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A REAL INVERSION FORMULA FOR A CLASS OF BILATERAL LAPLACE TRANSFORMS

WILLIAM ROBERT GAFFEY

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1. Introduction. The Post-Widder inversion formula for unilateral Laplace transformations [1] states that, under certain weak restrictions on $\phi(u)$,

$$\lim_{k \rightarrow \infty} \left(\frac{k}{c}\right)^{k+1} \frac{1}{k!} \int_0^{\infty} \phi(u) u^k \exp\left(-k \frac{u}{c}\right) du = \phi(c),$$

for any continuity point c of $\phi(u)$.

This formula applies when $\phi(u)$ is defined only for $u \geq 0$. A similar formula may be deduced if $\phi(u)$ is defined for $u \geq -a$, for some positive a . In such a case, we may let $\phi^*(u) = \phi(u-a)$, and we may then use the Post-Widder formula to determine $\phi^*(u)$ at the point $u = c+a$. The inversion formula then becomes

$$\lim_{k \rightarrow \infty} \left(\frac{k}{c+a}\right)^{k+1} \frac{1}{k!} \int_0^{\infty} \phi(u-a) u^k \exp\left(-k \frac{u}{c+a}\right) du = \phi(c),$$

or, if we make the transformation $z = u/(c+a)$,

$$(1) \quad \lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \int_0^{\infty} \phi[(c+a)z - a] z^k \exp(-kz) dz = \phi(c).$$

This suggests that, if $\phi(u)$ is defined for $-\infty < u < \infty$, some sort of limiting form of (1) applies. We shall prove that under suitable restrictions on ε and on the behavior of $\phi(u)$,

$$(2) \quad \lim_{k \rightarrow \infty} \frac{k^{k+1}}{k!} \int_{-\infty}^{\infty} \phi[(c+k^\varepsilon)z - k^\varepsilon] z^k \exp(-kz) dz = \phi(c).$$

2. Remarks. In the following sections $\phi(u)$ will be assumed to be integrable over the interval from $-\infty$ to ∞ , and c will be assumed to be a continuity point of $\phi(u)$. All limits should be understood to be for increasing values of k .

The expression $\delta/(c+k^\varepsilon)$, where δ and ε are positive numbers, occurs frequently. It will be denoted by $\delta(k, \varepsilon)$.

Finally, it may be noted that in terms of the Laplace transform of $\phi(u)$ for real t ,

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$$f(t) = \int_{-\infty}^{\infty} \phi(u) \exp(-tu) du,$$

the inversion formula (2) may be written in the form

$$\lim \frac{(-1)^k}{k!} \left(\frac{k}{c + k^\varepsilon} \right)^{k+1} \frac{d^k}{dt^k} [f(t) \exp(-tk^\varepsilon)]_{t=k/(c+k^\varepsilon)} = \phi(c).$$

3. Preliminary proofs. The results of the following four lemmas will be needed below. Proofs are given for the first two. The second two are proved in a similar way.

LEMMA 1. *If n is any fixed number and $1/3 < \varepsilon < 1/2$, then*

$$\lim k^n [1 + \delta(k, \varepsilon)]^k \exp[-k\delta(k, \varepsilon)] = 0.$$

Proof. If the logarithm of the expression under the limit sign is expanded in powers of $\delta(k, \varepsilon)$, the sum of two of the terms in the expansion approaches $-\infty$ as $k \rightarrow \infty$, while the sum of the rest of the terms is bounded.

LEMMA 2. *If $1/3 < \varepsilon < 1/2$, then*

$$\lim \frac{k^{k+1}}{k!} \int_1^{1+\delta(k, \varepsilon)} z^k \exp(-kz) dz = \frac{1}{2}.$$

Proof. It is well known [1] that

$$\lim \frac{k^{k+1}}{k!} \int_1^{\infty} z^k \exp(-kz) dz = \frac{1}{2}.$$

Therefore, it is sufficient to show that

$$\lim \frac{k^{k+1}}{k!} \int_{1+\delta(k, \varepsilon)}^{\infty} z^k \exp(-kz) dz = 0.$$

Since $z \exp(-z)$ is a decreasing function of z for $z > 1$, the above expression is, for fixed k , no larger than

$$\frac{k^{k+1}}{k!} [1 + \delta(k, \varepsilon)]^{k-1} \exp[-(k-1)(1 + \delta(k, \varepsilon))] \int_{1+\delta(k, \varepsilon)}^{\infty} z \exp(-z) dz.$$

By applying Stirling's formula and Lemma 1, we see that the upper bound approaches zero as k increases.

LEMMA 3. *If n is any fixed number and $0 < \varepsilon < 1/2$, then*

$$\lim k^n [1 - \delta(k, \epsilon)]^k \exp [k\delta(k, \epsilon)] = 0 ,$$

LEMMA 4. If $0 < \epsilon < 1/2$, then

$$\lim \frac{k^{k+1}}{k!} \int_{1-\delta(k, \epsilon)}^1 z^k \exp (-kz) dz = \frac{1}{2} .$$

4. The inversion formula.

THEOREM. If

$$(a) \quad \left| \int_{-\infty}^{-x} \phi(z) dz \right| \leq A \exp (-dx^{2+\alpha})$$

for some positive quantities A, d , and α , and if

$$(b) \quad \max (1/3, 1/(2 + \alpha)) < \epsilon < 1/2,$$

then

$$\lim I_k = \lim \frac{k^{k+1}}{k!} \int_{-\infty}^{\infty} \phi[(c + k^\epsilon)z - k^\epsilon] z^k \exp (-kz) dz = \phi(c) .$$

