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## **SOME TAUBERIAN THEOREMS**

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1. **Introduction.** The following Tauberian theorem is well known.

**THEOREM A.** *If the sequence  $\{s_n\}$ ,  $n=0, 1, 2, \dots$ , is summable Abel<sup>1</sup> to  $s$  and the sequence  $\{n(s_n - s_{n-1})\}$  is bounded on one side, then  $\{s_n\}$  is convergent to  $s$ .*

Another Tauberian theorem, proved in [4], is

**THEOREM B.** *If the series  $\sum_{n=0}^{\infty} a_n$  is summable Abel to  $s$  and the sequence  $\{n^2(a_{n-1} - a_n)\}$  is bounded on one side, then  $\lim_{n \rightarrow \infty} na_n = 0$ .*

An immediate consequence of Theorem B is the well known proposition that, for a convergent series  $\sum_{n=0}^{\infty} a_n$  with monotonically decreasing terms,  $\lim_{n \rightarrow \infty} na_n = 0$ .

By a well known theorem of Tauber, the series  $\sum_{n=0}^{\infty} a_n$  of Theorem B is convergent and hence the sequence  $\{s_n\}$  of partial sums of the series is summable  $(H, -1)$ , that is,  $\{s_n\}$  is summable by the Hölder method of order  $-1$ , as defined in § 2. Thus Theorem B is equivalent to the following

**THEOREM C.** *If the sequence  $\{s_n\}$ ,  $n=0, 1, 2, \dots$ , is summable Abel to  $s$  and the sequence  $\left\{\binom{n}{2}(s_{n-2} - 2s_{n-1} + s_n)\right\}$  is bounded on one side, then  $\{s_n\}$  is summable by the Hölder method of summability of order  $-1$ .*

As will be shown below both Theorem A and Theorem C are special cases of general results proved in § 5 of this paper.

The Tauberian conditions,

$$\binom{n}{1}(s_{n-1} - s_n) = O_L(1)$$

and

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<sup>1</sup> Concepts and propositions mentioned or used in this paper without definition or proof are to be found in Hardy's book [3].

$$\binom{n}{2}(s_{n-2} - 2s_{n-1} + s_n) = O_L(1),$$

belong to the general class of conditions of the form

$$\binom{n}{k} \Delta^k s_{n-k} = O_L(1),$$

where  $k$  is some fixed nonnegative integer and  $\Delta^k s_r$  is defined by

$$\Delta^k s_r = \sum_{p=0}^k (-1)^p \binom{k}{p} s_{r+p}.$$

In this paper we prove for the Abel transformation Tauberian theorems in which the Tauberian conditions are of the form

$$\binom{n}{k} \Delta^k s_{n-k} = o(1),$$

or  $O(1)$ , or  $O_L(1)$ , as  $n \rightarrow \infty$ . For these theorems see specially § 5.

**2. Some properties of Hausdorff and Hölder transforms.** For all sequences appearing in this paper the index denoting the order of the terms will assume the values  $0, 1, 2, \dots$ . If, in some formulae in this paper, a term appears with a negative value of the index denoting the order of the term, then we shall understand that this term assumes the value zero.

We say that a sequence  $\{t_n\}$  is a *Hausdorff transform*, generated by the sequence  $\{\mu_n\}$ , of the sequence  $\{s_n\}$ , if

$$(1) \quad t_n \equiv \sum_{m=0}^n \binom{n}{m} (\Delta^{n-m} \mu_m) s_m$$

for  $n=0, 1, 2, \dots$ . A Hausdorff transform generated by a sequence  $\{\mu_n\}$  will be called here, for shortness, a  $(\mathfrak{H}, \mu_n)$  transform.

It is known that a necessary and sufficient condition for a sequence  $\{t_n\}$  to be a  $(\mathfrak{H}, \mu_n)$  transform of  $\{s_n\}$  is the existence of

$$(2) \quad \Delta^n t_0 = \mu_n \cdot \Delta^n s_0$$

for  $n=0, 1, 2, \dots$ .

