NOTE ON A THEOREM OF HADWIGER

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Throughout this paper, $H$ denotes a Hilbert space over the real or complex numbers and $(x, y)$ denotes the inner product of the vectors $x$, $y$ of $H$. The only projections we consider are orthogonal ones.

Our starting point is the basic fact that, if $\{u_a\}$ is an orthonormal basis of $H$, then the Parseval relation

\[(x, y) = \sum (x, u_a)(u_a, y) \quad (1)\]

is valid for each pair of vectors $x$, $y$ of $H$. It is easy to see that (1) is also valid if $\{u_a\}$ is the projection of an orthonormal basis $\{w_a\}$ and if we restrict $x$ and $y$ to the range of the projection. Indeed, if $E$ is the projection, so that $w_aE = u_a$ for each $a$, then

\[(x, y) = \sum (x, w_a)(w_a, y) = \sum (xE, w_a)(w_a, yE) = \sum (x, w_aE)(w_aE, y) = \sum (x, u_a)(u_a, y),\]

The theorem referred to in the title deals with this result and also with the converse question:

**Theorem 1.** If the Parseval relation (1) is valid for each pair of vectors $x$ and $y$ of $H$, then the set $\{u_a\}$ is the projection of an orthonormal basis of a superspace $K$ of $H$.

This result was first proved by Hadwiger [1], and, then, by Julia [2]. We first give a simple proof of Theorem 1 that depends on a simple imbedding procedure, and then consider some related questions concerning projections of orthogonal sets of vectors.

**Proof of Theorem 1.** We choose as $K$ coordinate Hilbert space [4, p. 120] of dimension equal to the cardinality of the set $\{u_a\}$. We see from (1), with $x = u_\beta$, $y = u_\gamma$, that the matrix $U = ((u_a, u_\beta))$ is idempotent. Since $U$ is also Hermitian, it may be interpreted as a projection acting on $K$. We now imbed $H$ in $K$ by making correspond to $x$ in $H$ the (row) coordinate vector $x' = \{(x, u_a)\}$ in $K$. In particular, to the vector $u_\beta$ there corresponds the $\beta$th row of $U$ which is manifestly the image, under the projection $U$, of the $\beta$th coordinate basis vector. Finally, if $x' = \{(x, u_a)\}$ and $y' = \{(y, u_a)\}$, then $(x', y') = \sum (x, u_a)(y, u_a) = \sum (x, u_a)(u_a, y) = (x, y)$; thus the imbedding is isometric and we are done.

We next prove a related result which is due to Julia [2, (c)].

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**THEOREM 2.** If the Parseval relation (1) is valid relative to the set \( \{u_a\} \) of \( H \), and, if no \( u_a \) is in the closed subspace spanned by the others, then \( \{u_a\} \) is an orthonormal basis.

**Proof.** The second assumption implies the existence of a dual set \( \{v_\alpha\} \) in \( H \) such that \((u_a, v_\beta) = \delta_{a\beta} \) [see 3, p. 264]. Then, using (1), we get

\[
\delta_{a\beta} = (u_a, v_\beta) = \Sigma_j (u_a, u_j)(u_j, v_\beta) = \Sigma_j (u_a, u_j) \delta_{j\beta} = (u_a, u_\beta).
\]

We remark at this point that the methods of proof of Theorems 1 and 2 can be used to give proofs of the corresponding results about projections of biorthonormal bases of vectors \( \{u_a; v_\alpha\} \) for which \((u_a, v_\beta) = \delta_{a\beta} \). These methods are also used in our next proof [see 2, (b)].

**THEOREM 3.** A necessary and sufficient condition that a set of vectors \( \{u_a\} \) of \( H \) be the projection of an orthonormal set (not necessarily a basis) in some superspace \( K \) is that, for each \( x \in H \),

\[
\Sigma_j |(x, u_a)|^2 \leq (x, x).
\]

