

# Pacific Journal of Mathematics

**NOTE ON A THEOREM OF HADWIGER**

ROBERT STEINBERG

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Throughout this paper,  $H$  denotes a Hilbert space over the real or complex numbers and  $(x, y)$  denotes the inner product of the vectors  $x, y$  of  $H$ . The only projections we consider are orthogonal ones.

Our starting point is the basic fact that, if  $\{u_\alpha\}$  is an orthonormal basis of  $H$ , then the Parseval relation

$$(1) \quad (x, y) = \sum (x, u_\alpha)(u_\alpha, y)$$

is valid for each pair of vectors  $x, y$  of  $H$ . It is easy to see that (1) is also valid if  $\{u_\alpha\}$  is the projection of an orthonormal basis  $\{w_\alpha\}$  and if we restrict  $x$  and  $y$  to the range of the projection. Indeed, if  $E$  is the projection, so that  $w_\alpha E = u_\alpha$  for each  $\alpha$ , then

$$\begin{aligned} (x, y) &= \sum (x, w_\alpha)(w_\alpha, y) = \sum (xE, w_\alpha)(w_\alpha, yE) = \sum (x, w_\alpha E)(w_\alpha E, y) \\ &= \sum (x, u_\alpha)(u_\alpha, y). \end{aligned}$$

The theorem referred to in the title deals with this result and also with the converse question:

**THEOREM 1.** *If the Parseval relation (1) is valid for each pair of vectors  $x$  and  $y$  of  $H$ , then the set  $\{u_\alpha\}$  is the projection of an orthonormal basis of a superspace  $K$  of  $H$ .*

This result was first proved by Hadwiger [1], and, then, by Julia [2]. We first give a simple proof of Theorem 1 that depends on a simple imbedding procedure, and then consider some related questions concerning projections of orthogonal sets of vectors.

*Proof of Theorem 1.* We choose as  $K$  coordinate Hilbert space [4, p. 120] of dimension equal to the cardinality of the set  $\{u_\alpha\}$ . We see from (1), with  $x = u_\beta, y = u_\gamma$ , that the matrix  $U = ((u_\alpha, u_\beta))$  is idempotent. Since  $U$  is also Hermitian, it may be interpreted as a projection acting on  $K$ . We now imbed  $H$  in  $K$  by making correspond to  $x$  in  $H$  the (row) coordinate vector  $x' = \{(x, u_\alpha)\}$  in  $K$ . In particular, to the vector  $u_\beta$  there corresponds the  $\beta$ th row of  $U$  which is manifestly the image, under the projection  $U$ , of the  $\beta$ th coordinate basis vector. Finally, if  $x' = \{(x, u_\alpha)\}$  and  $y' = \{(y, u_\alpha)\}$ , then  $(x', y') = \sum (x, u_\alpha)(y, u_\alpha) = \sum (x, u_\alpha)(u_\alpha, y) = (x, y)$ ; thus the imbedding is isometric and we are done.

We next prove a related result which is due to Julia [2, (c)].

**THEOREM 2.** *If the Parseval relation (1) is valid relative to the set  $\{u_\alpha\}$  of  $H$ , and, if no  $u_\alpha$  is in the closed subspace spanned by the others, then  $\{u_\alpha\}$  is an orthonormal basis.*

*Proof.* The second assumption implies the existence of a dual set  $\{v_\alpha\}$  in  $H$  such that  $(u_\alpha, v_\beta) = \delta_{\alpha\beta}$  [see 3, p. 264]. Then, using (1), we get  $\delta_{\alpha\beta} = (u_\alpha, v_\beta) = \sum_\gamma (u_\alpha, u_\gamma)(u_\gamma, v_\beta) = \sum_\gamma (u_\alpha, u_\gamma)\delta_{\gamma\beta} = (u_\alpha, u_\beta)$ .

We remark at this point that the methods of proof of Theorems 1 and 2 can be used to give proofs of the corresponding results about projections of biorthonormal bases of vectors  $\{u_\alpha; v_\alpha\}$  for which  $(u_\alpha, v_\beta) = \delta_{\alpha\beta}$ . These methods are also used in our next proof [see 2, (b)].

**THEOREM 3.** *A necessary and sufficient condition that a set of vectors  $\{u_\alpha\}$  of  $H$  be the projection of an orthonormal set (not necessarily a basis) in some superspace  $K$  is that, for each  $x$  in  $H$ ,*

$$(2) \quad \Sigma |(x, u_\alpha)|^2 \leq (x, x) .$$

*Proof.* By the remarks preceding Theorem 1, the necessity is clear. In proving sufficiency, we may suppose  $\{u_\alpha\}$  is complete in  $H$ , since, otherwise, by adding to  $\{u_\alpha\}$  an orthonormal basis of the orthogonal complement of  $\{u_\alpha\}$  in  $H$ , we get a larger set which is complete, and for which the condition (2) is still valid. Next we show that, if  $U$  is the matrix  $((u_\alpha, u_\beta))$ , then  $0 \leq U \leq 1$ , in the sense that both  $U$  and  $1 - U$  are nonnegative [4, p. 213]. Let  $\xi_\alpha$  be any set of scalars of which all but a finite number are zero. Then, using Schwarz' inequality and (2), we get

