A REAL INVERSION FORMULA FOR A CLASS OF
BILATERAL LAPLACE TRANSFORMS

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1. Introduction. The Post-Widder inversion formula for unilateral Laplace transformations [1] states that, under certain weak restrictions on \( \phi(u) \),

\[
\lim_{k \to \infty} \left( \frac{k}{c} \right)^{k+1} \frac{1}{k!} \int_0^\infty \phi(u)u^k \exp \left(-\frac{u}{c} \right) du = \phi(c),
\]

for any continuity point \( c \) of \( \phi(u) \).

This formula applies when \( \phi(u) \) is defined only for \( u \geq 0 \). A similar formula may be deduced if \( \phi(u) \) is defined for \( u \geq -a \), for some positive \( a \). In such a case, we may let \( \phi^*(u) = \phi(u-a) \), and we may then use the Post-Widder formula to determine \( \phi^*(u) \) at the point \( u = c + a \). The inversion formula then becomes

\[
\lim_{k \to \infty} \left( \frac{k}{c+a} \right)^{k+1} \frac{1}{k!} \int_0^\infty \phi(u-a)u^k \exp \left(-\frac{u}{c+a} \right) du = \phi(c),
\]

or, if we make the transformation \( z = u/(c+a) \),

\[
(1) \quad \lim_{k \to \infty} \frac{k^{k+1}}{k!} \int_0^\infty \phi[(c+a)z-a]z^k \exp \left(-kz \right) dz = \phi(c).
\]

This suggests that, if \( \phi(u) \) is defined for \( -\infty < u < \infty \), some sort of limiting form of (1) applies. We shall prove that under suitable restrictions on \( \varepsilon \) and on the behavior of \( \phi(u) \),

\[
(2) \quad \lim_{k \to \infty} \frac{k^{k+1}}{k!} \int_{-\infty}^\infty \phi[(c+k)z-k^\varepsilon]z^k \exp \left(-kz \right) dz = \phi(c).
\]

2. Remarks. In the following sections \( \phi(u) \) will be assumed to be integrable over the interval from \( -\infty \) to \( \infty \), and \( c \) will be assumed to be a continuity point of \( \phi(u) \). All limits should be understood to be for increasing values of \( k \).

The expression \( \delta/(c+k^\varepsilon) \), where \( \delta \) and \( \varepsilon \) are positive numbers, occurs frequently. It will be denoted by \( \delta(k, \varepsilon) \).

Finally, it may be noted that in terms of the Laplace transform of \( \phi(u) \) for real \( t \),

Received December 7, 1955, and in revised form April 13, 1956.
the inversion formula (2) may be written in the form
\[ \lim \left( \frac{-1}{k!} \left( \frac{k}{c+k^s} \right)^{k+1} \right) \frac{d^k}{dt^k} \left[ f(t) \exp \left( -tk^s \right) \right]_{t=k/(c+k^s)} = \phi(c). \]

3. Preliminary proofs. The results of the following four lemmas will be needed below. Proofs are given for the first two. The second two are proved in a similar way.

**Lemma 1.** If \( n \) is any fixed number and \( 1/3 < \varepsilon < 1/2 \), then
\[ \lim k^n \left[ 1 + \delta(k, \varepsilon) \right]^k \exp \left[ -k\delta(k, \varepsilon) \right] = 0. \]

**Proof.** If the logarithm of the expression under the limit sign is expanded in powers of \( \delta(k, \varepsilon) \), the sum of two of the terms in the expansion approaches \( -\infty \) as \( k \to \infty \), while the sum of the rest of the terms is bounded.

**Lemma 2.** If \( 1/3 < \varepsilon < 1/2 \), then
\[ \lim k^{k+1} \int_{1}^{1+\delta(k, \varepsilon)} z^k \exp(-kz) \, dz = \frac{1}{2}. \]

**Proof.** It is well known [1] that
\[ \lim k^{k+1} \int_{1}^{\infty} z^k \exp(-kz) \, dz = \frac{1}{2}. \]

Therefore, it is sufficient to show that
\[ \lim k^{k+1} \int_{1+\delta(k, \varepsilon)}^{\infty} z^k \exp(-kz) \, dz = 0. \]

Since \( z \exp(-z) \) is a decreasing function of \( z \) for \( z > 1 \), the above expression is, for fixed \( k \), no larger than
\[ \frac{k^{k+1}}{k!} \left[ 1 + \delta(k, \varepsilon) \right]^{k-1} \exp \left[ -(k-1)(1 + \delta(k, \varepsilon)) \right] \int_{1+\delta(k, \varepsilon)}^{\infty} z \exp(-z) \, dz. \]

By applying Stirling’s formula and Lemma 1, we see that the upper bound approaches zero as \( k \) increases.

**Lemma 3.** If \( n \) is any fixed number and \( 0 < \varepsilon < 1/2 \), then
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\[
\lim k^n[1 - \delta(k, \varepsilon)]^k \exp [k\delta(k, \varepsilon)] = 0 ,
\]

**Lemma 4.** If \( 0 < \varepsilon < 1/2 \), then

\[
\lim \frac{k^{k+1}}{k!} \int_{1-\delta(k, \varepsilon)}^{1} z^k \exp (-kz)dz = \frac{1}{2} .
\]

4. The inversion formula.

**Theorem.** If

(a) \[
\left| \int_{-\infty}^{-z} \phi(z)dz \right| \leq A \exp (-dx^{2+\alpha})
\]
for some positive quantities \( A, d, \) and \( \alpha \), and if

