UNIQUENESS THEORY FOR ASYMPTOTIC EXPANSIONS IN GENERAL REGIONS

PHILIP DAVID
1. Introduction. Let $D$ be a simply connected region with an analytic boundary $C$. Assume that $z=0$ is an interior point while $z=1$ lies on the boundary. We assume further that the tangent to $C$ at $z=1$ is not parallel to the real axis. In this case, we shall be able to fit into $D$ small angles $\Gamma$ placed symmetrically about the real axis and with vertex at $z=1$. These angles will be of the form $-\delta \leq \theta \leq \delta$ or $\pi-\delta \leq \theta \leq \pi+\delta$, $\delta > 0$, depending upon the location of $z=1$. For a given $f(z)$ regular in $D$, we consider the following limits defined recursively

$$a_0 = \lim_{z \to 1} f(z)$$

$$a_1 = \lim_{z \to 1} (z-1)^{-1}[f(z)-a_0]$$

$$a_2 = \lim_{z \to 1} (z-1)^{-2}[f(z)-a_0-a_1(z-1)]$$

and so on.

If each limit in (1) exists and is independent of the manner in which $z \to 1$ through values in some angle $\Gamma$, then $f(z)$ is said to possess an asymptotic expansion at $z=1$ in the sense of Poincaré, and this is indicated by writing

$$f(z) \sim \sum_{n=0}^{\infty} a_n(z-1)^n.$$

We shall designate by $A(=A(D))$ the linear class of functions which are regular in $D$ and which possess asymptotic expansions at $z=1$ in the sense of Poincaré. The angle $\Gamma$ in which (1) is valid may depend upon the particular $f \in A$ selected.

Uniqueness theory is concerned with distinguishing nontrivial subclasses of $A$ within which the expansion $\sum_{n=0}^{\infty} a_n(z-1)^n$ determines the corresponding function uniquely. Write for the remainder

$$R_n(z) = f(z) - a_0 - a_1(z-1) - \cdots - a_{n-1}(z-1)^{n-1},$$

and consider the ratios

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For \( f \in A \), the functions \( f_n(z) \) are regular in \( D \) and are bounded as \( z \to 1 \) in \( I' \). For a given sequence of positive quantities \( \{m_n\} \), we consider the subset \( A(m_n) \) of \( A \) consisting of those functions which satisfy in addition

\[
\|f_n\|^2 < Mk^nm_n^2 \quad (n=0, 1, 2, \ldots)
\]

for some \( M > 0, k > 0 \). Here \( \| \cdot \| \) designates some conveniently chosen norm. The constants \( M \) and \( k \) may vary from function to function within the class. With the selection

\[
\|f\| = \max_{z \in D} |f(z)|,
\]

it has been shown by Watson [1] and F. Nevanlinna [5] that when \( D \) is a sector, we may produce uniqueness classes by restricting the growth of the sequence \( \{m_n\} \) sufficiently. When \( D \) is the unit circle, T. Carleman [2] has given necessary and sufficient conditions on \( \{m_n\} \) in order that the resulting subclass \( A(m_n) \) be a uniqueness class. At the same time Carleman raises the problem of giving necessary and sufficient conditions in the case of a more general region \( D \). This problem (with the norm (6)) has been known in the literature at the generalized problem of Watson. It has been treated by Mandelbrojt and MacLane [3] using the theory of distortion in conformal mapping. See also Meili [4]. In the present paper, we adopt the norm

\[
\|f\|^2 = \int_0^1 |f(z)|^2 \, ds,
\]

and show how it is possible to combine Carleman's idea of introducing an appropriate minimum problem with the techniques afforded by the theory of conformal kernel functions to arrive at a solution to this general problem. The class \( A(m_n) \) will henceforth refer to the norm (7). Thus the question which we are treating may beworded as follows: What are necessary and sufficient conditions on the sequence of constants \( \{m_n\} \) in order that

\[
\|f_n\|^2 = \int_0^1 |f_n(z)|^2 \, ds = \int_0^1 \left| \frac{f(z) - a_0 - a_1(z-1) - \cdots - a_{n-1}(z-1)^{n-1}}{(z-1)^n} \right|^2 \, ds < Mk^nm_n^2
\]

determine \( f(z) \) uniquely from the asymptotic coefficients \( a_n \).

