FAMILIES OF TRANSFORMATIONS IN THE FUNCTION SPACES $H^p$

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I. Introduction

Let the interior of the unit circle be denoted by $\mathcal{D}$; and let the set of functions single-valued and analytic in $\mathcal{D}$ be denoted by $\mathcal{A}$.

It is well known that certain subsets of $\mathcal{A}$ can be made into Banach spaces by the introduction of suitable norms. In particular, if $f \in \mathcal{A}$, and if, for $1 \leq p \leq \infty$,

$$
\mathcal{M}_p(f; r) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad p < \infty
$$

$$
\mathcal{M}_\infty(f; r) = \sup_{|z| < r} |f(z)|, \quad p = \infty
$$

and if $\sup_{r < 1} \mathcal{M}_p(f; r) < \infty$, then $f$ is said to be in the set $H^p$. Also, $H^p$ is a Banach space with

$$
\|f\|_{H^p} = \sup_{r < 1} \mathcal{M}_p(f; r)
$$

A proof of these statements, together with a discussion of many properties of the spaces $H^p$, can be found in [8].

This paper is concerned with certain transformations in the spaces $H^p$.

Let $\omega(z)$ be a function of $z$ which is analytic in $\mathcal{D}$ and such that $|\omega(z)| < 1$ for $z \in \mathcal{D}$. If $f \in \mathcal{A}$, then so is the function defined by $f[\omega(z)]$. For $f \in \mathcal{A}$, we define

$$
T_\omega f = g \iff f(\omega(z)) = g(z) \text{ for } z \in \mathcal{D}.
$$

$T_\omega$ is clearly an additive, homogeneous transformation.

It is well known [4] that if $f \in H^p$ and $\omega(0) = 0$, then $T_\omega f \in H^p$ and $\|T_\omega f\| \leq \|f\|$. In other words, if $\omega(0) = 0$, then $T_\omega \in [H^p]$ (the set of all linear bounded transformations on $H^p$ to $H^p$), and $\|T_\omega\| \leq 1$. Our first task is to prove the following.

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$^1$ In the following, all statements about $H^p$ refer to $1 \leq p \leq \infty$ unless further qualified.
THEOREM I.1. If $\omega \in \mathbb{A}$ and $|\omega(z)|<1$ for $z \in \Delta$, and if $|\omega(0)|=\alpha<1$, then $T_\omega \in [H^p]$ and $\|T_\omega\| \leq \left( \frac{1+\alpha}{1-\alpha} \right)^{1/p}$ There is at least one such $\omega$ for which the equality holds.

Proof. For $p=\infty$, the theorem is trivial. For $1 \leq p < \infty$, a simple proof (for which the author is indebted to the referee) is as follows.

For $f \in H^p$, let $u$ be the least harmonic majorant of $|f|^p$ in $\Delta$ (see [6]). Then $T_\omega u$ is a harmonic majorant of $|T_\omega f|^p$. Also,

$$\|f\| = \{u(0)\}^{1/p} \quad \text{and} \quad \|T_\omega f\| \leq \{(T_\omega u)(0)\}^{1/p} = \{u(\beta)\}^{1/p}$$

where $\beta=\omega(0)$. The Poisson integral for $u$ shows that

$$u(\beta) \leq u(0)\left( \frac{1+|\beta|}{1-|\beta|} \right)$$

Putting $\alpha=|\beta|$, it follows that

$$\|T_\omega f\| \leq \|f\| \left( \frac{1+\alpha}{1-\alpha} \right)^{1/p}$$

To complete the proof, we note that the following statement holds. Define the transformation $L_\alpha (0 \leq \alpha < 1)$ by

$$L_\alpha f(z) = f(z + \alpha)z$$

Then the function

$$f(z) = \left( \frac{z+1}{z-1} \right)^p$$

is an eigenfunction of $L_\alpha$: $L_\alpha f = \lambda f$, belonging to the eigenvalue

$$\lambda = \left( \frac{1+\alpha}{1-\alpha} \right)^p$$

provided $|\Re \gamma| < 1/p$. This follows trivially from the fact that $f \in H^p$ provided $|\Re \gamma| < 1/p$.

The result stated in Theorem I.1 can be sharpened as follows.