(b) \[
\max (1/3, 1/(2 + \alpha)) < \varepsilon < 1/2,
\]
then

\[
\lim I_k = \lim \frac{k^{k+1}}{k!} \int_{-\infty}^{\infty} \phi[(c + k^z)z - k^z]z^k \exp (-kz)dz = \phi(c) .
\]

**Proof.** For any \( \delta > 0 \), the infinite interval may be partitioned into the four subintervals \((-\infty, 1 - \delta(k, \varepsilon)), (1 - \delta(k, \varepsilon), 1), (1, 1 + \delta(k, z)), \) and \((1 + (k, \varepsilon), \infty)\). \( I_k \) may be considered as the sum of four integrals over these intervals, so that we may write

\[
I_k = I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)} .
\]

\( I_k^{(1)} \) is understood to represent the integral over \((-\infty, 1 - \delta(k, \varepsilon))\) etc.

\[
|I_k - \phi(c)| \leq |I_k^{(1)}| + \left| I_k^{(2)} - \frac{\phi(c)}{2} \right| + \left| I_k^{(3)} - \frac{\phi(c)}{2} \right| + |I_k^{(4)}| .
\]

We prove first that \( I_k^{(1)} \) and \( I_k^{(4)} \) approach zero as \( k \to \infty \). For \( I_k^{(1)} \), consider first the integral over the interval from 0 to \( 1 - \delta(k, \varepsilon) \). The function \( z \exp (-z) \) attains its maximum at the upper endpoint. Therefore an upper bound for the absolute value of this portion of the expression is

\[
\frac{k^{k+1}}{k!} \int_{1-\delta(k, \varepsilon)}^{1} [1 - \delta(k, \varepsilon)]^k \exp [-k + k\delta(k, \varepsilon)] \int_{0}^{1-\delta(k, \varepsilon)} |\phi[(c + k^z)z - k^z]|dz ,
\]

which approaches zero by Stirling's formula and Lemma 3.

Consider now the integral over the interval from \(-\infty \) to 0. Integrating by parts, we find that it is equal to


\[- \frac{1}{c+k^s} \frac{k^{k+2}}{k!} \int_{-\infty}^{z} F'[\{(c+k^s)z-k^s\}z^{k+1}(1-z)] \exp (-kz)dz , \]

where \( F'(z)=\int_{-\infty}^{z} \phi(u)du \). Note that, by the assumption on \( F(z) \),
\[ |F'[\{(c+k^s)z-k^s\}]| \leq A \exp \left\{ -d \{ -(c+k^s)z+k^s \}^{z+s} \right\} , \]
which is in turn equal to or less than
\[ A \exp \left\{ d(c+k^s)k^{z(1+\alpha)} - k \right\} . \]

The result of the integration by parts may be written as the difference between two integrals, the first containing \( z^{k-1} \) and the second containing \( z^k \). The first integral is no greater in absolute value than
\[ \frac{A}{(c+k^s)} \frac{k^{k+2}}{k!} \int_{-\infty}^{z} \left| z^{k-1} \right| \exp \left\{ d(c+k^s)k^{z(1+\alpha)} - k \right\} dz . \]

Since \( \varepsilon(2+\alpha)>1 \), the coefficient of \( z \) in the exponent above is positive for sufficiently large \( k \). Therefore, after some manipulation, this upper bound can be shown to be equal to
\[ \frac{A}{(c+k^s)} \frac{k^{k+2}}{k!} \cdot \frac{\Gamma(k)}{[d(c+k^s)k^{z(1+\alpha)} - k]^k} , \]
which approaches zero as \( k \to \infty \).

By the same argument, the second integral approaches zero, so that \( \lim I_k^{(3)}=0 \).

For \( I_k^{(3)} \), observe that since \( z \exp (-z) \) is a decreasing function of \( z \) for \( z>1 \), the expression has the following upper bound for its absolute value:
\[ \frac{k^{k+1}}{k!} [1+\delta(k, \varepsilon)]^k \exp \left\{ -k-k\delta(k, \varepsilon) \right\} \int_{1+\delta(k, \varepsilon)}^{\infty} |\phi((c+k^s)z-k^s)|dz . \]

Since the integral is bounded, the whole upper bound approaches zero by virtue of Stirling’s formula and Lemma 1.

We now prove that
\[ \left| \lim I_k^{(3)} - \frac{1}{2} \phi(c) \right| < \frac{\eta}{2} \]
for any \( \eta > 0 \). By Lemma 2, it is sufficient to show that
\[ \left| \lim \frac{k^{k+1}}{k!} \int_{1+\delta(k, \varepsilon)}^{\infty} \left\{ \phi((c+k^s)z-k^s) - \phi(c) \right\} z^k \exp (-kz)dz \right| < \frac{\eta}{2} . \]
Since \( c \) is a continuity point of \( \phi(u) \), there is a \( \delta > 0 \) such that if 
\[ |(c + k^s)z - k^s - c| < \delta, \]
that is, if \( |z - 1| < \delta(k, \varepsilon) \), then
\[ |\phi((c + k^s)z - k^s) - \phi(c)| < \eta. \]

For such a \( \delta \), the absolute value of the expression above is equal to or less than
\[
\eta \lim_{k \to \infty} \frac{k^{k+1}}{k!} \int_1^{1+\delta(k, \varepsilon)} z^k \exp(-kz)dz = \frac{\eta}{2}.
\]

By the use of Lemma 4, it may be shown in a similar way that
\[
\left| \lim I_k^{(2)} - \frac{1}{2} \phi(c) \right| < \frac{1}{2} \eta.
\]

Putting together these results, we have \( |\lim I_k - \phi(c)| < \eta \) for any \( \eta > 0 \), which proves the theorem.

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