2. Preliminary observations. We must first explain the sense in
which we shall understand the expression

\[ \int_0^\infty |f(z)|^2 \, ds \]

when \( f(z) \) is regular in \( D \) but not necessarily in its closure. Let \( w = m(z) \) map \( D \) conformally onto the unit circle with \( m(0) = 0 \) and \( m(1) = 1 \). The images of \( |w| = r \) will be designated by \( C_r, 0 < r < 1 \). It is well known that the set of functions

\[
\phi_n(z) = \frac{1}{\sqrt{2\pi}} \frac{[m'(z)]^{1/2}}{r^{n+1/2}} [m(z)]^n \quad (n = 0, 1, 2, \ldots)
\]

is complete and orthonormal over each \( C_r, 0 < r < 1 \), relative to the inner product

\[ (f, g)=\int_{C_r} f \bar{g} \, ds. \]

Suppose then that we are given a function \( f(z) \) which is regular in \( D \). Then for any fixed \( 0 < r < 1 \), \( f(z) \) is continuous on \( C_r \). Hence we can write

\[
f(z) = \sum_{n=0}^\infty a_n \phi_n(z)
\]

holding uniformly and absolutely in the interior of \( C_r \). The coefficients \( a_n \) are given by

\[
a_n = \int_{C_r} f(z) \bar{\phi}_n(z) \, ds \quad (n = 0, 1, \ldots).
\]

Hence, for \( r^* < r \), we have from (9) and (10),

\[
\int_{C_r^*} |f(z)|^2 \, ds = \sum_{n=0}^\infty |a_n|^2 \frac{r^{*2n+1}}{r^{2n+1}}.
\]

This equation tells us that

\[
\int_{C_r^*} |f(z)|^2 \, ds
\]

is an increasing function of \( r^* \) and hence

\[
\lim_{r^* \to 1^-} \int_{C_{r^*}} |f(z)|^2 \, ds
\]

exists (or equals \( +\infty \)). For \( f(z) \) regular in \( D \) we shall therefore agree that
LEMMA. Given an arbitrary sequence of positive constants \( \{m_n\} \); the class \( A(m_n) \) is not a uniqueness class for asymptotic expansions at \( z=1 \) if and only if there exists an \( f \not\equiv 0 \) regular in \( D \) and constants \( M>0, k>0, \) for which

\[
\left\| \frac{f(z)}{(z-1)^n} \right\| < M k^n m_n^2 \quad (n=0, 1, 2, \cdots).
\]

Proof. If \( A(m_n) \) is not a uniqueness class, there will exist two functions \( g(z), y(z) \in A(m_n), g \not\equiv h, \) possessing the same asymptotic expansion, say \( \sum_{n=0}^{\infty} a_n(z-1)^n, \) and satisfying

\[
\int_{c} \left| \frac{g(z) - \sum_{k=0}^{n-1} a_k(z-1)^k}{(z-1)^n} \right| ds < M_1 k_1^n m_n^2 \quad (n=0, 1, \cdots)
\]

(14)

\[
\int_{c} \left| \frac{h(z) - \sum_{k=0}^{n-1} a_k(z-1)^k}{(z-1)^n} \right| ds < M_2 k_2^n m_n^2
\]

with \( k_1 \leq k_2. \) Therefore, by Minkowski’s inequality,

\[
\int_{c} \left| \frac{g(z)-h(z)}{(z-1)^n} \right| ds < (M_1^{1/2} k_1^{n/2} + M_2^{1/2} k_2^{n/2}) m_n^2
\]

\[
= (M_1^{1/2} (k_1/k_2)^{n/2} + M_2^{1/2}) k_2^n m_n^2
\]

\[
< (M_1^{1/2} + M_2^{1/2}) k_2^n m_n^2
\]

so that \( g - h \) does not vanish identically and satisfies (13) with \( M = (M_1^{1/2} + M_2^{1/2}) \) and \( k = k_2. \)

