UNIQUENESS THEORY FOR ASYMPTOTIC EXPANSIONS IN GENERAL REGIONS

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1. Introduction. Let $D$ be a simply connected region with an analytic boundary $C$. Assume that $z=0$ is an interior point while $z=1$ lies on the boundary. We assume further that the tangent to $C$ at $z=1$ is not parallel to the real axis. In this case, we shall be able to fit into $D$ small angles $\Gamma$ placed symmetrically about the real axis and with vertex at $z=1$. These angles will be of the form $-\delta \leq \theta \leq \delta$ or $\pi - \delta \leq \theta \leq \pi + \delta$, $\delta > 0$, depending upon the location of $z=1$. For a given $f(z)$ regular in $D$, we consider the following limits defined recursively

$$a_0 = \lim_{z \to 1} f(z)$$

$$a_1 = \lim_{z \to 1} (z-1)^{-1}[f(z) - a_0]$$

$$a_2 = \lim_{z \to 1} (z-1)^{-2}[f(z) - a_0 - a_1(z-1)]$$

$$\cdots$$

If each limit in (1) exists and is independent of the manner in which $z \to 1$ through values in some angle $\Gamma$, then $f(z)$ is said to possess an asymptotic expansion at $z=1$ in the sense of Poincaré, and this is indicated by writing

$$f(z) \sim \sum_{n=0}^{\infty} a_n(z-1)^n.$$ 

We shall designate by $A(=A(D))$ the linear class of functions which are regular in $D$ and which possess asymptotic expansions at $z=1$ in the sense of Poincaré. The angle $\Gamma$ in which (1) is valid may depend upon the particular $f \in A$ selected.

Uniqueness theory is concerned with distinguishing nontrivial subclasses of $A$ within which the expansion $\sum_{n=0}^{\infty} a_n(z-1)^n$ determines the corresponding function uniquely. Write for the remainder

$$R_n(z) = f(z) - a_0 - a_1(z-1) - \cdots - a_{n-1}(z-1)^{n-1},$$

and consider the ratios
(4) \[ f_n(z) = (z-1)^{-n}R_n(z) \quad (n=1, 2, \cdots), \quad f_0 = f. \]

For \( f \in A \), the functions \( f_n(z) \) are regular in \( D \) and are bounded as \( z \to 1 \) in \( \Gamma \). For a given sequence of positive quantities \( \{m_n\} \), we consider the subset \( A(m_n) \) of \( A \) consisting of those functions which satisfy in addition

(5) \[ \|f_n\|^2 < Mk^nm_n^2 \quad (n=0, 1, 2, \cdots) \]

for some \( M > 0, \ k > 0 \). Here \( \| \| \) designates some conveniently chosen norm. The constants \( M \) and \( k \) may vary from function to function within the class. With the selection

(6) \[ \| f \| = \max_{z \in D} |f(z)|, \]

it has been shown by Watson [1] and F. Nevanlinna [5] that when \( D \) is a sector, we may produce uniqueness classes by restricting the growth of the sequence \( \{m_n\} \) sufficiently. When \( D \) is the unit circle, T. Carleman [2] has given necessary and sufficient conditions on \( \{m_n\} \) in order that the resulting subclass \( A(m_n) \) be a uniqueness class. At the same time Carleman raises the problem of giving necessary and sufficient conditions in the case of a more general region \( D \). This problem (with the norm (6)) has been known in the literature at the generalized problem of Watson. It has been treated by Mandelbrojt and MacLane [3] using the theory of distortion in conformal mapping. See also Meili [4]. In the present paper, we adopt the norm

(7) \[ \| f \|^2 = \int_0^\sigma |f(z)|^2 \, ds, \]

and show how it is possible to combine Carleman's idea of introducing an appropriate minimum problem with the techniques afforded by the theory of conformal kernel functions to arrive at a solution to this general problem. The class \( A(m_n) \) will henceforth refer to the norm (7). Thus the question which we are treating may be worded as follows: What are necessary and sufficient conditions on the sequence of constants \( \{m_n\} \) in order that

(8) \[ \|f_n\|^2 = \int_0^\sigma |f_n(z)|^2 \, ds \]

\[ = \int_0^\sigma \left| \frac{f(z) - a_0 - a_1(z-1) - \cdots - a_{n-1}(z-1)^{n-1}}{(z-1)^n} \right|^2 \, ds < Mk^nm_n^2 \]

determine \( f(z) \) uniquely from the asymptotic coefficients \( a_n \).

