Pacific Journal of Mathematics

UNIQUENESS THEORY FOR ASYMPTOTIC EXPANSIONS IN GENERAL REGIONS

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Vol. 7, No. 1 January 1957

UNIQUENESS THEORY FOR ASYMPTOTIC EXPANSIONS IN GENERAL REGIONS

PHILIP DAVIS

1. Introduction. Let D be a simply connected region with an analytic boundary C. Assume that $z=0$ is an interior point while $z=1$ lies on the boundary. We assume further that the tangent to C at $z=1$ is not parallel to the real axis. In this case, we shall be able to fit into D small angles Γ placed symmetrically about the real axis and with vertex at $z=1$. These angles will be of the form $-\delta \leqq \theta \leqq \delta$ or $\pi-\delta \leqq \theta \leqq \pi+\delta$, $\delta>0$, depending upon the location of $z=1$. For a given $f(z)$ regular in D, we consider the following limits defined recursively

$$\begin{aligned} a_0 &= \lim_{z\to 1} f(z) \\ a_1 &= \lim_{z\to 1} (z-1)^{-1}[f(z)-a_0] \\ a_2 &= \lim_{z\to 1} (z-1)^{-2}[f(z)-a_0-a_1(z-1)] \\ &\cdots \end{aligned} \tag{1}$$

If each limit in (1) exists and is independent of the manner in which $z\to 1$ through values in some angle Γ, then $f(z)$ is said to possess an asymptotic expansion at $z=1$ in the sense of Poincaré, and this is indicated by writing

$$f(z) \sim \sum_{n=0}^{\infty} a_n(z-1)^n . \tag{2}$$

We shall designate by $A(=A(D))$ the linear class of functions which are regular in D and which possess asymptotic expansions at $z=1$ in the sense of Poincaré. The angle Γ in which (1) is valid may depend upon the particular $f\in A$ selected.

Uniqueness theory is concerned with distinguishing nontrivial subclasses of A within which the expansion $\sum_{n=0}^{\infty} a_n(z-1)^n$ determines the corresponding function uniquely. Write for the remainder

$$R_n(z)=f(z)-a_0-a_1(z-1)-\cdots-a_{n-1}(z-1)^{n-1}, \tag{3}$$

and consider the ratios

Received January 6, 1956. The preparation of this paper was sponsored by the Office of Scientific Research and Development of the Air Research and Development Command, USAF.

$$f_n(z)=(z-1)^{-n}R_n(z) \qquad (n=1, 2, \cdots),\ f_0=f. \tag{4}$$

For $f \in A$, the functions $f_n(z)$ are regular in D and are bounded as $z \to 1$ in Γ. For a given sequence of positive quantities $\{m_n\}$, we consider the subset $A(m_n)$ of A consisting of those functions which satisfy in addition

$$\|f_n\|^2 < Mk^n m_n^2 \qquad (n=0, 1, 2, \cdots) \tag{5}$$

for some $M>0$, $k>0$. Here $\| \ \|$ designates some conveniently chosen norm. The constants M and k may vary from function to function within the class. With the selection

$$\|f\| = \max_{z \in D} |f(z)|, \tag{6}$$

it has been shown by Watson [1] and F. Nevanlinna [5] that when D is a sector, we may produce uniqueness classes by restricting the growth of the sequence $\{m_n\}$ sufficiently. When D is the unit circle, T. Carleman [2] has given necessary and sufficient conditions on $\{m_n\}$ in order that the resulting subclass $A(m_n)$ be a uniqueness class. At the same time Carleman raises the problem of giving necessary and sufficient conditions in the case of a more general region D. This problem (with the norm (6)) has been known in the literature at the generalized problem of Watson. It has been treated by Mandelbrojt and MacLane [3] using the theory of distortion in conformal mapping. See also Meili [4]. In the present paper, we adopt the norm

$$\|f\|^2 = \int_C |f(z)|^2 \, ds, \tag{7}$$

and show how it is possible to combine Carleman's idea of introducing an appropriate minimum problem with the techniques afforded by the theory of conformal kernel functions to arrive at a solution to this general problem. The class $A(m_n)$ will henceforth refer to the norm (7). Thus the question which we are treating may be worded as follows: *What are necessary and sufficient conditions on the sequence of constants $\{m_n\}$ in order that*

$$\begin{aligned} \|f_n\|^2 &= \int_C |f_n(z)|^2 \, ds \\ &= \int_C \left| \frac{f(z)-a_0-a_1(z-1)-\cdots-a_{n-1}(z-1)^{n-1}}{(z-1)^n} \right|^2 ds < Mk^n m_n^2 \end{aligned} \tag{8}$$

determine $f(z)$ uniquely from the asymptotic coefficients a_n.