It is easy to see that, if  $\{\lambda_n\}$  is defined by

$$(3) \quad \lambda_n = \Delta^n \mu_0$$

for  $n=0, 1, 2, \dots$ , where  $\{\mu_n\}$  is an arbitrary sequence, then for each pair of nonnegative integers  $p$  and  $q$

$$(4) \quad \Delta^p \lambda_q = \Delta^q \mu_p.$$

If  $\{\lambda_n\}$  is defined by (3) then (2) might be written in the form

$$(5) \quad \Delta^n t_0 = \Delta^n \lambda_0 \cdot \Delta^n s_0$$

for  $n=0, 1, 2, \dots$ . Equation (2) now shows that

$$t_n \equiv \sum_{m=0}^n \binom{n}{m} (\Delta^{n-m} \mu_m) s_m, \quad n=0, 1, 2, \dots,$$

is, by (4) and (5), equal to

$$\sum_{m=0}^n \binom{n}{m} (\Delta^m \lambda_{n-m}) s_m$$

which, by the symmetry of (5) in  $\{\lambda_n\}$  and  $\{s_n\}$ , is equal to

$$\begin{aligned} & \sum_{m=0}^n \binom{n}{m} (\Delta^m s_{n-m}) \lambda_m \\ &= \sum_{m=0}^n \binom{n}{m} (\Delta^m \mu_0) (\Delta^m s_{n-m}), \end{aligned}$$

for  $n=0, 1, 2, \dots$ .

Thus the  $(\mathfrak{S}, \mu_n)$  transform of  $\{s_n\}$  might be defined equivalently by

$$(6) \quad t_n \equiv \sum_{m=0}^n \binom{n}{m} (\Delta^m \mu_0) (\Delta^m s_{n-m}),$$

for  $n=0, 1, 2, \dots$ ; a fact which we use later.

We shall denote, in this paper, by  $\{\mu_n^{(\alpha)}\}$ , where  $\alpha$  is an arbitrary fixed real number, the sequence  $\{(n+1)^{-\alpha}\}$ . The *Hölder transform of order  $\alpha$* ,  $\{h_n^{(\alpha)}\}$  (or, in short, the  $(H, \alpha)$  transform) of a sequence, where  $\alpha$  is a real number, is defined as the  $(\mathfrak{S}, \mu_n^{(\alpha)})$  transform of the original sequence. We say that a sequence  $\{s_n\}$  is *summable Hölder* to  $s$  if it is summable  $(H, \alpha)$  to  $s$  for some real number  $\alpha$ . We say that  $\{s_n\}$  is *bounded Hölder* if it is bounded  $(H, \alpha)$  for some real number  $\alpha$ .

Let  $k$  be a fixed nonnegative integer. It is known that

$$(7) \quad \Delta^{k+1} \mu_n^{(-k)} = 0$$

$$(8) \quad \Delta^k \mu_n^{(-k)} = (-1)^k \cdot k!$$

for  $n=0, 1, 2, \dots$ ; therefore, by (6),

$$(9) \quad h_n^{(-k)} = \sum_{m=0}^k (\Delta^m \mu_0^{(-k)}) \cdot \binom{n}{m} \cdot (\Delta^m s_{n-m})$$

for  $n=0, 1, 2, \dots$ . Equations (9) and (8) immediately yield the identity

$$(10) \quad \binom{n}{k} \cdot \Delta^k s_{n-k} = (-1)^{\binom{k+1}{2}} \cdot \left\{ \prod_{p=0}^k p! \right\}^{-1} \cdot \begin{vmatrix} \mu_0^{(0)} & 0 & 0 & \dots & 0 & h_n^{(0)} \\ \mu_0^{(-1)} & \Delta \mu_0^{(-1)} & 0 & \dots & 0 & h_n^{(-1)} \\ \mu_0^{(-2)} & \Delta \mu_0^{(-2)} & \Delta^2 \mu_0^{(-2)} & \dots & 0 & h_n^{(-2)} \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \mu_0^{(-k)} & \Delta \mu_0^{(-k)} & \Delta^2 \mu_0^{(-k)} & \dots & \Delta^{k-1} \mu_0^{(-k)} & h_n^{(-k)} \end{vmatrix}$$

for  $n=0, 1, 2, \dots$ . If the determinant on the right side of (10) is expanded then we obtain

$$(11) \quad \binom{n}{k} \cdot \Delta^k s_{n-k} = \sum_{p=0}^k a_p^{(k)} \cdot h_n^{(-k+p)}$$

for  $n=0, 1, 2, \dots$ ; where, as is easy to see,

$$(12) \quad \sum_{p=0}^k a_p^{(k)} = 0; \quad a_0^{(k)} \neq 0,$$

for  $k=0, 1, 2, \dots$ . In the rest of this paper we shall denote by  $a_0^{(k)}, \dots, a_k^{(k)}$  the coefficients which appear in (11).