$$0 \leq (\Sigma_\alpha \xi_\alpha u_\alpha, \Sigma_\beta \xi_\beta u_\beta) = \Sigma_{\alpha,\beta} \xi_\alpha \bar{\xi}_\beta (u_\alpha, u_\beta) \\ \leq (\Sigma_\beta |\xi_\beta|^2)^{1/2} (\Sigma_\alpha |\xi_\alpha u_\alpha, u_\beta|^2)^{1/2} \leq (\Sigma_\beta |\xi_\beta|^2)^{1/2} (\Sigma_\alpha \xi_\alpha u_\alpha, \Sigma_\gamma \xi_\gamma u_\gamma)^{1/2} .$$

Thus  $0 \leq \Sigma_{\alpha,\beta} \xi_\alpha \bar{\xi}_\beta (u_\alpha, u_\beta) \leq \Sigma_\beta |\xi_\beta|^2$ ; so that  $0 \leq U \leq 1$ ,  $U^2$  exists and  $0 \leq U - U^2$  [4, p. 217]. Consider now the matrix  $E = \begin{pmatrix} U & \sqrt{U - U^2} \\ \sqrt{U - U^2} & 1 - U \end{pmatrix}$ . [See

4, pp. 215, 224]. This is Hermitian and idempotent and hence represents a projection in coordinate Hilbert space  $K$  of the appropriate dimension. As in Theorem 1, the (row) vectors given by the upper half of  $E$  not only are the images, under  $E$ , of "half" of the coordinate basis vectors of  $K$ , but also constitute an isometric imbedding of the set  $\{u_\alpha\}$  in  $K$ . Since  $\{u_\alpha\}$  is complete in  $H$ , the imbedding can be extended to all of  $H$ ; and the proof is complete.

At this stage, we introduce the following *definition*: A set of vectors  $\{u_\alpha\}$  in  $H$  has the property  $P$  if each  $x$  in  $H$  is orthogonal to all but a countable number of  $u_\alpha$ .

LEMMA (1) Any orthogonal set has property  $P$ . (2) Property  $P$  is invariant under projection: if  $\{u_\alpha\}$  has property  $P$  and  $E$  is a projection, then so does  $\{u_\alpha E\}$ .

*Proof.* The statement (1) is a classical result [4, p. 114]. To prove (2) we select any  $x$  in  $H$ . Then  $(x, u_\alpha E) = (xE, u_\alpha)$  which is zero for all but countably many  $u_\alpha$ .

This lemma leads us to the following conjecture: A necessary and sufficient condition that  $\{u_\alpha\}$  be the projection of an orthogonal set (not necessarily normal) is that  $\{u_\alpha\}$  has property  $P$ .

The lemma proves necessity. We have been unable to prove sufficiency. However, we can prove the following special case:

THEOREM 4. A necessary and sufficient condition for the set of non-zero vectors  $\{u_\alpha\}$  in a separable Hilbert space  $H$  to be the projection of an orthogonal set is that the set be countable.

*Proof.* Suppose first that  $\{u_\alpha\}$  is the projection of an orthogonal set. Then, by the lemma, it has property  $P$ . Let  $\{x_i\}$  be a (countable) basis for  $H$ . Then all but a countable number of  $u_\alpha$  are orthogonal to each  $x_i$  and hence to their union  $\{x_i\}$ . That is, all but countably many  $u_\alpha$  are 0. This proves the necessity. To prove sufficiency, we suppose that  $\{u_\alpha\}$  is countable and indexed by the positive integers. We then define  $v_\alpha = 2^{-\alpha} u_\alpha / (u_\alpha, u_\alpha)^{1/2}$ , for each  $\alpha$ . Then, if  $x$  is any vector of  $H$ , it follows, by Schwarz' inequality, that  $\sum |(x, v_\alpha)|^2 \leq (x, x) \sum (v_\alpha, v_\alpha) = (x, x) \sum 2^{-2\alpha} \leq (x, x)$ . Thus, by Theorem 3,  $\{v_\alpha\}$  is the projection of an orthogonal set and so is  $\{u_\alpha\}$ .

We close with an example of a set  $\{u_\alpha\}$  which is not the projection of an orthogonal set. Let  $\{x_\alpha\}$  be an uncountable orthonormal set in nonseparable Hilbert space and set  $u_\alpha = x_1 + x_\alpha$ , for each  $\alpha$ . Then  $\{u_\alpha\}$  does not have property  $P$  and hence, by the lemma, is not the projection of an orthogonal set. It is to be noted that Theorem 4 cannot be used to prove this result since every uncountable subset of  $\{u_\alpha\}$  spans a nonseparable subspace of  $H$ .

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