COROLLARY I.1. For any $\omega$ ($\omega \in \mathbb{A}$, mapping $\Delta$ into or onto itself),

$$\|T_\omega\| \leq \inf_{\gamma \in \Delta} \left\{ \left( \frac{1+|\zeta|}{1-|\zeta|} \right) \left( \frac{1+|\gamma|}{1-|\gamma|} \right) \left( \frac{1+|\Im \omega(\gamma, \zeta)|}{1-|\Im \omega(\gamma, \zeta)|} \right) \right\}^{1/p}$$
where

\[ \Gamma_\omega(\eta, \zeta) = \frac{\omega(\eta) + \zeta}{1 + \zeta \omega(\eta)} \]

Proof. For \( \zeta \in J \), define \( L_\zeta \) by

\[ L_\zeta f(z) = f\left(\frac{z + \zeta}{1 + \zeta z}\right) \]

Then

\[ T_\omega = L_{-\eta} L_\omega L_\eta L_{-\zeta} \]

where

\( \eta \in J, \ \zeta \in J \)

so that

\[ \| T_\omega \| \leq \| L_{-\eta} \| \| L_{-\zeta} \| \| L_\eta T_\omega L_\zeta \| \]

Now, \( \frac{z - \zeta}{1 - \zeta z} \) takes \( 0 \) into \( -\zeta \); \( \frac{z - \eta}{1 - \eta z} \) takes \( 0 \) into \( -\eta \);

and

\[ \omega\left(\frac{z + \eta}{1 + \eta z}\right) + \zeta \left/ 1 + \frac{\omega\left(\frac{z + \eta}{1 + \eta z}\right)}{1 + \zeta \omega(\eta)} \right. \]

Applying Theorem I.1, we obtain (1.4).

We are thus assured that a transformation \( T_\omega \) defined by \( T_\omega f(z) = f[\omega(z)] \) is a member of \( [H^p] \), \( 1 \leq p \leq \infty \). § II is devoted to a study of semigroups and groups of these transformations. Section III contains a discussion of two examples which illustrate the theorems of § II.

II. Families of Transformations in \( H^p \)

A. Definitions and preliminary results. Consider a family of functions \( \{\omega(z; t)\} \) —also denoted by \( \{\omega_t(z)\} \) —where \( z \in J \) and \( t \) belongs to a set \( \mathcal{J} \) of complex numbers. The individual functions will be denoted by \( \omega(z; t) \) or by \( \omega_t(z) \), according to convenience.

Let the set \( \mathcal{J} \) satisfy the following conditions.

(CII.1) (i) If \( t_1, t_2 \in \mathcal{J} \), then \( t_1 + t_2 \in \mathcal{J} \).

(ii) \( \mathcal{J} \) contains the origin and some ray originating at the origin.

(iii) Every two points in \( \mathcal{J} \) can be connected by a path\(^3\) in \( \mathcal{J} \).

\(^3\) Here a path is defined to mean a finite number of rectifiable Jordan arcs joined together; see [3, pp 13, 14].
Let the family \( \{\omega(z; t)\} \) satisfy the following conditions:

(CII.2) (i) For each \( t \in \mathcal{T} \), \( \omega_t \in \mathfrak{A} \), and \( \omega_t \) maps \( \Delta \) into (or onto) itself.

(ii) For \( t_1, t_2 \in \mathcal{T} \), and \( z \in \Delta \),
\[
\omega_{t_2} \omega_{t_1}(z) = \omega_{t_1}(\omega_{t_2}(z)) = \omega_{t_1 + t_2}(z)
\]

(iii) \( \omega(z; 0) = z \) for \( z \in \Delta \).

(iv) For each \( z \in \Delta \), \( \omega(z; t) \) is differentiable with respect to \( t \) for \( t \in \mathcal{T} \). Also, \( \omega(z; t) \) is differentiable with respect to \( t \) for \( t \in \mathcal{T} \).

Then \( P(z) = \frac{\partial}{\partial t} \omega(z; t)|_{t=0} \).

We can immediately state the following.

**Lemma II.1.** For fixed \( z \in \Delta \),

\[(II.1) \quad \frac{\partial}{\partial t} [\omega(z; t)] = P[\omega(z; t)] \]

**Proof.** \( \omega[\omega(z; t); h] = \omega(z; t + h) \) for \( t, h \in \mathcal{T} \).

Therefore
\[
\omega(z; t + h) - \omega(z; t) = \omega[\omega(z; t); h] - \omega(z; t)
\]
\[
= \omega[\omega(z; t); h] - \omega[\omega(z; t); 0]
\]

Letting \( h \to 0 \) (in \( \mathcal{T} \)), we obtain (II.1).

The family of transformations \( \{T_{\omega_t}\} \) defined by (I.3) with \( \omega = \omega_t \) will henceforth be denoted simply by \( \{T_t\} \). This family forms a semi-group (possibly a group) of linear bounded transformations in the spaces \( H^p \).