Conversely, let \( f \not\equiv 0 \) satisfy (13). We shall show that (13) implies

\[
\lim_{z \to 1} \frac{f(z)}{(z-1)^n} = 0 \quad (n=0, 1, 2, \cdots)
\]

(16)

as \( z \to 1 \) through values in some angle \( \Gamma. \) Assuming, for the moment, that this is so, (16) and (1) imply that

\[
f(z) \sim 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \cdots.
\]

(17)

That is, \( f(z) \) possesses an identically zero asymptotic expansion at \( z=1. \) Furthermore \( f_n = f(z)(z-1)^{-n}, \) so that (13) implies that \( f \in A(m_n). \) Thus, \( A(m_n) \) is not a uniqueness class for asymptotic expansions at \( z=1. \)
We show now that (13) implies (16). Given any $g(z)$ regular in $D$. Select any $0 < r < 1$. We have from (9), (10), (11), and the Schwarz inequality

$$|g(z)|^2 < K_{\sigma_r}(z, \bar{z}) \int_{\sigma_r} |g(z)|^2 \, ds,$$

for all $z$ interior to $C_r$. $K_{\sigma_r}$ is the so-called Szegö kernel function for $C_r$ whose explicit expression is (Szegö [6], Bergman [1])

$$K_{\sigma_r}(z, \bar{z}) = \sum_{n=0}^{\infty} \phi_n(z)\bar{\phi}_n(z) = \frac{1}{2\pi} \frac{r|m'(z)|}{r^2 - |m(z)|^2}.$$

Writing $f(z)/(z-1)^n$ in place of $g(z)$ in (18), and using (13) and the monotonicity with $r$ of

$$\int_{\sigma_r} |f(z)|^2 \, ds,$$

we find for $j \leq n$ and $z$ interior to $C_r$,

$$\left| \frac{f(z)}{(z-1)^j} \right| \leq \frac{|(z-1)^{n-j}|^2 r|m'(z)|}{(2\pi)(r^2 - |m(z)|^2)} Mk^n m_n^2 \quad (n=0, 1, 2, \cdots).$$

For each $z$ in $D$ we select an $r = r(z) = |m(z)| + \varepsilon(z) < 1$ where $\varepsilon(z)$ is defined by

$$\varepsilon(z) = \frac{1}{2} \left( 1 - |m(z)| \right).$$

Thus,

$$\lim_{z \to 1} \varepsilon(z) = 0.$$

Here, $z \to 1$ through values in $D$. From (20), (21), and $r < 1$,

$$\left| \frac{f(z)}{(z-1)^j} \right| \leq \frac{|(z-1)^{n-j}|^2 |m'(z)|}{2\pi} \frac{Mk^n m_n^2}{2|m(z)|\varepsilon(z) + \varepsilon(z)}$$

$$\leq \frac{|(z-1)^{n-j}|^2 |m'(z)|}{4\pi |m(z)|\varepsilon(z)} Mk^n m_n^2.$$

We are now ready to consider the limit of (23) as $z \to 1$. First consider

$$\frac{\varepsilon(z)}{|z-1|} = \frac{1 - |m(z)|}{2|z-1|} = \frac{1}{2} \left( 1 + |m(z)| \right)^{-1} \frac{1 - |m(z)|^2}{|z-1|}.$$

Since $m(z)$ is by assumption analytic at $z=1$, we have in a neighborhood of $z=1$,
(25) \[ m(z) = 1 + (z-1)R(z), \]
where \( R(z) \) is analytic there. Note that \( R(1) = m'(1) \neq 0 \), and write \( R(z) = a(z)e^{i\alpha(z)} \), \( a(z) > 0 \). We have \( a(1) \neq 0 \) and \( \alpha(1) \neq \pi/2, 3\pi/2 \), inasmuch as the tangent to \( C \) at \( z=1 \) is assumed not parallel to the real axis. Furthermore, write \( z = 1 + pe^{i\theta} \). Then, from (25),