2. Preliminary observations. We must first explain the sense in which we shall understand the expression

$$\int_C |f(z)|^2 \, ds$$

when $f(z)$ is regular in D but not necessarily in its closure. Let $w = m(z)$ map D conformally onto the unit circle with $m(0)=0$ and $m(1)=1$. The images of $|w|=r$ will be designated by C_r, $0<r<1$. It is well known that the set of functions

$$\phi_n(z) = \frac{1}{\sqrt{2\pi}} \frac{[m'(z)]^{1/2}}{r^{n+1/2}} [m(z)]^n \qquad (n=0, 1, 2, \cdots) \tag{9}$$

is complete and orthonormal over each C_r, $0<r<1$, relative to the inner product

$$(f, g) = \int_{C_r} f\bar{g} \, ds .$$

Suppose then that we are given a function $f(z)$ which is regular in D. Then for any fixed $0<r<1$, $f(z)$ is continuous on C_r. Hence we can write

$$f(z) = \sum_{n=0}^{\infty} a_n \phi_n(z) \tag{10}$$

holding uniformly and absolutely in the interior of C_r. The coefficients a_n are given by

$$a_n = \int_{C_r} f(z) \overline{\phi_n(z)} ds \qquad (n=0, 1, \cdots) . \tag{11}$$

Hence, for $r^* < r$, we have from (9) and (10),

$$\int_{C_{r^*}} |f(z)|^2 \, ds = \sum_{n=0}^{\infty} |a_n|^2 \frac{r^{*2n+1}}{r^{2n+1}} . \tag{12}$$

This equation tells us that

$$\int_{C_{r^*}} |f(z)|^2 \, ds$$

is an increasing function of r^* and hence

$$\lim_{r^* \to 1^-} \int_{C_{r^*}} |f(z)|^2 \, ds$$

exists (or equals $+\infty$). For $f(z)$ regular in D we shall therefore agree that

$$\int_C |f(z)|^2\,ds = \lim_{r\to 1-}\int_{C_r} |f(z)|^2\,ds\ .$$

LEMMA. *Given an arbitrary sequence of positive constants* $\{m_n\}$; *the class* $A(m_n)$ *is not a uniqueness class for asymptotic expansions at* $z=1$ *if and only if there exists an* $f\not\equiv 0$ *regular in* D *and constants* $M>0$, $k>0$, *for which*

$$\left\|\frac{f(z)}{(z-1)^n}\right\|^2 < Mk^n m_n^2 \qquad (n=0,\,1,\,2,\,\cdots). \tag{13}$$

Proof. If $A(m_n)$ is not a uniqueness class, there will exist two functions $g(z)$, $y(z)\in A(m_n)$, $g\not\equiv h$, possessing the same asymptotic expansion, say $\sum_{n=0}^{\infty} a_n(z-1)^n$, and satisfying

$$\int_C \left|\frac{g(z)-\sum_{k=0}^{n-1} a_k(z-1)^k}{(z-1)^n}\right|^2 ds < M_1k_1^n m_n^2 \qquad (n=0,\,1,\,\cdots) \tag{14}$$

$$\int_C \left|\frac{h(z)-\sum_{k=0}^{n-1} a_k(z-1)^k}{(z-1)^n}\right|^2 ds < M_2k_2^n m_n^2$$

with $k_1 \leqq k_2$. Therefore, by Minkowski's inequality,

$$\begin{aligned}\int_C \left|\frac{g(z)-h(z)}{(z-1)^n}\right|^2 ds &< (M_1^{1/2}k_1^{n/2}+M_2^{1/2}k_2^{n/2})^2 m_n^2 \\ &= (M_1^{1/2}(k_1/k_2)^{n/2}+M_2^{1/2})^2 k_2^n m_n^2 \\ &< (M_1^{1/2}+M_2^{1/2})^2 k_2^n m_n^2\end{aligned} \tag{15}$$

so that $g-h$ does not vanish identically and satisfies (13) with $M=(M_1^{1/2}+M_2^{1/2})^2$ and $k=k_2$.

