Pacific Journal of Mathematics

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Vol. 6, No. 4 1956

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Throughout this paper, H denotes a Hilbert space over the real or complex numbers and (x, y) denotes the inner product of the vectors x, y of H. The only projections we consider are orthogonal ones.

Our starting point is the basic fact that, if $\{u_\alpha\}$ is an orthonormal basis of H, then the Parseval relation

$$(1) \qquad (x, y)=\Sigma(x, u_\alpha)(u_\alpha, y)$$

is valid for each pair of vectors x, y of H. It is easy to see that (1) is also valid if $\{u_\alpha\}$ is the projection of an orthonormal basis $\{w_\alpha\}$ and if we restrict x and y to the range of the projection. Indeed, if E is the projection, so that $w_\alpha E=u_\alpha$ for each α, then

$$\begin{aligned}(x, y)&=\Sigma(x, w_\alpha)(w_\alpha, y)=\Sigma(xE, w_\alpha)(w_\alpha, yE)=\Sigma(x, w_\alpha E)(w_\alpha E, y)\\&=\Sigma(x, u_\alpha)(u_\alpha, y)\ .\end{aligned}$$

The theorem referred to in the title deals with this result and also with the converse question:

THEOREM 1. *If the Parseval relation* (1) *is valid for each pair of vectors x and y of H, then the set $\{u_\alpha\}$ is the projection of an orthonormal basis of a superspace K of H.*

This result was first proved by Hadwiger [1], and, then, by Julia [2]. We first give a simple proof of Theorem 1 that depends on a simple imbedding procedure, and then consider some related questions concerning projections of orthogonal sets of vectors.

Proof of Theorem 1. We choose as K coordinate Hilbert space [4, p. 120] of dimension equal to the cardinality of the set $\{u_\alpha\}$. We see from (1), with $x=u_\beta$, $y=u_\gamma$, that the matrix $U=((u_\alpha, u_\beta))$ is idempotent. Since U is also Hermitian, it may be interpreted as a projection acting on K. We now imbed H in K by making correspond to x in H the (row) coordinate vector $x'=\{(x, u_\alpha)\}$ in K. In particular, to the vector u_β there corresponds the βth row of U which is manifestly the image, under the projection U, of the βth coordinate basis vector. Finally, if $x'=\{(x, u_\alpha)\}$ and $y'=\{(y, u_\alpha)\}$, then $(x', y')=\Sigma(x, u_\alpha)\overline{(y, u_\alpha)}=\Sigma(x, u_\alpha)(u_\alpha, y)$ $=(x, y)$; thus the imbedding is isometric and we are done.

We next prove a related result which is due to Julia [2, (c)].

Received November 23, 1955.

THEOREM 2. *If the Parseval relation* (1) *is valid relative to the set* $\{u_\alpha\}$ *of* H, *and, if no* u_α *is in the closed subspace spanned by the others, then* $\{u_\alpha\}$ *is an orthonormal basis.*

Proof. The second assumption implies the existence of a dual set $\{v_\alpha\}$ in H such that $(u_\alpha, v_\beta)=\delta_{\alpha\beta}$ [see 3, p. 264]. Then, using (1), we get $\delta_{\alpha\beta}=(u_\alpha, v_\beta)=\Sigma_\gamma(u_\alpha, u_\gamma)(u_\gamma, v_\beta)=\Sigma_\gamma(u_\alpha, u_\gamma)\delta_{\gamma\beta}=(u_\alpha, u_\beta)$.

We remark at this point that the methods of proof of Theorems 1 and 2 can be used to give proofs of the corresponding results about projections of biorthonormal bases of vectors $\{u_\alpha; v_\alpha\}$ for which $(u_\alpha, v_\beta)=\delta_{\alpha\beta}$. These methods are also used in our next proof [see 2, (b)].

THEOREM 3. *A necessary and sufficient condition that a set of vectors* $\{u_\alpha\}$ *of* H *be the projection of an orthonormal set* (*not necessarily a basis*) *in some superspace* K *is that, for each* x *in* H,

$$\Sigma|(x, u_\alpha)|^2\leqq(x, x) . \tag{2}$$

Proof. By the remarks preceding Theorem 1, the necessity is clear. In proving sufficiency, we may suppose $\{u_\alpha\}$ is complete in H, since, otherwise, by adding to $\{u_\alpha\}$ an orthonormal basis of the orthogonal complement of $\{u_\alpha\}$ in H, we get a larger set which is complete, and for which the condition (2) is still valid. Next we show that, if U is the matrix $((u_\alpha, u_\beta))$, then $0\leqq U\leqq 1$, in the sense that both U and $1-U$ are nonnegative [4, p. 213]. Let ξ_α be any set of scalars of which all but a finite number are zero. Then, using Schwarz' inequality and (2), we get