(26) \[
\frac{1 - |m(z)|^2}{|z-1|} = -2 \Re \left\{ \frac{(z-1)R(z)}{|z-1|} \right\} - |z-1|^2 |R(z)|^2
\]
\[ = -2 \Re \{ e^{i\theta}a(z)e^{i\alpha(z)} \} - |z-1|^2 |R(z)|^2
\]
\[ = -2a(z)\cos(\theta + \alpha(z)) - |z-1|^2 |R(z)|^2 .
\]

If \( z \to 1 \) through some angle \( \Gamma' : -\delta \leq \theta \leq \delta \) or \( \pi - \delta \leq \theta \leq \pi + \delta \), then, since \( a(1) \neq \pi/2, 3\pi/2 \), it follows from the above that for \( \delta \) sufficiently small, the expression (26) will be bounded away from 0. In view of (24) we will have

(27) \[
\frac{\epsilon(z)}{|z-1|} \geq \tau > 0 ; \quad z \to 1
\]
for \( z \) in some \( \Gamma' \). From (23), we have,

(28) \[
|f(z)|^p < |z-1|^{2n-2j-1} |m'(z)|MK^a m_n^2 / |z-1|^{4p|m(z)| \epsilon(z)} .
\]

Thus, for \( 2n-2j-1 > 1 \) it is now clear from (28) and (27) that

\[
\lim_{z \to 1} \frac{f(z)}{(z-1)^j} = 0 .
\]

For each \( j \) considered we need only use an \( n > j + 1 \). This completes the proof of the lemma.

3. The uniqueness theorem.

**Theorem.** Given an arbitrary sequence of positive constants \( m_n \). The class \( A(m_n) \) is a uniqueness class for asymptotic expansions at \( z=1 \) if and only if for all \( t > 0 \),

(20) \[
\limsup_{n \to \infty} \int_C \log \left\{ \sum_{k=0}^n \frac{t^k}{m_k} |(z-1)^{n-k}|^p \right\} \frac{d \log |m(z)|}{dn} ds = \infty .
\]

Here \( \partial/\partial n \) designates normal differentiation in the positive sense.

The above statement is equivalent to saying that \( A(m_n) \) is not a uniqueness class if and only if there exists a \( t > 0 \) and a \( K > 0 \) such
that

\[
\int_0 \log \left\{ \sum_{k=0}^n \frac{t^k}{m_k^2} |(z-1)^n| \right\} \frac{\partial}{\partial n} \log |m(z)| ds < K, \quad n=0, 1, 2, \ldots
\]

\(K\) may depend upon \(t\), but is independent of \(n\).

In view of the lemma of the preceding section, we shall prove that (30) is a necessary and sufficient condition for the existence of an \(f(z)\) \(\neq 0\), and \(M\), and a \(k\) which satisfy (13).

Consider the following sequence of integrals

\[
I_n(f) = \sum_{k=0}^n \frac{t^k}{m_k^2} \int |f(z)|^k (z-1)^k ds;
\]

where we have written

\[
= \sum_{k=0}^n \frac{t^k}{m_k^2} \|f\|_k^k, \quad n=0, 1, \ldots,
\]

where we have written

\[
\|f\|_k = \int_0 |f(z)|^k (z-1)^k ds; \quad k=0, 1, \ldots.
\]

We can also write (31) in the form

\[
I_n(f) = \left\| \rho_n(z) \frac{f(z)}{(z-1)^n} \right\|^2
\]

where \(\rho_n(z)\) is an analytic function which is regular in \(D\), continuous on \(C\) and is such that

\[
|\rho_n(z)| = \left\{ \sum_{k=0}^n \frac{t^k}{m_k^2} |(z-1)^n|^2 \right\}^{1/2}, \quad \text{for } z \text{ on } C.
\]

We shall show below how a \(\rho_n(z)\) may be constructed which has these properties and has, in addition, the property that

\[
\rho_n(z) \neq 0 \quad \text{for } z \text{ in } D.
\]