Conversely, let $f\not\equiv 0$ satisfy (13). We shall show that (13) implies

$$\lim_{z\to 1}\frac{f(z)}{(z-1)^n}=0 \qquad (n=0,\,1,\,2,\,\cdots) \tag{16}$$

as $z\to 1$ through values in some angle Γ. Assuming, for the moment, that this is so, (16) and (1) imply that

$$f(z)\sim 0+0\cdot(z-1)+0\cdot(z-1)^2+\cdots\ . \tag{17}$$

That is, $f(z)$ possesses an identically zero asymptotic expansion at $z=1$. Furthermore $f_n=f(z)(z-1)^{-n}$, so that (13) implies that $f\in A(m_n)$. Thus, $A(m_n)$ is not a uniqueness class for asymptotic expansions at $z=1$.

We show now that (13) implies (16). Given any $g(z)$ regular in D. Select any $0<r<1$. We have from (9), (10), (11), and the Schwarz inequality

$$|g(z)|^2 < K_{C_r}(z, \bar{z}) \int_{C_r} |g(z)|^2 \, ds, \tag{18}$$

for all z interior to C_r. K_{C_r} is the so-called Szegö kernel function for C_r whose explicit expression is (Szegö [6], Bergman [1])

$$K_{C_r}(z, \bar{z}) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(z)} = \frac{1}{2\pi} \frac{r|m'(z)|}{r^2 - |m(z)|^2}. \tag{19}$$

Writing $f(z)/(z-1)^n$ in place of $g(z)$ in (18), and using (13) and the monotonicity with r of

$$\int_{C_r} |f(z)|^2 \, ds,$$

we find for $j \leqq n$ and z interior to C_r,

$$\left|\frac{f(z)}{(z-1)^j}\right|^2 \leqq \frac{|(z-1)^{n-j}|^2 r|m'(z)|}{(2\pi)(r^2 - |m(z)|^2)} Mk^n m_n^2 \qquad (n=0, 1, 2, \cdots). \tag{20}$$

For each z in D we select an $r = r(z) = |m(z)| + \varepsilon(z) < 1$ where $\varepsilon(z)$ is defined by

$$\varepsilon(z) = \frac{1}{2}(1 - |m(z)|). \tag{21}$$

Thus,

$$\lim_{z \to 1} \varepsilon(z) = 0. \tag{22}$$

Here, $z \to 1$ through values in D. From (20), (21), and $r<1$,

$$\left|\frac{f(z)}{(z-1)^j}\right|^2 \leqq \frac{|(z-1)^{n-j}|^2}{2\pi} \cdot \frac{|m'(z)| Mk^n m_n^2}{2|m(z)|\varepsilon(z) + \varepsilon^2(z)} \tag{23}$$

$$< \frac{|(z-1)^{a-j}|^2 |m'(z)| Mk^n m_n^2}{4\pi |m(z)| \varepsilon(z)}.$$

We are now ready to consider the limit of (23) as $z \to 1$. First consider

$$\frac{\varepsilon(z)}{|z-1|} = \frac{1 - |m(z)|}{2|z-1|} = \frac{1}{2}(1 + |m(z)|)^{-1} \frac{(1 - |m(z)|^2)}{|z-1|}. \tag{24}$$