2. Preliminary observations. We must first explain the sense in
which we shall understand the expression
\[ \int_0^1 |f(z)|^2 \, ds \]
when \( f(z) \) is regular in \( D \) but not necessarily in its closure. Let \( w = m(z) \) map \( D \) conformally onto the unit circle with \( m(0) = 0 \) and \( m(1) = 1 \). The images of \( |w| = r \) will be designated by \( C_r, \, 0 < r < 1 \). It is well known that the set of functions
\[ \phi_n(z) = \frac{1}{\sqrt{2\pi}} \frac{[m'(z)]^{1/2}}{r^{n+1/2}} [m(z)]^n \quad (n = 0, 1, 2, \ldots) \]
is complete and orthonormal over each \( C_r, \, 0 < r < 1 \), relative to the inner product
\[ (f, g) = \int_{C_r} f \overline{g} \, ds. \]
Suppose then that we are given a function \( f(z) \) which is regular in \( D \). Then for any fixed \( 0 < r < 1 \), \( f(z) \) is continuous on \( C_r \). Hence we can write
\[ f(z) = \sum_{n=0}^{\infty} a_n \phi_n(z) \]
holding uniformly and absolutely in the interior of \( C_r \). The coefficients \( a_n \) are given by
\[ a_n = \int_{C_r} f(z) \overline{\phi_n(z)} \, ds \quad (n = 0, 1, \ldots). \]
Hence, for \( r^* < r \), we have from (9) and (10),
\[ \int_{C_{r^*}} |f(z)|^2 \, ds = \sum_{n=0}^{\infty} |a_n|^2 \frac{r^{*2n+1}}{r^{2n+1}}. \]
This equation tells us that
\[ \int_{C_{r^*}} |f(z)|^2 \, ds \]
is an increasing function of \( r^* \) and hence
\[ \lim_{r^* \to 1} \int_{C_{r^*}} |f(z)|^2 \, ds \]
exists (or equals +\( \infty \)). For \( f(z) \) regular in \( D \) we shall therefore agree that
LEMMA. Given an arbitrary sequence of positive constants \( \{m_n\} \); the class \( A(m_n) \) is not a uniqueness class for asymptotic expansions at \( z=1 \) if and only if there exists an \( f \not\equiv 0 \) regular in \( D \) and constants \( M > 0, k > 0 \), for which

\[
\left\| \frac{f(z)}{(z-1)^n} \right\|^2 < Mk^n m_n^2 \quad (n=0, 1, 2, \ldots).
\]

Proof. If \( A(m_n) \) is not a uniqueness class, there will exist two functions \( g(z), \ y(z) \in A(m_n) \), \( g \not\equiv h \), possessing the same asymptotic expansion, say \( \sum a_n(z-1)^n \), and satisfying

\[
\int_c \left| \frac{g(z) - \sum_{k=0}^{n-1} a_k(z-1)^k}{(z-1)^n} \right|^2 ds < M_1 k_1^n m_n^2 \quad (n=0, 1, \ldots)
\]

\[
\int_c \left| \frac{h(z) - \sum_{k=0}^{n-1} a_k(z-1)^k}{(z-1)^n} \right|^2 ds < M_2 k_2^n m_n^2
\]

with \( k_1 \leq k_2 \). Therefore, by Minkowski’s inequality,

\[
\int_c \left| \frac{g(z) - h(z)}{(z-1)^n} \right|^2 ds < (M_1^{1/2} k_1^{n/2} + M_2^{1/2} k_2^{n/2})^2 m_n^2
\]

\[
= (M_1^{1/2}(k_1/k_2)^{n/2} + M_2^{1/2})^2 k_2^n m_n^2
\]

\[
< (M_1^{1/2} + M_2^{1/2})^2 k_2^n m_n^2
\]

so that \( g - h \) does not vanish identically and satisfies (13) with \( M = (M_1^{1/2} + M_2^{1/2})^2 \) and \( k = k_2 \).