Let \(n\) be fixed, and consider the following minimum problem \(P_n\). Determine that function \(f(z)\) regular in \(D\) with \(f(0)=1\) and such that

\[
I_n(f) = \text{minimum}.
\]

This problem can be solved by passing to a related problem \(P_n'\). Determine that function \(g(z)\) regular in \(D\) with \(g(0)=1\) and such that

\[
\|g\|^2 = \text{minimum}
\]

The solution of the problem \(P_n'\) is given by the function (see, for ex-
ample Szegö [6], Bergman [1])

\[ g^*(z) = K_\beta(z, 0)/K_\beta(0, 0) \]

where \( K_\beta(z, \bar{w}) \) is the Szegö kernel function of the region \( D \). The minimum value of the integral (37) is \( 1/K_\beta(0, 0) \). If we write

\[ I_n(f) = |\rho_n(0)|^2 \left\| \frac{\rho_n(z)f(z)}{\rho_n(0)(1-z)^n} \right\|^2, \]

we see, in view of (35) that the function \( \rho_n(z)f(z)/\rho_n(0)(1-z)^n \) can play the role of \( g(z) \) in the problem \( P_n \). The minimizing function \( f_n^* \) of the problem \( P_n \) is therefore

\[ f_n^*(z) = \frac{K_\beta(z, 0)(1-z)^n\rho_n(0)}{\rho_n(z)K_\beta(0, 0)}, \]

and the minimum value of the integral is

\[ I_n(f_n^*) = \frac{|\rho_n(0)|^2}{K_\beta(0, 0)}. \]

We now assert: a necessary and sufficient condition in order that there exist an \( f(z) \neq 0 \) and constants \( M > 0, k > 0 \) such that

\[ \| f \|^2_n = \left\| \frac{f(z)}{(z-1)^n} \right\|^2 < Mk^n m_n^2 \quad (n = 0, 1, \ldots) \]

is that there exists a \( t > 0 \) and a \( K > 0 \) such that

\[ I_n(f_n^*) \leq K \quad n = 0, 1, 2, \ldots. \]

Referring to (41), this is equivalent to asserting that there exist a \( t > 0 \) and a \( K' \) such that

\[ |\rho_n(0)| \leq K' \quad n = 0, 1, 2, \ldots. \]

We can prove this as follows. Suppose first that \( q(z) \) is such that (42) holds for it. This function \( q(z) \) may have a zero of the \( p \)th order at \( z = 0 \). The function \( f(z) = q(z)/z^p \) is then regular in \( D \) and is such that \( f(0) \neq 0 \). Now,

\[ I_n(f(z)/f(0)) = \sum_{j=0}^{n} \frac{t^j}{m_j^3} \int_{0}^{1} \left\| \frac{q(z)}{f(0)z^p(z-1)^j} \right\|^2 ds \]

\[ \leq \sum_{j=0}^{n} \frac{t^j}{m_j^3} \frac{1}{|f(0)|^2} \frac{1}{d^p} M \cdot m_j^2 k^j \]

\[ \leq \frac{M}{d^p|f(0)|^2} \sum_{j=0}^{n} t^j k^j \leq \frac{M}{d^p|f(0)|^2(1-tk)}, \]
provided we select $0 < t < 1/k$. Here $d$ designates the minimum distance from $z=0$ to $C$. Now since

$$ I_n(f^*_n) \leq I_n(f(z)/f(0)) \leq \frac{M}{M} d^n |f(0)|^p (1-tk), \quad (n=0, 1, \cdots) $$

then (43) is satisfied with $K$ equal to the right hand constant in (46).