It is known that the Hölder transform of order  $\alpha$  of the Hölder transform of order  $\beta$  of a sequence  $\{s_n\}$  is identical with the Hölder transform of order  $\alpha + \beta$  of  $\{s_n\}$ .

Let  $\{\mu_n\}$  be defined by  $\mu_n = \binom{n}{k}$ ,  $n=0, 1, 2, \dots$ , where  $k$  is a fixed nonnegative integer. It is easy to see that

$$(13) \quad \Delta^p \mu_n = \begin{cases} (-1)^p \binom{n}{k-p} & \text{for } 0 \leq p \leq k \\ 0 & \text{for } p > k. \end{cases}$$

A consequence of (13) is that the sequence  $\left\{ \binom{n}{k} \Delta^k s_{n-k} \right\}$ ,  $n=0, 1, 2, \dots$ , is a Hausdorff transform, generated by  $\left\{ (-1)^k \binom{n}{k} \right\}$ , of the sequence  $\{s_n\}$ .

It is known that the product of two Hausdorff transformations is commutative; therefore, taking one the transformations to be that given by  $\{h_n^{(\alpha)}\}$  and the other to be that given by  $\left\{ \binom{n}{k} \Delta^k s_{n-k} \right\}$  we obtain the following consequence of (11).

LEMMA 1. *Let  $\alpha$  be a real number and  $k$  a nonnegative integer; then, for any sequence  $\{s_n\}$ ,*

$$\binom{n}{k} \Delta^k h_{n-k}^{(\alpha)} = \sum_{p=0}^k \alpha_p^{(k)} \cdot h_n^{(\alpha-k+p)}$$

for  $n=0, 1, 2, \dots$ .

3. A proposition concerning the product of two summability methods and three Tauberian theorems. We shall use later the following proposition (proved by O. Szász in [7]).

**THEOREM D:** *If  $\{s_n\}$  is summable Abel to  $s$  and  $\{t_n\}$  is a regular Hausdorff transform of  $\{s_n\}$ ; then  $\{t_n\}$  is summable Abel to  $s$  too,*

and the three theorems

**THEOREM E.** *If  $\{s_n\}$  is summable Abel to  $s$  and  $\{s_n\}$  is bounded, then  $\{s_n\}$  is summable  $(H, \varepsilon)$  to  $s$  for each  $\varepsilon > 0$ .*

**THEOREM F.** *If  $\{s_n\}$  is summable Abel to  $s$  and  $\{s_n\}$  is bounded on one side, then  $\{s_n\}$  is summable  $(H, 1)$  to  $s$ .*

Theorem E may be deduced from Theorem 92 and Theorem 70 of [3], while Theorem F is Theorem 94 of the same book.

**THEOREM G.** *If  $f(x)$  possesses a finite  $n$ th derivative,  $n \geq 2$ , in the interval  $0 < x < 1$ , and if for some real number  $\alpha$*

$$\begin{aligned} f(x) &= o((1-x)^\alpha), & x \uparrow 1, \\ f^{(n)}(x) &= O_L((1-x)^{\alpha-n}), & x \uparrow 1, \end{aligned}$$

then for all integers  $k$  satisfying  $1 < k < n$ ,

$$f^{(k)}(x) = o((1-x)^{\alpha-k}), \quad x \uparrow 1.$$

If, in Theorem G, we put  $1-x=y^{-1}$ , the theorem becomes a result first proved by N. Obrechhoff in [5] and subsequently generalized by M. Parthasarathy and C. T. Rajagopal in Theorems B and C of [6].

We shall now show the following proposition to be a consequence of Theorem G.

