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**CERTAIN GENERALIZED HYPERGEOMETRIC IDENTITIES
OF THE ROGERS-RAMANUJAN TYPE**

V. N. SINGH

CERTAIN GENERALIZED HYPERGEOMETRIC IDENTITIES OF THE ROGERS-RAMANUJAN TYPE

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1. Introduction. In a recent paper H. L. Alder [1] has obtained a generalization of the well-known Rogers-Ramanujan identities. In this paper I have deduced the above generalizations as simple limiting cases of a general transformation in the theory of hypergeometric series given by Sears [5]. This method, besides being much simpler than that of Alder, also gives a simple form for the polynomials $G_{k,\mu}(x)$ given by him. In Alder's proof the polynomials $G_{k,\mu}(x)$ had to be calculated for every fixed k with the help of certain difference equations but in the present case we get directly the general form of these polynomials.

2. Notation. I have used the following notation throughout the paper. Assuming $|x| < 1$, let

$$(a)_s \equiv (a; s) = (1-a)(1-ax)\cdots(1-ax^{s-1}), \quad (a)_0 = 1$$

$$\prod^s (a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_t) = \frac{(a_1; s)(a_2; s)\cdots(a_r; s)}{(b_1; s)(b_2; s)\cdots(b_t; s)}$$

$$\prod(a) = \prod_{n=0}^{\infty} (1-ax^n)$$

$$\prod(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_t) = \frac{\prod(a_1)\prod(a_2)\cdots\prod(a_r)}{\prod(b_1)\prod(b_2)\cdots\prod(b_t)}$$

$$K_s = \frac{(k; s)(x\sqrt{k}; s)(-x\sqrt{k}; s)}{(x; s)(\sqrt{k}; s)(-\sqrt{k}; s)}$$

$$K_{s,r} = K_s^r \frac{(x^{-r}; s)}{(kx^{r+1}; s)} x^{rs}$$

$$S_{n,n-1} = \sum_{r_n=0}^{r_{n-1}-1} \frac{k^{r_n} x^{r_n^2} (x^{r_{n-1}-r_n+1}; r_n)}{(x; r_n)}, \quad S_{1,0} = \sum_{r_1=0}^r \frac{k^{r_1} x^{r_1^2} (x^{r-r_1+1}; r_1)}{(x; r_1)}$$

$$T_{n,M} = \sum_{t_n=0}^{\left[\frac{M-n-1}{M-n} - t_{n-1} \right]} \frac{(x^{t_{n-1}-2t_n+1}; 2t_n) x^{-2t_n(t_{n-1}-t_n)}}{(x; t_n)(x^{t_{n-1}-2t_n+1}; t_n)}, \quad (M=3, 4, 5, \dots)$$

where $[a]$ denotes the integral part of a .

The numbers $s, r, r_1, r_2, \dots, t, t_1, t_2, \dots$ are either zero or positive

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integers. r_0 and t_0 , wherever they occur, have been replaced simply by r and t respectively. Empty products are to mean unity.

3. Sears [5, § 4] has proved the following theorem :

$$(3.1) \quad \sum_{s=0}^{\infty} x^{\frac{1}{2}s(s-1)} (kx/a_1 a_2)^s \prod (a_1, a_2; x, kx/a_1, kx/a_2) \theta_s$$

$$= \prod (kx, kx/a_1 a_2; kx/a_1, kx/a_2) \sum_{r=0}^{\infty} (kx/a_1 a_2)^r \prod (a_1, a_2; x, kx)$$

$$\times \sum_{t=0}^r \frac{(x^{-r}; t)(-1)^t x^{rt}}{(kx^{r+1}; t)(x; t)} \theta_t,$$

wrere $|kx/a_1 a_2| < 1$, $|x| < 1$ and θ_s is any sequence. The theorem holds provided only that the series on the left converges.