(II.2) \( A f = \lim_{t \to 0} \frac{T_t f - f}{t} \), \( f \in H^p \)

the limit taken in the strong sense in \( H^p \). The domain of \( A \), denoted
by $\mathcal{D}(A)$, is defined to be the subset of $H^p$ for which the limit in (II.2) exists as $t \to 0$, $t \in \mathcal{T}$ (the limit to be the same for all modes of approach within $\mathcal{T}$ to 0).

It follows from (II.2) that, for $f \in \mathcal{D}(A)$, and each $z \in \Delta$,

$$A(f(z) = \lim_{t \to t_0} f(z; t) - f(z)$$

This is true since, for fixed $z \in \Delta$, $f(z)$ is a bounded linear functional of $f$, [7].

Now

$$A(f(z) = \lim_{t \to t_0} f(z; t) - f(z)$$

$$= \lim_{t \to t_0} f(z; t) - f(z; 0)$$

$$= \frac{\partial}{\partial t} f(z; t)_{t=0} = f'(z; t) \frac{\partial}{\partial t} w(z; t)_{t=0}$$

or

$$A(f(z) = P(z) f'(z)$$

It is thus clear that $\mathcal{D}(A)$ is contained in the subset of $H^p$ consisting of those elements $f$ for which $f'(z) P(z)$ defines an element of $H^p$.

B. Differentiability properties of the family $\{T_t\}$

**Theorem II.1.** Let $f$ be in $H^p$, and $t_0$ be in $\mathcal{T}$; let $g(z) = P(z) f'(z)$ and suppose that

(i) There exists a neighborhood $\mathcal{N}_{t_0}$ of $t_0$ and a positive constant $M$ such that every point $t$ of $\mathcal{N}_{t_0}$ can be connected to $t_0$ by a polygonal line in $\mathcal{N}_{t_0} \cap \mathcal{T}$ of length $\leq M |t_0 - t|$;

(ii) $T_t g \in H^p$ for $t \in \mathcal{N}_{t_0} \cap \mathcal{T}$;

(iii) $\|T_t g - T_{t_0} g\| \to 0$ as $t \to t_0$ ($t \in \mathcal{T}$).

Then, $T_t f$ is strongly differentiable with respect to $t$ at $t_0$ and

$$\frac{d}{dt} T_t f|_{t=t_0} = T_{t_0} g$$

Before giving the proof, the following formal derivation might be of interest.
This is however not a rigorous proof, even when \( f \in \mathcal{D}(A) \), since \( s=t_{0} \) may not be in \( \mathcal{M} \) for all \( t \in \mathcal{M}_{t_{0}} \). A rigorous proof is as follows.

Let \( f[\omega(z; t)] = h(z; t) \) and let

\[
D(z; t; t_{0}) = \frac{h(z; t) - h(z; t_{0}) - T_{t_{0}}g(z)}{t - t_{0}}
\]

If \( z = re^{\theta} \), and if \( \frac{\partial}{\partial t} h(z; t) \) is denoted by \( h_{t}(z; t) \), then, from (II.1),

\[
D(z; t; t_{0}) = \frac{h(re^{\theta}; t) - h(re^{\theta}; t_{0}) - h_{t}(re^{\theta}; t_{0})}{t - t_{0}}
\]

\[
= \frac{1}{t - t_{0}} \int_{t_{0}}^{t} [h_{t}(re^{\theta}; \tau) - h_{t}(re^{\theta}; t_{0})] d\tau
\]

where \( t \) is chosen in \( \mathcal{M}_{t_{0}} \) and the integral is taken along a polygonal line in \( \mathcal{M}_{t_{0}} \) connecting \( t \) and \( t_{0} \) and of length \( \leq M|t - t_{0}| \).

First suppose that \( 1 \leq \rho < \infty \). Then

\[
\mathcal{M}_{\rho}(D; r) = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \left[ D(re^{\theta}; t; t_{0}) \right]^{\rho} d\theta \right\}^{1/\rho}
\]

\[
= \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \frac{1}{t - t_{0}} \int_{t_{0}}^{t} [h_{t}(re^{\theta}; \tau) - h_{t}(re^{\theta}; t_{0})] d\tau \right]^{\rho} d\theta \right\}^{1/\rho}
\]