Since $m(z)$ is by assumption analytic at $z=1$, we have in a neighborhood of $z=1$,

$$m(z)=1+(z-1)R(z)\,, \tag{25}$$

where $R(z)$ is analytic there. Note that $R(1)=m'(1)\neq 0$, and write $R(z)=\sigma(z)e^{i\alpha(z)}$, $\sigma(z)>0$. We have $\sigma(1)\neq 0$ and $\alpha(1)\neq\pi/2$, $3\pi/2$, inasmuch as the tangent to C at $z=1$ is assumed not parallel to the real axis. Furthermore, write $z=1+\rho e^{i\theta}$. Then, from (25),

$$\begin{aligned}\frac{1-|m(z)|^2}{|z-1|}&=\frac{-2\mathscr{R}\{(z-1)R(z)\}}{|z-1|}-\frac{|z-1|^2|R(z)|^2}{|z-1|}\\ &=-2\mathscr{R}\{e^{i\theta}\sigma(z)^{i\alpha(z)}\}-|z-1||R(z)|^2\\ &=-2\sigma(z)\cos(\theta+\alpha(z))-|z-1||R(z)|^2\,.\end{aligned} \tag{26}$$

If $z\to 1$ through some angle Γ: $-\delta\leqq\theta\leqq\delta$ or $\pi-\delta\leqq\theta\leqq\pi+\delta$, then, since $\alpha(1)\neq\pi/2$, $3\pi/2$, it follows from the above that for δ sufficiently small, the expression (26) will be bounded away from 0. In view of (24) we will have

$$\frac{\varepsilon(z)}{|z-1|}\geqq\tau>0\;;\; z\to 1 \tag{27}$$

for z in some Γ. From (23), we have,

$$\left|\frac{f(z)}{(z-1)^j}\right|^2<|z-1|^{2n-2j-1}|m'(z)|Mk^n m_n^2\Big/\frac{4\pi|m(z)|\cdot\varepsilon(z)}{|z-1|}\,. \tag{28}$$

Thus, for $2n-2j-1>1$ it is now clear from (28) and (27) that

$$\lim_{z\to 1}\frac{f(z)}{(z-1)^j}=0\,.$$

For each j considered we need only use an $n>j+1$. This completes the proof of the lemma.

3. The uniqueness theorem.

THEOREM. *Given an arbitrary sequence of positive constants m_n. The class $A(m_n)$ is a uniqueness class for asymptotic expansions at $z=1$ if and only if for all $t>0$,*

$$\limsup_{n\to\infty}\int_C\log\left\{\sum_{k=0}^{n}\frac{t^k}{m_k^2}|(z-1)^{n-k}|^2\right\}\frac{\partial}{\partial n}\log|m(z)|\,ds=\infty\,. \tag{20}$$

Here $\partial/\partial n$ designates normal differentiation in the positive sense.

The above statement is equivalent to saying that $A(m_n)$ is not a uniqueness class if and only if there exists a $t>0$ and a $K>0$ such that

$$\int_C \log\left\{\sum_{k=0}^{n} \frac{t^k}{m_k^2}\,|(z-1)^{n-k/2}\right\}\frac{\partial}{\partial n}\log|m(z)|\,ds<K\,, \qquad n=0,\,1,\,2,\,\cdots \tag{30}$$

K may depend upon t, but is independent of n.

In view of the lemma of the preceding section, we shall prove that (30) is a necessary and sufficient condition for the existence of an $f(z) \neq 0$, and M, and a k which satisfy (13).

Consider the following sequence of integrals

$$I_n(f)=\sum_{k=0}^{n}\frac{t^k}{m_k^2}\int_C\left|\frac{f(z)}{(z-1)^k}\right|^2 ds\,; \tag{31}$$

$$=\sum_{k=0}^{n}\frac{t^k}{m_k^2}\,\|f\|_k^2 \qquad n=0,\,1,\,\cdots\,,$$

where we have written

$$\|f\|_k^2=\int_C\left|\frac{f(z)}{(z-1)^k}\right|^2 ds; \qquad k=0,\,1,\,\cdots\,. \tag{32}$$

We can also write (31) in the form

$$I_n(f)=\left\|\frac{\rho_n(z)f(z)}{(z-1)^n}\right\|^2 \tag{33}$$

where $\rho_n(z)$ is an analytic function which is regular in D, continuous on C and is such that

$$|\rho_n(z)|=\left\{\sum_{k=0}^{n}\frac{t^k}{m_k^2}\,|(z-1)^{n-k}|^2\right\}^{1/2}, \quad \text{for } z \text{ on } C\,. \tag{34}$$