Proof. For any $\delta > 0$, the infinite interval may be partitioned into the four subintervals $(-\infty, 1 - \delta(k, \epsilon))$, $(1 - \delta(k, \epsilon), 1)$, $(1, 1 + \delta(k, \epsilon))$, and $(1 + \delta(k, \epsilon), \infty)$. I_k may be considered as the sum of four integrals over these intervals, so that we may write

$$I_k = I_k^{(1)} + I_k^{(2)} + I_k^{(3)} + I_k^{(4)} .$$

$I_k^{(1)}$ is understood to represent the integral over $(-\infty, 1 - \delta(k, \epsilon))$ etc.

$$|I_k - \phi(c)| \leq |I_k^{(1)}| + \left| I_k^{(2)} - \frac{\phi(c)}{2} \right| + \left| I_k^{(3)} - \frac{\phi(c)}{2} \right| + |I_k^{(4)}| .$$

We prove first that $I_k^{(1)}$ and $I_k^{(4)}$ approach zero as $k \rightarrow \infty$. For $I_k^{(1)}$, consider first the integral over the interval from 0 to $1 - \delta(k, \epsilon)$. The function $z \exp (-z)$ attains its maximum at the upper endpoint. Therefore an upper bound for the absolute value of this portion of the expression is

$$\frac{k^{k+1}}{k!} [1 - \delta(k, \epsilon)]^k \exp [-k + k\delta(k, \epsilon)] \int_0^{1-\delta(k, \epsilon)} |\phi[(c + k^\epsilon)z - k^\epsilon]| dz ,$$

which approaches zero by Stirling's formula and Lemma 3.

Consider now the integral over the interval from $-\infty$ to 0. Integrating by parts, we find that it is equal to

$$-\frac{1}{c+k^\varepsilon} \frac{k^{k+2}}{k!} \int_{-\infty}^0 F'[(c+k^\varepsilon)z-k^\varepsilon] z^{k+1} (1-z) \exp(-kz) dz,$$

where $F(z) = \int_{-\infty}^z \phi(u) du$. Note that, by the assumption on $F(z)$,

$$|F'[(c+k^\varepsilon)z-k^\varepsilon]| \leq A \exp[-d\{-(c+k^\varepsilon)z+k^\varepsilon\}^{2+\alpha}],$$

which is in turn equal to or less than

$$A \exp[dz(c+k^\varepsilon)k^{\varepsilon(1+\alpha)}].$$

The result of the integration by parts may be written as the difference between two integrals, the first containing z^{k-1} and the second containing z^k . The first integral is no greater in absolute value than

$$\frac{A}{(c+k^\varepsilon)} \frac{k^{k+2}}{k!} \int_{-\infty}^0 |z^{k-1}| \exp[z\{d(c+k^\varepsilon)k^{\varepsilon(1+\alpha)}-k\}] dz.$$

Since $\varepsilon(2+\alpha) > 1$, the coefficient of z in the exponent above is positive for sufficiently large k . Therefore, after some manipulation, this upper bound can be shown to be equal to

$$\frac{A}{(c+k^\varepsilon)} \frac{k^{k+2}}{k!} \cdot \frac{\Gamma(k)}{[d(c+k^\varepsilon)k^{\varepsilon(1+\alpha)}-k]^k},$$

which approaches zero as $k \rightarrow \infty$.

By the same argument, the second integral approaches zero, so that $\lim I_k^{(2)} = 0$.

For $I_k^{(4)}$, observe that since $z \exp(-z)$ is a decreasing function of z for $z > 1$, the expression has the following upper bound for its absolute value:

$$\frac{k^{k+1}}{k!} [1 + \delta(k, \varepsilon)]^k \exp[-k - k\delta(k, \varepsilon)] \int_{1+\delta(k, \varepsilon)}^{\infty} |\phi[(c+k^\varepsilon)z-k^\varepsilon]| dz.$$

Since the integral is bounded, the whole upper bound approaches zero by virtue of Stirling's formula and Lemma 1.

We now prove that

$$\left| \lim I_k^{(3)} - \frac{1}{2} \phi(c) \right| < \frac{1}{2} \eta$$

for any $\eta > 0$. By Lemma 2, it is sufficient to show that

$$\left| \lim \frac{k^{k+1}}{k!} \int_1^{1+\delta(k, \varepsilon)} \{\phi[(c+k^\varepsilon)z-k^\varepsilon] - \phi(c)\} z^k \exp(-kz) dz \right| < \frac{\eta}{2}.$$

Since c is a continuity point of $\phi(u)$, there is a $\delta > 0$ such that if $|(c+k^\varepsilon)z-k^\varepsilon-c| < \delta$, that is, if $|z-1| < \delta(k, \varepsilon)$, then

$$|\phi[(c+k^\varepsilon)z-k^\varepsilon]-\phi(c)| < \eta.$$

For such a δ , the absolute value of the expression above is equal to or less than

$$\eta \lim \frac{k^{k+1}}{k!} \int_1^{1+\delta(k, \varepsilon)} z^k \exp(-kz) dz = \frac{\eta}{2}.$$

By the use of Lemma 4, it may be shown in a similar way that

$$\left| \lim I_k^{(2)} - \frac{1}{2} \phi(c) \right| < \frac{1}{2} \eta.$$

Putting together these results, we have $|\lim I_k - \phi(c)| < \eta$ for any $\eta > 0$, which proves the theorem.

REFERENCE

1. C. V. Widder, *Inversion of the Laplace transform and the related moment problem*, Trans. Amer. Math. Soc. **36** (1934), 107-200.

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