Conversely, let \( f \not\equiv 0 \) satisfy (13). We shall show that (13) implies

\[
\lim_{z \to 1} \frac{f(z)}{(z-1)^n} = 0 \quad (n=0, 1, 2, \ldots)
\]

as \( z \to 1 \) through values in some angle \( \Gamma \). Assuming, for the moment, that this is so, (16) and (1) imply that

\[
f(z) \sim 0 + 0 \cdot (z-1) + 0 \cdot (z-1)^2 + \cdots.
\]

That is, \( f(z) \) possesses an identically zero asymptotic expansion at \( z=1 \). Furthermore \( f_n = f(z)(z-1)^{-n} \), so that (13) implies that \( f \in A(m_n) \). Thus, \( A(m_n) \) is not a uniqueness class for asymptotic expansions at \( z=1 \).
We show now that (13) implies (16). Given any \(g(z)\) regular in \(D\). Select any \(0 < r < 1\). We have from (9), (10), (11), and the Schwarz inequality

\[
|g(z)|^2 < K_{\varrho_r}(z, \bar{z}) \int_{\varrho_r} |g(z)|^2 \, ds,
\]

for all \(z\) interior to \(C_r\). \(K_{\varrho_r}\) is the so-called Szegö kernel function for \(C_r\) whose explicit expression is (Szegö [6], Bergman [1])

\[
K_{\varrho_r}(z, \bar{z}) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(z)} \frac{1}{2\pi r^2 - |m(z)|^2}.
\]

Writing \(f(z)/(z-1)^n\) in place of \(g(z)\) in (18), and using (13) and the monotonicity with \(r\) of

\[
\int_{\varrho_r} |f(z)|^2 \, ds,
\]

we find for \(j \leq n\) and \(z\) interior to \(C_r\),

\[
\left| \frac{f(z)}{(z-1)^j} \right| \leq \frac{|(z-1)^n-j| r |m'(z)|}{(2\pi)(r^2 - |m(z)|^2)} \cdot M k^m m^2_n \quad (n=0, 1, 2, \ldots).
\]

For each \(z\) in \(D\) we select an \(r=r(z)=|m(z)| + \varepsilon(z) < 1\) where \(\varepsilon(z)\) is defined by

\[
\varepsilon(z) = \frac{1}{2} (1 - |m(z)|).
\]

Thus,

\[
\lim_{z \to 1} \varepsilon(z) = 0.
\]

Here, \(z \to 1\) through values in \(D\). From (20), (21), and \(r < 1\),

\[
\left| \frac{f(z)}{(z-1)^j} \right| \leq \frac{|(z-1)^n-j|}{2\pi} \cdot \frac{|m'(z)| M k^m m^2_n}{2|m(z)|\varepsilon(z) + \varepsilon^2(z)} \leq \frac{|(z-1)^n-j| |m'(z)| M k^m m^2_n}{4\pi |m(z)| \varepsilon(z)}.
\]

We are now ready to consider the limit of (23) as \(z \to 1\). First consider

\[
\frac{\varepsilon(z)}{|z-1|} = \frac{1 - |m(z)|}{2 |z-1|} = \frac{1}{2} \left( 1 + |m(z)| \right)^{-1} \frac{1 - |m(z)|}{|z-1|}.
\]