Conversely, suppose that there exists a $t > 0$ and $K > 0$ such that (43) holds. Then from (31),

$$ \sum_{k=0}^{n} \frac{t^k}{m_k^2} \| f^*_n \| \leq K \quad (n=0, 1, 2, \cdots). $$

In particular, taking the first term of (47) we obtain

$$ \frac{1}{m_0^2} \| f^*_n \| < K \quad n=0, 1, 2, \cdots. $$

Hence we have

$$ \| f^*_n \| < \text{const.} \quad (n=0, 1, 2, \cdots). $$

The inequalities (49) imply that the sequence of minimizing functions $\{f^*_n\}$ form a normal family and therefore there exist indices $n_i, n_2, \cdots$ such that $f^*_{n_k} \to F(z)$ uniformly in any closed region interior to $D$.

Again, using (47) we have, for fixed $j$ and for all $n \geq j$

$$ \frac{t^j}{m_j^2} \| f^*_n \| \leq K. $$

Now for any $0 < \rho < 1$, we have

$$ \| f^*_n \|^p = \int_c \left| \frac{f^*_n(z)}{(z-1)^j} \right|^p ds \geq \int_{c_\rho} \left| \frac{f^*_n(z)}{(z-1)^j} \right|^p ds, $$

so that from (50) and (51),

$$ \int_{c_\rho} \left| \frac{f^*_n(z)}{(z-1)^j} \right|^p ds < Km^j t^{-j} \quad (k=0, 1, 2, \cdots). $$

Let $n$ take on the values $n_i$ in (52) and let $j$ be fixed. Then since $f^*_{n_k}(z) \to F(z)$ uniformly in and on $C_\rho$,

$$ \int_{c_\rho} \left| \frac{F(z)}{(z-1)^j} \right|^p ds \leq Km^j t^{-j}. $$

This result is independent of $\rho$ and hence we may allow $\rho \to 1$. Thus,
Since obviously $F(0) = 1$, we have exhibited in $F(z)$ a function regular in $D$, which does not vanish identically, a constant $M (= K)$ and a constant $k (= t^{-1})$ for which (42) holds.

It remains to construct $\rho_n(z)$, to show that it does not vanish, and to compute $\rho_n(0)$. Designate by $t_n(z)$ the positive function
\begin{equation}
(55)\quad t_n(z) = \left\{ \sum_{k=0}^{n} \frac{t^k}{m_k^2} |(z-1)^{n-k}|^2 \right\}^{1/2}
\end{equation}
defined on $C$. Now $\log t_n(z)$ is continuous on $C$ and hence
\begin{equation}
(56)\quad u_n(z) = \frac{1}{2\pi} \int_0^1 \log t_n(w) \frac{\partial g(z, w)}{\partial n} \, ds
\end{equation}
where $g(z, w)$ is the Green’s function for $D$, is harmonic in $D$ and assumes on $C$ the boundary values $\log t_n(z)$. Designate by $v_n$ the harmonic conjugate of $u_n$. Then $u_n(z) + iv_n(z)$ is regular and single valued in $D$, as is
\begin{equation}
(57)\quad p_n(z) = \exp \left[ u_n(z) + iv_n(z) \right].
\end{equation}
Now, $|p_n(z)| = e^{u_n}$, so that on $C$, $|p_n(z)| = t_n(z)$. Furthermore $p_n(z) \neq 0$, as is clear from (57). Thus we may use $\rho_n(z) = p_n(z)$. The condition (44) then becomes: there exists a $t > 0$ and a $K' > 0$ such that
\begin{equation}
(58)\quad u_n(0) \leq K' \quad \text{ (} n \to \infty \text{ )}.
\end{equation}
Finally, using the representation
\begin{equation}
(59)\quad g(z, w) = \log \left| \frac{m(z) - m(w)}{1 - m(z)m(w)} \right|
\end{equation}
with $z = 0$ in (56), we obtain the stated condition (29).

4. Concluding remarks. Norms other than (6) might be contemplated. In particular, we might have used
\begin{equation}
(60)\quad \| f \|^2 = \iint_\rho |f(z)|^2 \, dA.
\end{equation}
However (60) has the disadvantage that the solution of the corresponding minimum problem $P_n$ can not be so elegantly expressed in terms of an analytic function $\rho_n(z)$ and so the role of the sequence $\{m_n\}$ is not immediately evident as with (29).
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