CONTINUOUS SPECTRA AND UNITARY EQUIVALENCE

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1. Introduction. In the differential equation

\[ (px')' + (\lambda + f(t))x = 0, \]

let \( \lambda \) denote a real parameter and let \( p(t) > 0 \) and \( f(t) \) be continuous real-valued functions on \( 0 \leq t < \infty \). Suppose that (1) is of the limit-point type, so that (1) and a linear homogeneous boundary condition

\[ x(0) \cos \alpha + x'(0) \sin \alpha = 0, \quad 0 \leq \alpha < \pi, \]

determine a boundary value problem with a spectrum \( S = S_\alpha \) on the half-line \( 0 \leq t < \infty \); cf. [7]. The continuous spectrum \( C_\alpha \) (if it exists) is determined by a continuous monotone nondecreasing basis function \( \rho_\alpha(\lambda) \). It is known that the set of cluster points, \( S' \), of \( S_\alpha \) is independent of \( \alpha \), [7, p. 251]; the question as to whether the corresponding assertion for \( C_\alpha \) is also true was raised by Weyl [7, 7. 252] but is still undecided.

Consider the self-adjoint operators \( H_\alpha = \int \lambda dE_\alpha(\lambda) \) (all of which are extensions of the same symmetric operator) belonging to the various boundary value problems determined by (1) and (2); cf. for example, [2]. The object of this note is to shown that any two \( H_\alpha \) possessing purely continuous (hence, in view of the above remark concerning \( S' \), necessarily identical) spectra are unitarily equivalent, at least if certain conditions concerning the nature of the sets \( C_\alpha \) and the basis functions \( \rho_\alpha(\lambda) \) are met. In fact there will be proved the following.

**THEOREM (\(*\)).** Suppose that there exist two (distinct) values \( \alpha_1 \) and \( \alpha_2 \) \( (0 \leq \alpha_1 < \alpha_2 < \pi) \) such that, for each of the two boundary value problems determined by (1) and (2), the following three conditions are satisfied:

(i) \( S_{a_1} \neq (-\infty, \infty) \),

(ii) the point spectrum is empty, and

(iii) \( \rho_{a_2}(\lambda) \) is absolutely continuous. Then \( H_{a_1} \) and \( H_{a_2} \) are unitarily equivalent.

The condition (i) of \( (*) \) surely holds if, for instance, \( f \) is bounded or even bounded from below on \( 0 \leq t < \infty \). It should be noted however that every (real) \( \lambda \) belongs to an \( S_\alpha \) for some \( \alpha \) (depending on \( \lambda \)); [1].

For other results on the continuous spectra of boundary value pro-

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problems with absolutely continuous basis functions (on certain intervals), see [4].

The proof of (*) in § 2 will depend upon the following result of M. Rosenblum [5] concerning perturbations of self-adjoint operators: Let the self-adjoint operators \( A_k = \int \lambda dE(\lambda) \) (for \( k=1, 2, 3 \)) satisfy \( A_1 - A_2 = A_3 \). Suppose, in addition, that \( A_3 \) is completely continuous and such that \( \int |\lambda| dE_3(\lambda) \) has a finite trace while \((E_1x, y)\) and \((E_2x, y)\) are absolutely continuous functions of \( \lambda \) for arbitrary \( x \) and \( y \) in Hilbert space. Then \( A_1 \) and \( A_2 \) are unitarily equivalent.

2. Proof of (*). In the sequel, the index \( \alpha_k \) will be replaced by \( k \). It is clear from the assumptions that there exists some real value \( \lambda = \lambda^* \) not belonging to \( S_k \) for \( k=1, 2 \). Consequently, since \( f(t) \) can be replaced in (1) by \( f(t) + \lambda^* \), it can be supposed without loss of generality that \( \lambda = 0 \) is not in either of the sets \( S_k \). Then the operators \( H_k^{-1} \), where

\[
H_k^{-1} = \int A_k dE_k(\lambda) = \int dF_k(\lambda) \quad (F_k(\lambda) = E_k(\lambda^{-1}))
\]

are bounded, self-adjoint integral operators with kernels \( G_k(s, t) \) on \( 0 \leq s, t < \infty \); cf. for example, [2], [7]. Furthermore,

\[
G_1(s, t) - G_2(s, t) = cg(s)g(t),
\]

where \( c \) denotes a constant and \( g(t) \) is a function of class \( L^2[0, \infty) \); cf. [7, p. 251]. Thus \((H_1^{-1} - H_2^{-1})x\) is a multiple of \( g \) for every element \( x \) of class \( L^2[0, \infty) \). Hence \( H_1^{-1} - H_2^{-1} \) is a multiple of a one-dimensional projection operator; in particular, \( H_1^{-1} - H_3^{-1} \), corresponding to \( A_3 \), satisfies the trace condition on that operator mentioned at the end of § 1.

In view of the assumptions (ii) and (iii) of (*), it follows from the formulas relating the basis functions \( \rho_k(\lambda) \) to the projections \( E_k(\lambda) \) (cf., for example, [2]) that \( ||E(\lambda)x|| \) is an absolutely continuous function of \( \lambda \) for every \( x \) in the Hilbert space; therefore \((E_k(\lambda)x, y)\), hence also \((F_k(\lambda)x, y)\), is absolutely continuous for every pair \( x, y \) of this space. According to the above mentioned theorem of Rosenblum, it now follows that the operators \( H_1^{-1} \) (hence also the \( H_k \)) are unitarily equivalent, and the proof of (*) is now complete.

3. Consider the special case of (1) in which \( f \equiv 0 \). It is readily seen that there are no eigenvalues for either of the boundary value problems determined by \( x'' + \lambda x = 0 \) and the respective boundary conditions \( x(0) = 0 \) and \( x'(0) = 0 \). (These boundary conditions correspond to \( \alpha = 0 \),
π/2 in \((2^\circ)\); in a somewhat more general connection, cf. [3, p. 792]). Thus, in each case, there is a purely continuous spectrum consisting of the half-line \(0 \leq \lambda < \infty\). Moreover, the basis functions, which, in this instance, are even known explicitly [6, p. 59] are absolutely continuous. Consequently, Theorem (*) is applicable and shows that the self-adjoint operators belonging to the above mentioned boundary value problems are unitarily equivalent.

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