Since \(m(z)\) is by assumption analytic at \(z=1\), we have in a neighborhood of \(z=1\),
where $R(z)$ is analytic there. Note that $R(1)=m'(1)\neq 0$, and write

$$R(z)=\sigma(z)e^{i\alpha(z)}, \quad \sigma(z) > 0.$$ 
We have $\sigma(1) \neq 0$ and $\alpha(1) \neq \pi/2, \ 3\pi/2$, inasmuch as the tangent to $C$ at $z=1$ is assumed not parallel to the real axis. Furthermore, write $z=1+\rho e^{i\theta}$. Then, from (25),

$$m(z)=1+(z-1)R(z),$$

$$m(z) = 1 + (z-1)R(z),$$

where $R(z)$ is analytic there. Note that $R(1) = m'(1) \neq 0$, and write

$$R(z) = \sigma(z)e^{i\alpha(z)}, \quad \sigma(z) > 0.$$ 
We have $\sigma(1) \neq 0$ and $\alpha(1) \neq \pi/2, \ 3\pi/2$, inasmuch as the tangent to $C$ at $z=1$ is assumed not parallel to the real axis. Furthermore, write $z=1+\rho e^{i\theta}$. Then, from (25),

$$\frac{1-|m(z)|^2}{|z-1|} = -2R \left\{ (z-1)R(z) \right\} - \frac{|z-1|^2|R(z)|^2}{|z-1|}$$

$$= -2R \left\{ e^{i\alpha(z)} \right\} - |z-1||R(z)|^2$$

$$= -2\sigma(z)\cos (\theta + \alpha(z)) - |z-1||R(z)|^2.$$ 
If $z \to 1$ through some angle $\Gamma$: $-\delta \leq \theta \leq \delta$ or $\pi - \delta \leq \theta \leq \pi + \delta$, then, since $\alpha(1) \neq \pi/2, \ 3\pi/2$, it follows from the above that for $\delta$ sufficiently small, the expression (26) will be bounded away from 0. In view of (24) we will have

$$\frac{\varepsilon(z)}{|z-1|} \geq \tau > 0; \quad z \to 1$$

for $z$ in some $\Gamma$. From (23), we have,

$$\left| \frac{f(z)}{(z-1)^j} \right|^2 < |z-1|^{2n-2j-1}|m'(z)|MK^\alpha_m^2 / 4\pi |m(z)| |\varepsilon(z)|.$$ 

Thus, for $2n-2j-1 > 1$ it is now clear from (28) and (27) that

$$\lim_{z \to 1} \frac{f(z)}{(z-1)^j} = 0.$$ 
For each $j$ considered we need only use an $n > j+1$. This completes the proof of the lemma.

3. The uniqueness theorem.

**Theorem.** Given an arbitrary sequence of positive constants $m_n$. The class $A(m_n)$ is a uniqueness class for asymptotic expansions at $z=1$ if and only if for all $t > 0$,

$$\lim_{n \to \infty} \sup_{t>0} \int_0^t \log \left\{ \sum_{k=0}^n \frac{t^k}{m_k^2} |(z-1)^{n-k}|^2 \right\} \partial_n \log |m(z)| ds = \infty.$$ 

Here $\partial/\partial n$ designates normal differentiation in the positive sense.

The above statement is equivalent to saying that $A(m_n)$ is not a uniqueness class if and only if there exists a $t > 0$ and a $K > 0$ such
that

\[ \int_0 \log \left\{ \sum_{k=0}^{n} \frac{t^k}{m_k^2} |(z-1)^{n-k}|^2 \right\} \frac{\partial}{\partial n} \log |m(z)| \, ds \leq K, \quad n=0, 1, 2, \ldots \]

\( K \) may depend upon \( t \), but is independent of \( n \).

In view of the lemma of the preceding section, we shall prove that (30) is a necessary and sufficient condition for the existence of an \( f(z) \neq 0 \), and \( M \), and a \( k \) which satisfy (13).

Consider the following sequence of integrals

\[ I_n(f) = \sum_{k=0}^{n} \frac{t^k}{m_k^2} \int_0 \left| f(z) \right|^2 ds ; \]

\[ = \sum_{k=0}^{n} \frac{t^k}{m_k^2} \| f \|_k^2 \]

where we have written

\[ \| f \|_k^2 = \int_0 \left| f(z) \right|^2 ds ; \]

\[ k=0, 1, \ldots . \]

We can also write (31) in the form

\[ I_n(f) = \left\| \rho_n(z) f(z) \right\| \]

where \( \rho_n(z) \) is an analytic function which is regular in \( D \), continuous on \( C \) and is such that

\[ |\rho_n(z)| = \left\{ \sum_{k=0}^{n} \frac{t^k}{m_k^2} |(z-1)^{n-k}|^2 \right\}^{1/2} , \quad \text{for } z \text{ on } C . \]