**LFMMA 2.** *Let the real sequence  $\{s_n\}$  be summable Abel to  $s$ , that is*

$$(14) \quad \lim_{x \uparrow 1} (1-x) \sum_{n=0}^{\infty} s_n x^n = s.$$

*If for some nonnegative integer  $k$*

$$(15) \quad \binom{n}{k} \Delta^k s_{n-k} = O_L(1), \quad n \rightarrow \infty,$$

then for all integers  $p$  satisfying  $0 \leq p \leq k$ ,

$$(16) \quad \lim_{x \uparrow 1} (1-x)^{-(k-p-1)} \cdot \sum_{n=p}^{\infty} \binom{n}{p} (\Delta^k s_{n-k}) x^{n-p} = (-1)^{k-p} \binom{k-1}{p} s.$$

*Proof.* The identity

$$\sum_{n=0}^{\infty} s_n x^n = (-1)^r (1-x)^{-r} \cdot \sum_{n=0}^{\infty} (\Delta^r s_{n-r}) x^n$$

for  $r=0, 1, 2, \dots$  combined with (14) yields (16) with  $p=0$ ; that is

$$(17) \quad \sum_{n=0}^{\infty} (\Delta^k s_{n-k}) x^n \asymp (-1)^k s (1-x)^{k-1}, \quad x \uparrow 1.$$

Taking the  $k$ th derivative of the left side of (17) and using (15) we obtain

$$(18) \quad \begin{aligned} \frac{d^k}{dx^k} \left\{ \sum_{n=0}^{\infty} (\Delta^k s_{n-k}) x^n \right. &= k! \cdot \sum_{n=k}^{\infty} \binom{n}{k} (\Delta^k s_{n-k}) x^{n-k} \\ &= O_L \left( \sum_{n=k}^{\infty} x^{n-k} \right), & x \uparrow 1, \\ &= O_L((1-x)^{-1}), & x \uparrow 1. \end{aligned}$$

The validity of (16), for all integers  $p$  satisfying  $0 < p < k$ , follows now from (17) and (18) by an appeal to Theorem G with

$$f(x) = \sum_{n=0}^{\infty} \Delta^k s_{n-k} x^n - (-1)^k s (1-x)^{k-1}, \quad \alpha = k-1, n=k, k=p.$$

**4. A Tauberian inequality for power series.** In this section we prove one of the fundamental steps used in proving the main results of this paper. This step is the following.

**THEOREM 1.** *Let  $p$  be a fixed nonnegative integer. If for some real or complex sequence  $\{s_n\}$ ,*

$$\overline{\lim}_{n \rightarrow \infty} \left| \binom{n}{p+1} \cdot \Delta^{p+1} s_{n-p-1} \right| \equiv S^{(p+1)} < +\infty,$$

then, for  $x=1-(m+1)^{-1}$ ,

$$(19) \quad \begin{aligned} \overline{\lim}_{m \rightarrow \infty} \left| -(1-x)^{-p} \cdot \Delta^p s_{m-p} - \sum_{r=0}^p (1-x)^{r-p} \cdot \sum_{n=0}^{\infty} \binom{n}{r} x^{n-r} \cdot \Delta^{p-1} s_{n-p-1} \right| \\ \leq \rho_p \cdot \overline{\lim}_{n \rightarrow \infty} \left| \binom{n}{p+1} \cdot \Delta^{p+1} s_{n-p-1} \right|, \end{aligned}$$

where  $\rho_p$  is independent of  $\{s_n\}$ .

The case  $p=0$  of Theorem 1 is well known. See for instance, inequality (15) of H. Hadwiger's paper [2].

The proof of Theorem 1 requires the following auxiliary proposition.

LEMMA 3. For any pair  $m, n$  of integers satisfying  $m \geq 1, n \geq 0$ , and for  $0 \leq x \leq 1$ , we have

$$0 \leq 1 - \sum_{p=0}^{m-1} \binom{n}{p} (1-x)^p x^{n-p} \leq \binom{n}{m} (1-x)^m$$

where we suppose  $\binom{n}{p} = 0$  if  $p > n$ .