Take

$$\theta_s = \prod \left[\begin{matrix} k, x\sqrt{k}, -x\sqrt{k}, a_3, a_4, \dots, a_{2M+1}; \\ \sqrt{k}, -\sqrt{k}, kx/a_3, kx/a_4, \dots, kx/a_{2M+1} \end{matrix} \right]$$

$$\times \frac{(k^{M-1} x^{M-1})^s}{(a_3 a_4 \dots a_{2M+1})^s} x^{\frac{1}{2}s(1-s)}, \quad (M=1, 2, 3, \dots)$$

Then

$$(3.2) \quad \sum_{s=0}^{\infty} K_s \frac{(a_1; s)(a_2; s) \dots (a_{2M+1}; s)}{(kx/a_1; s)(kx/a_2; s) \dots (kx/a_{2M+1}; s)} \frac{(k^M x^M)^s}{(a_1 a_2 \dots a_{2M+1})^s}$$

$$= \prod (kx, kx/a_1 a_2; kx/a_1, kx/a_2) \sum_{r=0}^{\infty} (kx/a_1 a_2)^r \prod (a_1, a_2; x, kx)$$

$$\times \sum_{t=0}^r K_{t,r} \frac{(a_3; t)(a_4; t) \dots (a_{2M+1}; t)(-1)^t x^{\frac{1}{2}t(1-t)} (k^{M-1} x^{M-1})^t}{(kx/a_3; t)(kx/a_4; t) \dots (kx/a_{2M+1}; t)(a_3 a_4 \dots a_{2M+1})^t}.$$

Now let $a_1, a_2, \dots, a_{2M+1} \rightarrow \infty$ in (3.2). Then we get

$$(3.3) \quad \sum_{s=0}^{\infty} K_s (-1)^s k^M x^{\frac{1}{2}s\{2(M+1)s-1\}}$$

$$= \prod (kx) \sum_{r=0}^{\infty} \frac{k^r x^{r^2}}{(x; r)(kx; r)} \sum_{t=0}^r K_{t,r} k^{(M-1)t} x^{(M-1)t^2}.$$

And in (3.2) if we take $(M-1)$ for M , $a_1 = x^{-r}$ and let $a_2, a_3, \dots, a_{2M-1}$ tend to ∞ , we have

$$(3.4) \quad \sum_{t=0}^r K_{t,r} k^{(M-1)t} x^{(M-1)t^2}$$

$$= (kx; r) \sum_{t=0}^r \frac{k^t x^{t^2} (x^{r-t+1}; t)}{(x; t)} \sum_{s=0}^t K_{s,t} k^{(M-2)s} x^{(M-2)s^2}.$$

On repeated application of (3.4) on the right-hand side of (3.3) it follows that

$$\{\prod (kx)\}^{-1} \sum_{s=0}^{\infty} K_s(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}} = \sum_{r=0}^{\infty} \frac{k^r x^{r^2}}{(x; r)^{M-2}} \prod_{n=1}^{M-2} S_{n,n-1},$$

there being $(M-2)$ terminating series on the right since

$$(3.5) \quad \sum_{s=0}^t K_{s,t} = 0$$

by Watson's transformation [(2); § 8.5 (2)] of a terminating ${}_8\phi_7$ into a Saalschützian ${}_4\phi_3$.

Now it is easily verified that

$$\prod_{n=1}^{M-2} S_{n,n-1}$$

can, by suitable rearrangements, be simplified to

$$\sum_{t_1=0}^{(M-2)r} \frac{k^{t_1} x^{t_1^2}}{(x; t_1)} (x^{r-t_1+1}; t_1)^{\left[\begin{smallmatrix} M-3 \\ M-2 \end{smallmatrix} t_1 \right]} \frac{(x^{t_1-2t_2+1}; 2t_2) x^{-2t_2(t_1-t_2)}}{(x; t_2)(x^{r-t_1+1}; t_2)^{M-2}} \prod_{n=3}^{M-2} T_{n,M},$$

where $t_h = r_h + r_{h+1} + \dots + r_{M-2}$, $(h=1, 2, \dots, M-2)$.