**Proof.** By the remarks preceding Theorem 1, the necessity is clear. In proving sufficiency, we may suppose \( \{u_a\} \) is complete in \( H \), since, otherwise, by adding to \( \{u_a\} \) an orthonormal basis of the orthogonal complement of \( \{u_a\} \) in \( H \), we get a larger set which is complete, and for which the condition (2) is still valid. Next we show that, if \( U \) is the matrix \(((u_a, u_\beta))\), then \( 0 \leq U \leq 1 \), in the sense that both \( U \) and \( I - U \) are nonnegative [4, p. 213]. Let \( \xi_\alpha \) be any set of scalars of which all but a finite number are zero. Then, using Schwarz' inequality and (2), we get

\[
0 \leq (\Sigma_\alpha \xi_\alpha u_a, \Sigma_\beta \xi_\beta u_\beta) = \Sigma_{a,\beta} \xi_a \xi_\beta (u_a, u_\beta) \leq \Sigma_\beta |\xi_\beta|^2 (\Sigma_\alpha |\xi_\alpha|^2)^{1/2} \leq \Sigma_\beta |\xi_\beta|^2 (\Sigma_\alpha |\xi_\alpha|^2)^{1/2} (
\Sigma_a \xi_a u_a, \Sigma_\gamma \xi_\gamma u_\gamma)^{1/2}.
\]

Thus \( 0 \leq \Sigma_{a,\beta} \xi_a \xi_\beta (u_a, u_\beta) \leq \Sigma_\beta |\xi_\beta|^2 \); so that \( 0 \leq U \leq 1 \), \( U^2 \) exists and \( 0 \leq U - U^2 \) [4, p. 217]. Consider now the matrix \( E = \begin{pmatrix} U & \sqrt{U-U^2} \\ \sqrt{U-U^2} & 1-U \end{pmatrix} \). [See 4, pp. 215, 224]. This is Hermitian and idempotent and hence represents a projection in coordinate Hilbert space \( K \) of the appropriate dimension. As in Theorem 1, the (row) vectors given by the upper half of \( E \) not only are the images, under \( E \), of “half” of the coordinate basis vectors of \( K \), but also constitute an isometric imbedding of the set \( \{u_a\} \) in \( K \). Since \( \{u_a\} \) is complete in \( H \), the imbedding can be extended to all of \( H \); and the proof is complete.

At this stage, we introduce the following definition: A set of vectors \( \{u_a\} \) in \( H \) has the property \( P \) if each \( x \in H \) is orthogonal to all but a countable number of \( u_a \).
**Theorem of Hadwiger**

**Lemma** (1) Any orthogonal set has property P. (2) Property P is invariant under projection: if \( \{u_a\} \) has property P and \( E \) is a projection, then so does \( \{u_aE\} \).

**Proof.** The statement (1) is a classical result [4, p. 114]. To prove (2) we select any \( x \) in \( H \). Then \( (x, u_aE) = (xE, u_a) \) which is zero for all but countably many \( u_a \).

This lemma leads us to the following conjecture: A necessary and sufficient condition that \( \{u_a\} \) be the projection of an orthogonal set (not necessarily normal) is that \( \{u_a\} \) has property P.

The lemma proves necessity. We have been unable to prove sufficiency. However, we can prove the following special case:

**Theorem 4.** A necessary and sufficient condition for the set of non-zero vectors \( \{u_a\} \) in a separable Hilbert space \( H \) to be the projection of an orthogonal set is that the set be countable.

**Proof.** Suppose first that \( \{u_a\} \) is the projection of an orthogonal set. Then, by the lemma, it has property P. Let \( \{x_i\} \) be a (countable) basis for \( H \). Then all but a countable number of \( u_a \) are orthogonal to each \( x_i \) and hence to their union \( \{x_i\} \). That is, all but countably many \( u_a \) are 0. This proves the necessity. To prove sufficiency, we suppose that \( \{u_a\} \) is countable and indexed by the positive integers. We then define \( v_a = 2^{-a}u_a|u_a, u_a|^{1/2} \), for each \( \alpha \). Then, if \( x \) is any vector of \( H \), it follows, by Schwarz' inequality, that \( \Sigma[(x, v_a)]^2 \leq (x, x)\Sigma(v_a, v_a) = (x, x)\Sigma 2^{-2a} \leq (x, x) \). Thus, by Theorem 3, \( \{v_a\} \) is the projection of an orthogonal set and so is \( \{u_a\} \).

We close with an example of a set \( \{u_a\} \) which is not the projection of an orthogonal set. Let \( \{x_a\} \) be an uncountable orthonormal set in nonseparable Hilbert space and set \( u_a = x_1 + x_a \), for each \( \alpha \). Then \( \{u_a\} \) does not have property P and hence, by the lemma, is not the projection of an orthogonal set. It is to be noted that Theorem 4 cannot be used to prove this result since every uncountable subset of \( \{u_a\} \) spans a nonseparable subspace of \( H \).

**References**


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