*Proof.* By the Taylor expansion

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^n}{n!} f^{(n)}(a + \theta(b-a)),$$

$$0 < \theta < 1,$$

we obtain, by choosing  $b=1, a=1-x$  ( $0 \leq x \leq 1$ ) and  $f(t) = t^n$ ,

$$1 = \sum_{p=0}^{m-1} \binom{n}{p} (1-x)^p x^{n-p} + \binom{n}{m} (1-x)^m (x + \theta(1-x))^{n-m}, \quad 0 < \theta < 1.$$

Hence, for the stated values (in the theorem) of  $m, n$  and  $x$ ,

$$0 \leq 1 - \sum_{p=0}^{m-1} \binom{n}{p} (1-x)^p x^{n-p} \leq \binom{n}{m} (1-x)^m.$$

*Proof of Theorem 1.* We have

$$(20) \quad - (1-x)^{-p} \cdot \Delta^p s_{m-p} - \sum_{r=0}^p (1-x)^{r-p} \cdot \sum_{n=0}^{\infty} \binom{n}{r} x^{n-r} \cdot \Delta^{p+1} s_{n-p-1}$$

$$= (1-x)^{-p} \sum_{n=0}^m \left\{ 1 - \sum_{r=0}^p \binom{n}{r} (1-x)^r x^{n-r} \right\} \cdot \Delta^{p+1} s_{n-p-1}$$

$$- (1-x)^{-p} \sum_{n=m+1}^{\infty} \left\{ \sum_{r=0}^p \binom{n}{r} (1-x)^r x^{n-r} \right\} \cdot \Delta^{p+1} s_{n-p-1}$$

$$\equiv I_1 + I_2.$$

Lemma 3 yields

$$|I_1| \leq (1-x)^{-p} \sum_{n=0}^m (1-x)^{p+1} \left| \binom{n}{p+1} \cdot \Delta^{p+1} s_{n-p-1} \right|.$$

Now, for  $x = 1 - (m+1)^{-1}$ ,



$$(21) \quad \overline{\lim}_{m \rightarrow \infty} |I_1| \leq \overline{\lim}_{n \rightarrow \infty} \left| \binom{n}{p+1} \cdot \Delta^{p+1} s_{n-p-1} \right|.$$

For each  $\varepsilon > 0$  there exists an integer  $m_0(\varepsilon)$  such that, for every  $m > m_0(\varepsilon)$ ,

$$\left| \binom{m}{p+1} \cdot \Delta^{p+1} s_{m-p-1} \right| < S^{(p+1)} + \varepsilon.$$

We suppose now  $m > m_0(\varepsilon)$ ; then

$$(22) \quad |I_2| \leq (S^{(p+1)} + \varepsilon) \cdot \sum_{n=m+1}^{\infty} \left\{ \sum_{r=0}^n \binom{n}{r} (1-x)^{-(p-r)} x^{n-r} \right\} \cdot \binom{n}{p+1}^{-1} \\ = (p+1)(S^{(p+1)} + \varepsilon) \sum_{r=0}^p \binom{p}{r} (1-x)^{-(p-r)} \cdot \sum_{n=m+1}^{\infty} x^{n-r} \cdot \Delta^{p-r} (n-p)^{-1}.$$

It is easy to show that for  $0 \leq r \leq p$  we have

$$(23) \quad (1-x)^{-(p-r)} \sum_{n=m+1}^{\infty} x^{n-r} \cdot \Delta^{p-r} (n-p)^{-1} \\ = x^{m+1-r} \sum_{q=0}^{p-r-1} (-1)^q (1-x)^{-(p-r-q)} \cdot \Delta^{p-r-q-1} (m+1-q-p)^{-1} \\ + (-1)^{p-r} \sum_{n=m+1+p-r}^{\infty} (n-p)^{-1} x^{n-p},$$

and for  $r=p$

$$(24) \quad (1-x)^{-(p-r)} \sum_{n=m+1}^{\infty} x^{n-r} \cdot \Delta^{p-r} (n-p)^{-1} = \sum_{n=m+1}^{\infty} (n-p)^{-1} x^{n-p}.$$