$$\begin{aligned}0&\leqq(\Sigma_\alpha\xi_\alpha u_\alpha, \Sigma_\beta\xi_\beta u_\beta)=\Sigma_{\alpha,\beta}\xi_\alpha\bar{\xi}_\beta(u_\alpha, u_\beta)\\ &\leqq(\Sigma_\beta|\xi_\beta|^2)^{1/2}(\Sigma_\beta|(\Sigma_\alpha\xi_\alpha u_\alpha, u_\beta)|^2)^{1/2}\leqq(\Sigma_\beta|\xi_\beta|^2)^{1/2}(\Sigma_\alpha\xi_\alpha u_\alpha, \Sigma_\gamma\xi_\gamma u_\gamma)^{1/2} .\end{aligned}$$

Thus $0\leqq\Sigma_{\alpha,\beta}\xi_\alpha\bar{\xi}_\beta(u_\alpha, u_\beta)\leqq\Sigma_\beta|\xi_\beta|^2$; so that $0\leqq U\leqq 1$, U^2 exists and $0\leqq U-U^2$ [4, p. 217]. Consider now the matrix $E=\begin{pmatrix}U & \sqrt{U-U^2}\\ \sqrt{U-U^2} & 1-U\end{pmatrix}$. [See 4, pp. 215, 224]. This is Hermitian and idempotent and hence represents a projection in coordinate Hilbert space K of the appropriate dimension. As in Theorem 1, the (row) vectors given by the upper half of E not only are the images, under E, of "half" of the coordinate basis vectors of K, but also constitute an isometric imbedding of the set $\{u_\alpha\}$ in K. Since $\{u_\alpha\}$ is complete in H, the imbedding can be extended to all of H; and the proof is complete.

At this stage, we introduce the following *definition*: *A set of vectors* $\{u_\alpha\}$ *in* H *has the property* P *if each* x *in* H *is orthogonal to all but a countable number of* u_α.

LEMMA (1) *Any orthogonal set has property P.* (2) *Property P is invariant under projection: if* $\{u_\alpha\}$ *has property P and E is a projection, then so does* $\{u_\alpha E\}$.

Proof. The statement (1) is a classical result [4, p. 114]. To prove (2) we select any x in H. Then $(x, u_\alpha E)=(xE, u_\alpha)$ which is zero for all but countably many u_α.

This lemma leads us to the following *conjecture*: *A necessary and sufficient condition that* $\{u_\alpha\}$ *be the projection of an orthogonal set (not necessarily normal) is that* $\{u_\alpha\}$ *has property P.*

The lemma proves necessity. We have been unable to prove sufficiency. However, we can prove the following special case:

THEOREM 4. *A necessary and sufficient condition for the set of nonzero vectors* $\{u_\alpha\}$ *in a separable Hilbert space H to be the projection of an orthogonal set is that the set be countable.*

Proof. Suppose first that $\{u_\alpha\}$ is the projection of an orthogonal set. Then, by the lemma, it has property P. Let $\{x_i\}$ be a (countable) basis for H. Then all but a countable number of u_α are orthogonal to each x_i and hence to their union $\{x_i\}$. That is, all but countably many u_α are 0. This proves the necessity. To prove sufficiency, we suppose that $\{u_\alpha\}$ is countable and indexed by the positive integers. We then define $v_\alpha=2^{-\alpha}u_\alpha/(u_\alpha, u_\alpha)^{1/2}$, for each α. Then, if x is any vector of H, it follows, by Schwarz' inequality, that $\Sigma|(x, v_\alpha)|^2\leqq(x, x)\Sigma(v_\alpha, v_\alpha)=(x, x)\Sigma 2^{-2\alpha}\leqq(x, x)$. Thus, by Theorem 3, $\{v_\alpha\}$ is the projection of an orthogonal set and so is $\{u_\alpha\}$.

We close with an example of a set $\{u_\alpha\}$ which is not the projection of an orthogonal set. Let $\{x_\alpha\}$ be an uncountable orthonormal set in nonseparable Hilbert space and set $u_\alpha=x_1+x_\alpha$, for each α. Then $\{u_\alpha\}$ does not have property P and hence, by the lemma, is not the projection of an orthogonal set. It is to be noted that Theorem 4 cannot be used to prove this result since every uncountable subset of $\{u_\alpha\}$ spans a nonseparable subspace of H.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Pacific Journal of Mathematics

Vol. 6, No. 4 , 1956