We shall show below how a $\rho_n(z)$ may be constructed which has these properties and has, in addition, the property that

$$\rho_n(z)\neq 0 \quad \text{for} \quad z \text{ in } D. \tag{35}$$

Let n be fixed, and consider the following minimum problem P_n. *Determine that function $f(z)$ regular in D with $f(0)=1$ and such that*

$$I_n(f)=\textit{minimum}\,. \tag{36}$$

This problem can be solved by passing to a related problem P_n'. *Determine that function $g(z)$ regular in D with $g(0)=1$ and such that*

$$\|g\|^2=\textit{minimum} \tag{37}$$

The solution of the problem P_n' is given by the function (see, for example Szegö [6], Bergman [1])

$$g^*(z) = K_D(z, 0)/K_D(0, 0) \tag{38}$$

where $K_D(z, \bar{w})$ is the Szegö kernel function of the region D. The minimum value of the integral (37) is $1/K_D(0, 0)$. If we write

$$I_n(f) = |\rho_n(0)|^2 \left\| \frac{\rho_n(z) f(z)}{\rho_n(0)(1-z)^n} \right\|^2, \tag{39}$$

we see, in view of (35) that the function $\rho_n(z)f(z)/\rho_n(0)(1-z)^n$ can play the role of $g(z)$ in the problem P_n'. The minimizing function f_n^* of the problem P_n is therefore

$$f_n^*(z) = \frac{K_D(z, 0)(1-z)^n \rho_n(0)}{\rho_n(z) K_D(0, 0)}, \tag{40}$$

and the minimum value of the integral is

$$I_n(f_n^*) = \frac{|\rho_n(0)|^2}{K_D(0, 0)}. \tag{41}$$

We now assert: a necessary and sufficient condition in order that there exist an $f(z) \not\equiv 0$ and constants $M > 0$, $k > 0$ such that

$$\| f \|_n^2 = \left\| \frac{f(z)}{(z-1)^n} \right\|^2 < M k^n m_n^2 \qquad (n = 0, 1, \cdots) \tag{42}$$

is that there exists a $t > 0$ and a $K > 0$ such that

$$I_n(f_n^*) \leqq K \qquad n = 0, 1, 2, \cdots. \tag{43}$$

Referring to (41), this is equivalent to asserting that there exist a $t > 0$ and a K' such that

$$|\rho_n(0)| \leqq K' \qquad n = 0, 1, 2, \cdots. \tag{44}$$

We can prove this as follows. Suppose first that $q(z)$ is such that (42) holds for it. This function $q(z)$ may have a zero of the pth order at $z=0$. The function $f(z) = q(z)/z^p$ is then regular in D and is such that $f(0) \neq 0$. Now,

$$\begin{aligned} I_n(f(z)/f(0)) &= \sum_{j=0}^{n} \frac{t^j}{m_j^2} \int_C \left| \frac{q(z)}{f(0) z^p (z-1)^j} \right|^2 ds \\ &\leqq \sum_{j=0}^{n} \frac{t^j}{m_j^2} \frac{1}{|f(0)|^2} \frac{1}{d^{2p}} M \cdot m_j^2 k^j \\ &\leqq \frac{M}{d^{2p} |f(0)|^2} \sum_{j=0}^{n} t^j k^j \leqq \frac{M}{d^{2p} |f(0)|^2 (1 - tk)}, \end{aligned} \tag{45}$$

provided we select $0<t<1/k$. Here d designates the minimum distance from $z=0$ to C. Now since

$$I_n(f_n^*)\leqq I_n(f(z)/f(0))\leqq \frac{M}{d^{2p}|f(0)|^2(1-tk)}, \qquad (n=0,1,\cdots) \tag{46}$$

then (43) is satisfied with K equal to the right hand constant in (46).