If we choose  $x=1-(m+1)^{-1}$  and apply (23) and (24) to (22) we infer easily that, for  $p \geq 0$ , there exists a positive constant  $\lambda_p$  which is independent of the sequence  $\{s_n\}$  and such that

$$\overline{\lim}_{m \rightarrow \infty} |I_2| \leq \lambda_p \cdot (S^{(p+1)} + \varepsilon).$$

Since  $\varepsilon > 0$  is chosen arbitrarily we infer that, for  $x=1-(m+1)^{-1}$ ,

$$(25) \quad \overline{\lim}_{m \rightarrow \infty} |I_2| \leq \lambda_p \cdot S^{(p+1)}.$$

Combining (20), (21) and (25) we see that our proposition is proved.

A consequence of Theorem 1 which will be used later is the following proposition.

**LEMMA 4.** *Let  $\{s_n\}$  be summable Abel to  $s$ , and let there be a fixed positive integer  $k$  such that*

$$(26) \quad \binom{n}{k} \Delta^k s_{n-k} = o(1), \quad n \rightarrow \infty.$$

Then (i)  $\binom{n}{p} \Delta^p s_{n-p} = o(1)$ ,  $n \rightarrow \infty$ , for  $1 \leq p \leq k$ , (ii)  $\{s_n\}$  is convergent to  $s$ .

*Proof.* If  $k=1$ , we have to prove conclusion (ii) alone, and this follows from Theorem 1 with  $p=0$ . If  $k \geq 2$ , then, by Theorem 1 and (26), for  $x=1-(m+1)^{-1}$ ,

$$(27) \quad \overline{\lim}_{m \rightarrow \infty} \left| -(1-x)^{-k+1} \cdot \Delta^{k-1} s_{m-k+1} - \sum_{r=0}^{k-1} (1-x)^{r-k+1} \sum_{n=0}^{\infty} \binom{n}{r} x^{n-r} \cdot \Delta^k s_{n-k} \right| = 0.$$

The Abel summability of  $\{s_n\}$ , (26) and Lemma 3 show that

$$(28) \quad \lim_{x \uparrow 1} \sum_{r=0}^{k-1} (1-x)^{r-k+1} \sum_{n=0}^{\infty} \binom{n}{r} x^{n-r} \cdot \Delta^k s_{n-k} = \sum_{r=0}^{k-1} (-1)^{k-r} \binom{k-1}{r} \cdot 0 \\ = (-1)^k \cdot 0 \cdot (1-1)^{k-1} \\ = 0.$$

(28) and (27) show, for  $x=1-(m+1)^{-1}$ , that

$$\lim_{m \rightarrow \infty} |(1-x)^{-(k-1)} \cdot \Delta^{k-1} s_{m-(k-1)}| = 0.$$

The last fact shows, immediately, that

$$\binom{n}{k-1} \cdot \Delta^{k-1} s_{n-(k-1)} = o(1), \quad n \rightarrow \infty.$$

Thus we reduced  $k$  in (26) by one, and by such a reduction (repeated if necessary) prove conclusion (i). Finally we derive conclusion (ii) from conclusion (i) as already stated.

**5. Some Tauberian theorems.** The main result of this paper is the following.

**THEOREM 2.** *Be  $k$  some fixed positive integer. A necessary and sufficient condition for  $\{s_n\}$  to be summable  $(H, k)$  is that  $\{s_n\}$  should be summable Abel to  $s$  and  $\lim_{n \rightarrow \infty} \binom{n}{k} \cdot \Delta^k s_{n-k} = 0$ .*

*Proof.* Proof of the sufficiency part. From the convergence of  $\{s_n\}$  to  $s$  and the relations  $\binom{n}{p} \Delta^p s_{n-p} = o(1)$ ,  $n \rightarrow \infty$ , for  $p=1, \dots, k$  (from Lemma 4) rewritten in the form (11), we get

$$\lim_{n \rightarrow \infty} h_n^{(-p)} = s$$

for  $p=1, 2, \dots, k$ , successively; which proves the sufficiency part of the

theorem. The proof of the necessity part of our proposition follows from (11) and the fact that the limits

$$\lim_{n \rightarrow \infty} h_n^{(-k)}, \lim_{n \rightarrow \infty} h_n^{(-k+1)}, \dots, \lim_{n \rightarrow \infty} h_n^{(0)}$$

exist and are all equal to  $s$ .