We shall show below how a \( \rho_n(z) \) may be constructed which has these properties and has, in addition, the property that

\[ \rho_n(z) \neq 0 \quad \text{for } z \text{ in } D. \]

Let \( n \) be fixed, and consider the following minimum problem \( P_n \). Determine that function \( f(z) \) regular in \( D \) with \( f(0) = 1 \) and such that

\[ I_n(f) = \text{minimum} . \]

This problem can be solved by passing to a related problem \( P_n' \). Determine that function \( g(z) \) regular in \( D \) with \( g(0) = 1 \) and such that

\[ \| g \| = \text{minimum} \]

The solution of the problem \( P_n' \) is given by the function (see, for ex-
ample Szegö [6], Bergman [1])

\[ g^*(z) = K_\rho(z, 0)/K_\rho(0, 0) \]

where \( K_\rho(z, \bar{w}) \) is the Szegö kernel function of the region \( D \). The minimum value of the integral (37) is \( 1/K_\rho(0, 0) \). If we write

\[ I_n(f) = \left| \rho_n(0) \right|^2 \left\| \frac{\rho_n(z)f(z)}{\rho_n(0)(1-z)^n} \right\|^2, \]

we see, in view of (35) that the function \( \rho_n(z)f(z)/\rho_n(0)(1-z)^n \) can play the role of \( g(z) \) in the problem \( P_n' \). The minimizing function \( f_n^* \) of the problem \( P_n \) is therefore

\[ f_n^*(z) = \frac{K_\rho(z, 0)(1-z)^n\rho_n(0)}{\rho_n(z)K_\rho(0, 0)}, \]

and the minimum value of the integral is

\[ I_n(f_n^*) = \frac{\left| \rho_n(0) \right|^2}{K_\rho(0, 0)}. \]

We now assert: a necessary and sufficient condition in order that there exist an \( f(z) \neq 0 \) and constants \( M > 0, k > 0 \) such that

\[ \| f \|_n = \left\| \frac{f(z)}{(z-1)^n} \right\| < Mk^n m_n^2 \quad (n = 0, 1, \cdots) \]

is that there exists a \( t > 0 \) and a \( K > 0 \) such that

\[ I_n(f_n^*) \leq K \quad n = 0, 1, 2, \cdots. \]

Referring to (41), this is equivalent to asserting that there exist a \( t > 0 \) and a \( K' > 0 \) such that

\[ |\rho_n(0)| \leq K' \quad n = 0, 1, 2, \cdots. \]

We can prove this as follows. Suppose first that \( q(z) \) is such that (42) holds for it. This function \( q(z) \) may have a zero of the \( p \)th order at \( z = 0 \). The function \( f(z) = q(z)/z^p \) is then regular in \( D \) and is such that \( f(0) \neq 0 \). Now,

\[ I_n(f(z)/f(0)) = \frac{M}{d^{2p} f(0)^2} \sum_{j=0}^n t^j m_j^2 \int_c \left| \frac{q(z)}{|f(0)z^p(z-1)^j|} \right|^2 ds \]

\[ \leq \frac{M}{d^{2p} f(0)^2} \sum_{j=0}^n t^j m_j^2 M \cdot m_j^2 \]

\[ \leq \frac{M}{d^{2p} f(0)^2} \sum_{j=0}^n t^j k^j \leq \frac{M}{d^{2p} f(0)^2(1-tk)}, \]
provided we select $0 < t < 1/k$. Here $d$ designates the minimum distance from $z=0$ to $C$. Now since
\begin{equation}
I_n(f^*_n) \leq I_n(f(z)|f'(0)) \leq \frac{M}{d^{2n}|f'(0)|^n(1-t^k)}, \quad (n=0, 1, \ldots)
\end{equation}
then (43) is satisfied with $K$ equal to the right hand constant in (46).