Let \( \tau = \tau(s) \), \( 0 \leq s \leq 1 \), \( \tau(0) = t_{0} \), \( \tau(1) = t \). Here \( s \) is a constant times the arc length. Then [4], [1]

\[
\mathcal{M}_{\rho}(D; r) = \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \frac{1}{t - t_{0}} \int_{t_{0}}^{t} [h_{t}(re^{\theta}; \tau) - h_{t}(re^{\theta}; t_{0})] \tau'(s) ds \right]^{\rho} d\theta \right\}^{1/\rho}
\]

\[
\leq \frac{1}{|t - t_{0}|} \int_{0}^{1} \tau'(s) \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} \left[ h_{t}(re^{\theta}; \tau) - h_{t}(re^{\theta}; t_{0}) \right]^{\rho} d\theta \right\}^{1/\rho}
\]

Hence,

\[
\| D \| = \left\| \frac{T_{t}f - T_{t_{0}}f}{t - t_{0}} - T_{t_{0}}g \right\| = \sup_{\rho \leq 1} \mathcal{M}_{\rho}(D; r)
\]
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\[ \leq \frac{1}{|t-t_0|} \int_0^1 |\tau'(s)| \left[ \sup_{r \leq 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| h_r(re^{i\theta}; \tau) - h_r(re^{i\theta}; t_0) \right|^{1/p} \right\} \right] ds \]

\[ = \frac{1}{|t-t_0|} \int_0^1 |\tau'(s)| \| T_t g - T_{t_0} g \| ds \leq \sup_{0 \leq s \leq 1} |T_t g - T_{t_0} g| \]

Now, by (iii), as $t \to t_0$, the quantity $\sup_{0 \leq s \leq 1} |T_t g - T_{t_0} g|$ goes to zero. Thus $\| D \| \to 0$ as $t \to t_0$.

For $p = \infty$, the proof follows similar lines.

**COROLLARY II.1-1.** Let $f$ be in $H^p$, $t_0$ be in $\mathcal{J}$, and let $g(z) = P(z)f'(z)$. Suppose condition (i) of Theorem II.1 holds and in addition, suppose that

(a) \( |\omega(z; t_0)| < r < 1 \) for \( z \in \mathcal{A} \)

(b) \( \omega(z; t) \) is continuous with respect to $t$ at $t_0$, uniformly in $z$ for $z \in \mathcal{A}$.

Then, $T_t f$ is differentiable with respect to $t$ at $t_0$ and (II.5) holds.

**Proof.** By (b), there exists a neighborhood $\mathcal{N}_{r_0}$ of $t_0$ such that $|\omega(z; t)| < r' < 1$ for $z \in \mathcal{A}$, $t \in \mathcal{N}_{r_0} \cap \mathcal{J}$.

Now, $g(z)$ is analytic in $\mathcal{A}$. Therefore for $t \in \mathcal{N}_{r_0} \cap \mathcal{J}$, $T_t g(z) = g[\omega(z; t)]$ is bounded in $\mathcal{A}$ and therefore $T_t g \in H^p$.

Also, $T_t g(z)$ is continuous with respect to $t$ at $t_0$, uniformly in $z$ for $z \in \mathcal{A}$. Hence $\sup_{z \in \mathcal{A}} |T_t g(z) - T_{t_0} g(z)| \to 0$ as $t \to t_0$.

**THEOREM II.2.** Suppose

(i) Condition (i) of Theorem II.1 holds for $t_0 = 0$;

(ii) $\| T_t f - f \| \to 0$ as $t \to 0$ ($t \in \mathcal{J}$) for every $f \in H^p$.

Then, $\mathcal{D}(A)$, the domain of the generator $A$ (defined by II.2), is the set of elements $f \in H^p$ for which $g(z) = f'(z)P(z)$ defines an element $g$ of $H^p$.

**Proof.** Let $\mathcal{D}$ denote the set of elements $f \in H^p$ such that $g(z) = f'(z)P(z)$ defines an element $g$ of $H^p$. We already know (last paragraph of IIA) that $\mathcal{D}(A) \subset \mathcal{D}$. To show that $\mathcal{D} \subset \mathcal{D}(A)$, one must verify conditions (ii) and (iii) of Theorem II.1 for $f \in \mathcal{D}$, $t_0 = 0$.

Since $f \in \mathcal{D}$ implies $g \in H^p$, it follows from Theorem I.1 that $T_t g \in H^p$ for all $t \in \mathcal{J}$. Also, condition (iii) of Theorem II.1 is obtained for $t_0 = 0$ by applying condition (ii) of Theorem II.2 to the function $g$.

Equation (II.5) becomes
$Af=g$ where $g(z)=\frac{d}{dz}P(z)f(z)$.

**Theorem II.3.** Under conditions (i) and (ii) of Theorem II.2, $A$ is a closed transformation. Also $\mathcal{D}(A)$ is dense in $H^\alpha$.