Now we prove three interesting consequences of Theorem 2.

**THEOREM 3.** *A necessary and sufficient condition for a sequence  $\{s_n\}$  to be summable  $(H, \alpha)$ , for some real value of  $\alpha$ , is that  $\{s_n\}$  should be summable Abel and that the sequence*

$$\left\{ \binom{n}{k} \Delta^k s_{n-k} \right\}, \quad n=0, 1, 2, \dots$$

*should be summable  $(H, \alpha+k)$  to zero for some fixed positive integer  $k$ .*

*Proof.* The necessity of the Abel summability of  $\{s_n\}$  is obvious. The necessity of the  $(H, \alpha+k)$  summability of

$$\left\{ \binom{n}{k} \Delta^k s_{n-k} \right\}, \quad n=0, 1, 2, \dots,$$

to zero follows from Lemma 1 (if we replace  $\alpha$  there by  $\alpha+k$ ). Thus we have proved the necessity part of our theorem. The sufficiency part of our theorem is proved as follows. Suppose, first, that  $\alpha \geq -k$ . Then, by Theorem D, the sequence

$$\{h_n^{(\alpha+k)}\}, \quad n=0, 1, 2, \dots,$$

is summable Abel to the same sum as the original sequence  $\{s_n\}$ , hence, using Theorem 2 with  $\{h_n^{(\alpha+k)}\}$  instead of  $\{s_n\}$ , which is justified by Lemma 1 with  $\alpha$  replaced by  $\alpha+k$ ,  $\{s_n\}$  is summable  $(H, \alpha)$ ; which proves the sufficiency part of our theorem for  $\alpha \geq -k$ . In the case  $\alpha < -k$ ,

$$\left\{ \binom{n}{k} \Delta^k s_{n-k} \right\}$$

being summable  $(H, \alpha+k)$  to zero, is necessarily convergent to zero; and so, by Theorem 2,  $\{s_n\}$  is summable  $(H, -k)$ , or  $\{h_n^{(\alpha)}\}$  is summable  $(H, -\alpha-k)$ , and consequently summable Abel too. Thus, by Theorem D,  $\{h_n^{(\alpha+k)}\}$  is also summable Abel and the proof can be completed as in the case  $\alpha \geq -k$ .

The case  $k=1$  is a special case of Theorem (9.4) of [1], with  $\beta=\alpha+1$  there.

**THEOREM 4.** *Be  $k$  an arbitrary fixed nonnegative integer. If a*

sequence  $\{s_n\}$  is summable Abel to  $s$  and the sequence

$$\left\{ \binom{n}{k} \Delta^k s_{n-k} \right\}$$

is bounded  $(H, \alpha+k)$  for some real number  $\alpha$ , then  $\{s_n\}$  is summable  $(H, \alpha+\varepsilon)$  for each  $\varepsilon > 0$ .

The case  $k=1$  of the last theorem is the special case  $\beta=\alpha+1$  of Theorem (9.5) (for Abel summability) of [1].

*Proof.* First suppose  $\alpha \geq 0$ . Then, by Theorem D, (11) and (12),

$$v_n \equiv \sum_{p=0}^k a_p^{(k)} h_n^{(\alpha+p)} = O(1), \quad n \rightarrow \infty,$$

and  $\{v_n\}$  is summable Abel to zero. Therefore, by Theorem E,  $\{v_n\}$  is summable  $(H, \varepsilon)$ , for each  $\varepsilon > 0$ , to zero, or  $\left\{ \binom{n}{k} \Delta^k s_{n-k} \right\}$  is summable  $(H, \alpha+k+\varepsilon)$  to zero, and the conclusion follows by Theorem 3. If  $\alpha < 0$ , we apply the preceding argument to the  $(H, -\alpha)$  transform of  $\{v_n\}$  which is clearly  $O(1)$ , as  $n \rightarrow \infty$ , and summable Abel to zero. Thus we find that the  $(H, -\alpha)$  transform of  $\{v_n\}$  is summable  $(H, \varepsilon)$  to zero, for each  $\varepsilon > 0$ , or that  $\{v_n\}$  is summable  $(H, -\alpha+\varepsilon)$  to zero and hence summable Abel to zero. Since  $v_n = O(1)$ ,  $\{v_n\}$  is, by Theorem E, summable  $(H, \varepsilon)$  to zero and the proof is completed exactly as in the case  $\alpha \geq 0$ .