Conversely, suppose that there exists a $t > 0$ and $K > 0$ such that (43) holds. Then from (31),
\begin{equation}
\sum_{k=0}^{n} \frac{t^k}{m_k} \| f^*_n \| \leq K \quad (n=0, 1, 2, \ldots).
\end{equation}
In particular, taking the first term of (47) we obtain
\begin{equation}
\frac{1}{m_0} \| f^*_n \| \leq K \quad n=0, 1, 2, \ldots.
\end{equation}
Hence we have
\begin{equation}
\| f^*_n \| \leq \text{const.} \quad (n=0, 1, 2, \ldots).
\end{equation}
The inequalities (49) imply that the sequence of minimizing functions \{f^*_n\} form a normal family and therefore there exist indices $n_1, n_2, \ldots$ such that $f^*_{n_k} \to F(z)$ uniformly in any closed region interior to $D$. Again, using (47) we have, for fixed $j$ and for all $n \geq j$
\begin{equation}
\frac{t^j}{m_j^2} \| f^*_n \| \leq K.
\end{equation}
Now for any $0 < \rho < 1$, we have
\begin{equation}
\| f^*_n \| = \int_0 \left| \frac{f^*_n(z)}{(z-1)^j} \right|^2 \, ds \geq \int_{\rho} \left| \frac{f^*_n(z)}{(z-1)^j} \right|^2 \, ds,
\end{equation}
so that from (50) and (51),
\begin{equation}
\int_{\rho} \left| \frac{f^*_n(z)}{(z-1)^j} \right|^2 \, ds \leq Km_j^2t^{-j} \quad (k=0, 1, 2, \ldots).
\end{equation}
Let $n$ take on the values $n_i$ in (52) and let $j$ be fixed. Then since $f^*_n(z) \to F(z)$ uniformly in and on $C_\rho$,
\begin{equation}
\int_{\rho} \left| \frac{F(z)}{(z-1)^j} \right|^2 \, ds \leq Km_j^2t^{-j}.
\end{equation}
This result is independent of $\rho$ and hence we may allow $\rho \to 1$. Thus,
Since obviously $F(0)=1$, we have exhibited in $F(z)$ a function regular in $D$, which does not vanish identically, a constant $M(=K)$ and a constant $k(=t^{-1})$ for which (42) holds.

It remains to construct $\rho_n(z)$, to show that it does not vanish, and to compute $\rho_n(0)$. Designate by $t_n(z)$ the positive function

$$t_n(z)=\left\{\sum_{k=0}^{n} \frac{t^k}{m_k} \left| (z-1)^n - k \right| \right\}^{1/2}$$

defined on $C$. Now $\log t_n(z)$ is continuous on $C$ and hence

$$u_n(z)=\frac{1}{2\pi} \int_0^1 \log t_n(w) \frac{\partial g(z, w)}{\partial n} \, ds$$

where $g(z, w)$ is the Green's function for $D$, is harmonic in $D$ and assumes on $C$ the boundary values $\log t_n(z)$. Designate by $v_n$ the harmonic conjugate of $u_n$. Then $u_n(z)+iv_n(z)$ is regular and single valued in $D$, as is

$$p_n(z)=\exp[u_n(z)+iv_n(z)].$$

Now, $|p_n(z)|=e^{u_n}$, so that on $C$, $|p_n(z)|=t_n(z)$. Furthermore $p_n(z)\neq 0$, as is clear from (57). Thus we may use $p_n(z)=p_n(z)$. The condition (44) then becomes: there exists a $t>0$ and a $K'>0$ such that

$$u_n(0) \leq K' \quad (n \to \infty).$$

Finally, using the representation

$$g(z, w)=\log \left| \frac{m(z)-m(w)}{1-m(z)m(w)} \right|$$

with $z=0$ in (56), we obtain the stated condition (29).

4. Concluding remarks. Norms other than (6) might be contemplated. In particular, we might have used

$$\|f\|^2=\int_D |f(z)|^2 \, dA.$$ 

However (60) has the disadvantage that the solution of the corresponding minimum problem $P_n$ can not be so elegantly expressed in terms of an analytic function $\rho_n(z)$ and so the role of the sequence $\{m_n\}$ is not immediately evident as with (29).
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