$ be
\[
\begin{pmatrix}
 b \\
 1 \\
 b
\end{pmatrix},
\]
where $b=0$ or 2. Then we seek solutions of:
\[
\begin{align*}
 u &= x_1^2s_1 + x_2^2s_2 + \cdots + x_m^2s_m = b \pmod{8} \\
 v &= y_1^2s_1 + y_2^2s_2 + \cdots + y_m^2s = b_m \pmod{8} \\
 r &= x_1y_1s_1 + x_2y_2s_2 + \cdots + x_my_ms_m = 1 \pmod{8}.
\end{align*}
\]
Here
\[
8^3A_6(S, T) = \sum_{h, k, l, m} \omega^{h(-b)+k(z-b)+(r-1)}.
\]
Let $\omega^i = \omega_i$ and call
\[
f_i(h, k, l) = \sum x_iy_i \omega^{ixy+kxy^2}.
\]
Then
\[
8^3A_6(S, T) = \sum_{h, k, l} f_1f_2\cdots f_m \omega^{-hb-kb-l}.
\]
We calculate $f_i$, considering the value of $l \pmod{8}$, and note that as before we need consider only $l \equiv 0 \pmod{4}$. We get:
\[
\begin{array}{ccc}
 h & k & l \pmod{8} & f_i \\
 \hline
 \text{odd} & \text{odd} & 0 & c=16\omega_i^{h+k} \\
 & & 4 & -c=-16\omega_i^{h+k} \\
 \text{odd} & \text{even} & 0 & d=16\omega_i^{h+k} + 16\omega_i^h \\
 & & 4 & e=-16\omega_i^{h+k} + 16\omega_i^h \\
 \text{even} & \text{even} & 0 & p=16(\omega_i^{h+k} + \omega_i^h + \omega_i^k + 1) \\
 & & 4 & q=16(-\omega_i^{h+k} + \omega_i^h + \omega_i^k + 1).
\end{array}
\]
Then from (13), we get
\[
8^3A_6(S, T) = 2 \sum_{h, k \text{ odd}} \left( \prod_{i=1}^{m} d_i - \prod_{i=1}^{m} e_i \right) \omega^{-hb-kb} + (1 - (-1)^n) \left( \sum_{h, k \text{ odd}} \left( \prod_{i=1}^{m} c_i \right) \omega^{-hb-kb} \right) \\
+ \sum_{h, k \text{ even}} \left( \prod_{i=1}^{m} p_i - \prod_{i=1}^{m} q_i \right) \omega^{-hb-kb},
\]
where all the sum indices are taken modulo 8. Replacement of $k$ by $k+4$ in the first summand merely changes the sign of the expression, so the first sum is zero. The second sum is easily seen to be $16^{m+1} \alpha(1-(-1)^n)$, where $\alpha=1$ if $\Sigma s_i \equiv b \pmod{4} \text{ and } \alpha=0$ otherwise.
We consider particular contributions to the third sum, using \( \omega_3^h = i^h j^k \) and adjusting so that \( h \) and \( k \) run through a complete residue system modulo 4.

(a) For \( h \equiv 2 \pmod{4} \) and all \( k \pmod{4} \), we have contributed \(-4\alpha(32)^m\).

(b) For \( h \equiv k \equiv 2 \pmod{4} \), we get \(-(-32)^m\).

(c) For \( h \equiv 0 \pmod{4} \) and \( k \equiv 1, 3 \pmod{4} \), we obtain
\[
16^m \cdot 2^{m+1} \cdot 2^{-h(2^{m/2} \cos (\pi B/4) - 1)}, \quad \text{where } B = \sum_{j=1}^{m} (i)^{j-1} .
\]

(d) For \( h \) and \( k \) odd, with \( h \equiv k \pmod{4} \), we get
\[
16^m(-2^{m+1}2^{m/2} \cos (\pi B/4) + 2^{m+1} \cos (\pi B/2)).
\]

(e) For \( h \) and \( k \) odd, with \( h \equiv -k \pmod{4} \), we get \(2(32)^m\).

(f) For \( h \equiv k \equiv 0 \pmod{4} \), we have \(16^m(2^{2m} - 2^m)\).

Thus, here
\[
8^3 A_8(S, T) = 16^m \alpha(1 - (-1)^m) + 32^m(-8\alpha + (-1)^m + 4i^{-b}(2^{m/2} \cos (\pi B/4) - 1))
+ 32^m(2 \cos (\pi B/2) - 2^{1+(m/2)} \cos (\pi B/4) + 2 + 2^m - 1) .
\]

For \( n > 2 \), where \( S \) is even and \( T \) odd, we use the recursion formula with the results for \( n = 2 \). The successive diagonal forms of \( T \) are clear. From
\[
(14) \quad S_i + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv U_iSU_1 \pmod{8} ,
\]
we see firstly that \( S_i \) is even and secondly, that its determinant is determined modulo 8. Again, using (14) and the remarks of § 4, 1.2 b, we see from the following transformations that the number of 3's, modulo 4, in a diagonal form of \( S_i \) is one less than the number of 3's modulo 4, in a diagonal form of \( S \); hence, \( \lambda(S_i) \) is known:
\[
ax^2 + 2yz \rightarrow a(x+y)^2 + 2yz = ax^2 + ay^2 + 2y(ax + z) \rightarrow
ax^2 + ay^2 + 2yz = ax^2 + ay^2 + a(y + az)^2 - ax^2 \rightarrow ax^2 + ay^2 - az^2 ,
\]
where \( a \) is odd, the congruence is taken modulo 8, and \( \rightarrow \) indicates 2-adic equivalence. Thus \( S_i \) is classwise unique and easily determined.

References

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