**THEOREM 5.** *Be  $k$  an arbitrary fixed positive integer. If a sequence  $\{s_n\}$  is summable Abel to  $s$  and the sequence*

$$\left\{ \binom{n}{k} \Delta^k s_{n-k} \right\}$$

*is bounded  $(H, \alpha+k)$  on one side, then  $\{s_n\}$  is summable  $(H, \alpha+1)$  to  $s$ .*

The case  $k=1$  is the special case  $\beta=\alpha+1$  of Theorem (9.6) of [1].

The proof of Theorem 5 is exactly the same as that of Theorem 4. But now we have to use Theorem F in place of Theorem E.

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#### REFERENCES

1. A. Amir (Jakimovski), *Some relations between the methods of summability of Abel, Borel, Cesàro, Hölder and Hausdorff*, J. Analyse Math. **3** (1953/4). 346-381.
2. H. Hadwiger, *Über ein Distanztheorem Bei der A Limitierung*, Comm. Math. Helv.,

**16** (1943/4), 209–214.

3. G. H. Hardy, *Divergent Series*, Oxford University Press.

4. Kien Kwong Chen, *Some one-sided Tauberian theorems*, Anais Acad. Brasil, Ci **17** (1945), 249–259.

5. N. Obrechhoff, *Sur une formule pour les différences divisées et de leur dérivées*, C. R. Acad. Bulgare Sci., **2** (1949), No. 1, 5–8.

6. M. Parthasaraty and C. T. Rajagopal, *A theorem on the Riemann-Liouville integral*, Math. Z., **55** (1951), 84–91.

7. O. Szász, *On the product of two summability methods*, Ann. Soc. Pol. Math., **25** (1952), 75–84 (1953).

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Richard Davis Anderson, <i>Zero-dimensional compact groups of homeomorphisms</i> .....	797
Hans-Joachim Bremermann, <i>Holomorphic functionals and complex convexity in Banach spaces</i> .....	811
Hugh D. Brunk, G. M. Ewing and W. R. Utz, <i>Minimizing integrals in certain classes of monotone functions</i> .....	833
Philip David, <i>Uniqueness theory for asymptotic expansions in general regions</i> .....	849
Paul Erdős and Harold Nathaniel Shapiro, <i>On the least primitive root of a prime</i> .....	861
Watson Bryan Fulks, <i>Regular regions for the heat equation</i> .....	867
William Robert Gaffey, <i>A real inversion formula for a class of bilateral Laplace transforms</i> .....	879
Ronald Kay Gettoor, <i>On characteristic functions of Banach space valued random variables</i> .....	885
Louis Guttman, <i>Some inequalities between latent roots and minimax (maximin) elements of real matrices</i> .....	897
Frank Harary, <i>The number of dissimilar supergraphs of a linear graph</i> .....	903
Edwin Hewitt and Herbert S. Zuckerman, <i>Structure theory for a class of convolution algebras</i> .....	913
Annon Jakimovski, <i>Some Tauberian theorems</i> .....	943
C. T. Rajagopal, <i>Simplified proofs of "Some Tauberian theorems" of Jakimovski</i> .....	955
Paul Joseph Kelly, <i>A congruence theorem for trees</i> .....	961
Robert Forbes McNaughton, Jr., <i>On the measure of normal formulas</i> .....	969
Richard Scott Pierce, <i>Distributivity in Boolean algebras</i> .....	983
Calvin R. Putnam, <i>Continuous spectra and unitary equivalence</i> .....	993
Marvin Rosenblum, <i>Perturbation of the continuous spectrum and unitary equivalence</i> .....	997
V. N. Singh, <i>Certain generalized hypergeometric identities of the Rogers-Ramanujan type</i> .....	1011
Peter Swerling, <i>Families of transformations in the function spaces <math>H^p</math></i> .....	1015