Conversely, suppose that there exists a $t>0$ and $K>0$ such that (43) holds. Then from (31),

$$\sum_{k=0}^{n}\frac{t^k}{m_k^2}\|f_n^*\|_k^2\leqq K \qquad (n=0,1,2,\cdots). \tag{47}$$

In particular, taking the first term of (47) we obtain

$$\frac{1}{m_0^2}\|f_n^*\|_0^2<K \qquad n=0,1,2,\cdots. \tag{48}$$

Hence we have

$$\|f_n^*\|<\text{const.} \qquad (n=0,1,2,\cdots). \tag{49}$$

The inequalities (49) imply that the sequence of minimizing functions $\{f_n^*\}$ form a normal family and therefore there exist indices $n_1, n_2, \cdots$ such that $f_{n_k}^*\rightarrow F(z)$ uniformly in any closed region interior to D. Again, using (47) we have, for fixed j and for all $n\geqq j$

$$\frac{t^j}{m_j^2}\|f_n^*\|_j^2\leqq K. \tag{50}$$

Now for any $0<\rho<1$, we have

$$\|f_n^*\|_j^2=\int_C\left|\frac{f_n^*(z)}{(z-1)^j}\right|^2 ds\geqq\int_{C_\rho}\left|\frac{f_n^*(z)}{(z-1)^j}\right|^2 ds, \tag{51}$$

so that from (50) and (51),

$$\int_{C_\rho}\left|\frac{f_n^*(z)}{(z-1)^j}\right|^2 ds<Km_j^2t^{-j} \qquad (k=0,1,2,\cdots). \tag{52}$$

Let n take on the values n_i in (52) and let j be fixed. Then since $f_n^*(z)\rightarrow F(z)$ uniformly in and on C_ρ,

$$\int_{C_\rho}\left|\frac{F(z)}{(z-1)^j}\right|^2 ds\leqq Km_j^2t^{-j}. \tag{53}$$

This result is independent of ρ and hence we may allow $\rho\rightarrow 1$. Thus,

$$\int_C \left| \frac{F(z)}{(z-1)^j} \right|^2 ds < K m_j^2 t^{-j} \qquad (j=0, 1, 2, \cdots) . \tag{54}$$

Since obviously $F(0)=1$, we have exhibited in $F(z)$ a function regular in D, which does not vanish identically, a constant $M(=K)$ and a constant $k(=t^{-1})$ for which (42) holds.

It remains to construct $\rho_n(z)$, to show that it does not vanish, and to compute $\rho_n(0)$. Designate by $t_n(z)$ the positive function

$$t_n(z) = \left\{ \sum_{k=0}^{n} \frac{t^k}{m_k^2} |(z-1)^{n-k}|^2 \right\}^{1/2} \tag{55}$$

defined on C. Now $\log t_n(z)$ is continuous on C and hence

$$u_n(z) = \frac{1}{2\pi} \int_C \log t_n(w) \frac{\partial g(z, w)}{\partial n} ds \tag{56}$$

where $g(z, w)$ is the Green's function for D, is harmonic in D and assumes on C the boundary values $\log t_n(z)$. Designate by v_n the harmonic conjugate of u_n. Then $u_n(z)+iv_n(z)$ is regular and single valued in D, as is

$$p_n(z) = \exp [u_n(z) + iv_n(z)] . \tag{57}$$

Now, $|p_n(z)| = e^{u_n}$, so that on C, $|p_n(z)| = t_n(z)$. Furthermore $p_n(z) \neq 0$, as is clear from (57). Thus we may use $\rho_n(z) = p_n(z)$. The condition (44) then becomes: there exists a $t > 0$ and a $K' > 0$ such that

$$u_n(0) \leqq K' \qquad (n \to \infty) . \tag{58}$$

Finally, using the representation

$$g(z, w) = \log \left| \frac{m(z) - m(w)}{1 - m(z)\overline{m(w)}} \right| \tag{59}$$

with $z=0$ in (56), we obtain the stated condition (29).

4. Concluding remarks. Norms other than (6) might be contemplated. In particular, we might have used

$$\| f \|^2 = \iint_D |f(z)|^2 \, dA . \tag{60}$$

However (60) has the disadvantage that the solution of the corresponding minimum problem P_n can not be so elegantly expressed in terms of an analytic function $\rho_n(z)$ and so the role of the sequence $\{m_n\}$ is not immediately evident as with (29).

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