Thus on putting $r + t_1 = t$, we finally have

$$(3.6) \quad \{\prod (kx)\}^{-1} \sum_{s=0}^{\infty} K_s(-1)^s k^{Ms} x^{\frac{1}{2}s\{(2M+1)s-1\}} \\ = \sum_{t=0}^{\infty} \frac{k^t x^{t^2}}{(x; t)} \sum_{t_1=0}^{\left[\begin{smallmatrix} M-2 \\ M-1 \end{smallmatrix} t \right]} \frac{(x^{t-2t_1+1}; 2t_1) x^{-2t_1(t-t_1)}}{(x; t_1)^{M-2}} \prod_{n=2}^{M-2} T_{n,M}.$$

This is a k -cum- M generalization of the Rogers-Ramanujan identities. For any assigned values of M and t , the repeated terminating series can, by dividing out by the denominator factors, be evaluated as polynomials in x .

Let us now write

$$(3.7) \quad G_{M,t}(x) = x^{t^2} \sum_{t_1=0}^{\left[\begin{smallmatrix} M-2 \\ M-1 \end{smallmatrix} t \right]} \frac{(x^{t-2t_1+1}; 2t_1) x^{-2t_1(t-t_1)}}{(x; t_1)^{M-2}} \prod_{n=2}^{M-2} T_{n,M}.$$

Then, as usual, for $k=1$ and $k=x$ respectively, the left-hand side of (3.6) can be expressed as a product by means of Jacobi's classical identity

$$(3.8) \quad \sum_{n=-\infty}^{\infty} (-1)^n x^{n^2} z^n = \prod_{n=1}^{\infty} (1-x^{2n-1}z)(1-x^{2n-1}/z)(1-x^{2n})$$

and we get Alder's generalization of the first and second Rogers-

Ramanujan identities in the form

$$(3.9) \quad \prod_{n=0}^{\infty} \frac{(1-x^{(2M+1)n+M})(1-x^{(2M+1)n+M+1})}{(1-x^{(2M+1)n+1})(1-x^{(2M+1)n+2}) \dots (1-x^{(2M+1)n+2M})} = \sum_{t=0}^{\infty} \frac{G_{M,t}(x)}{(x; t)}$$

and

$$(3.10) \quad \prod_{n=0}^{\infty} \frac{1}{(1-x^{(2M+1)n+2})(1-x^{(2M+1)n+3}) \dots (1-x^{(2M+1)n+2M-1})} = \sum_{t=0}^{\infty} \frac{x^t G_{M,t}(x)}{(x; t)}$$

where $G_{M,t}(x)$ is given by (3.7). The polynomials $G_{M,t}(x)$ can be seen by easy verification to be identical with $G_{R,\mu}(x)$ of Alder.

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Added in Proof. If in (3.2) we take $a_1 = -\sqrt{kx}$, make $a_2, a_3, \dots, a_{2M+1}$ tend to ∞ , and proceed as in § 3, we get for $k=1$ and $k=x$ the respective identities

$$\begin{aligned} & \prod_{n=1}^{\infty} \frac{(1-x^{2Mn-(M-\frac{1}{2})})(1-x^{2Mn-(M+\frac{1}{2})})(1-x^{2Mn})}{(1-x^n)} \\ &= \{ \prod (-x^{\frac{1}{2}}) \}^{-1} \sum_{t=0}^{\infty} \frac{x^{\frac{1}{2}t^2} (-x^{\frac{1}{2}})_t}{(x)_t} \sum_{t_1=0}^{[M-\frac{1}{2}t]} x^{-t_1(t-\frac{3}{2}t_1)} \\ & \quad \times \frac{(x^{t-2t_1+1})_{2t_1}}{(-x^{\frac{1}{2}+t-t_1})_{t_1}} \prod_{n=2}^{M-2} T_{n,M} \end{aligned}$$

and

$$\begin{aligned} & \prod_{n=1}^{\infty} \frac{(1-x^{2Mn-1})(1-x^{2Mn-(2M-1)})(1-x^{2Mn})}{(1-x^n)} \\ &= \{ \pi(-x) \}^{-1} \sum_{t=0}^{\infty} \frac{x^{\frac{1}{2}t(t+1)} (-x)_t}{(x)_t} \sum_{t_1=0}^{[M-\frac{1}{2}t]} x^{\frac{1}{2}t_1} x^{-t_1(t-\frac{3}{2}t_1)} \\ & \quad \times \frac{(x^{t-2t_1+1})_{2t_1}}{(-x^{1+t-t_1})_{t_1}} \prod_{n=2}^{M-2} T_{n,M}. \end{aligned}$$

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