**Proof.** Let $f_n$ be in $\mathcal{D}(A)$; $f_n \to f$ (in the norm of $H^\alpha$) $Af_n \to g \in H^\alpha$ (in the norm of $H^\alpha$). Then [7]

$$f_n(z) \to f(z), \quad P(z)f_n'(z) \to g(z)$$

uniformly on compact subsets of $\Delta$, that is, $g(z)=P(z)f'(z)$ for $z \in \Delta$.

Therefore, since $g \in H^\alpha$, then, by Theorem II.2, $f \in \mathcal{D}(A)$ and $Af = g$. See [2, Chap. 11] for the fact that $\mathcal{D}(A)$ is dense in $H^\alpha$.

**C. The family of transformations generated by a given operator of the form** $Af(z)=P(z)f'(z)$. Suppose $P$ is a given function in $\mathcal{U}$. The following question arises: Is there a set $\mathcal{T}$ in the complex plane and a set of functions $\{\omega_i\}$ satisfying, respectively, conditions CII.1 and CII.2? If so, how, knowing just $P(z)$, can one determine the family $\{\omega_i\}$ and the maximum set $\mathcal{T}$?

To investigate these questions, additional conditions will be imposed on the given function $P(z)$. First,

(CII.3) $1/P(z)$ is analytic in $\Delta$ except, possibly, for a single pole.

Let the function $Q(z)$ be defined by

$$(\text{II.9}) \quad Q(z)=\int_{z_0}^{z} \frac{d\zeta}{z_0-P(\zeta)} \quad z_0, \ z \in \Delta$$

The path of integration is chosen in $\Delta$ so as not to pass through any singularity of $1/P(z)$; also, $z_0$ is chosen so as not to be a singularity of $1/P(z)$. $Q(z)$ may be a many-valued function.

$Q(z)$ depends on the choice of $z_0$; however, as will become clear below, it is not worthwhile to express this dependence in the notation. Clearly, all definitions of $Q$ (corresponding to different choices of $z_0$) differ from each other by additive constants.

The following property of $Q$ is worth noting.

Let $z_1$ and $z_2$ be in $\Delta$, and not singularities of $1/P(z)$; let $Q^{(1)}(z_1)$, $Q^{(2)}(z_1)$ be two values of $Q$ at $z=z_1$; and let $Q^{(1)}(z_2)-Q^{(2)}(z_2)=h$. Let $Q^{(1)}(z_2)$ be a value of $Q$ at $z=z_2$. There exists a value of $Q$ at $z=z_2$, which may be denoted by $Q^{(2)}(z_2)$, such that $Q^{(1)}(z_2)-Q^{(2)}(z_2)=h$. This is clear from the definition of $Q$ and from (CII.3).
We shall further assume:

(CII.4) If $z_1$ and $z_2$ are in $\Delta$, are not singularities of $1/P(z)$, and $z_1 \neq z_2$, then $Q(z_1) \neq Q(z_2)$.

This may, of course, be regarded as a condition on $P(z)$.

Now suppose $P \in \mathcal{P}$ is given satisfying (CII.3) and (CII.4), and that a set $\mathcal{S}$ and a family $\{\omega_t\}$ exist satisfying (CII.1) and (CII.2). From (II.1) and (CII.2-iii), regarding $z$ as fixed for the moment, one can write

$$\frac{d}{dt} \omega(z; t) = P[\omega(z; t)]$$
$$\omega(z; 0) = z$$
$$t \in \mathcal{S}$$
$$z \in \Delta$$

Let $z$ be fixed in $\Delta$ and not a singularity of $1/P(z)$. Then, from (II.10), $\omega(z; t)$ must satisfy

$$Q[\omega(z; t)] = Q(z) + t.$$  

Now, for fixed $t \in \mathcal{S}$, $\omega(z; t)$ must be an analytic function of $z$ in $\Delta$, mapping $\Delta$ into itself.

Let $I_q$ be the image under $Q$ of $\Delta$ (excluding the possible singularity of $1/P(z)$). The set $I_q$ includes all values of $Q(z)$ which can be obtained by integrating in (II.9) along paths which are entirely in $\Delta$. If $\omega(z; t)$, for fixed $t \in \mathcal{S}$, is defined for all $z \in \Delta$, and such that $|\omega(z; t)| < 1$, then (II.11) implies that this $t$ must translate $I_q$ into a subset of itself: $I_q + t \subset I_q$.

Let $\mathcal{S}_q$ be the set of translations of $I_q$ into or onto itself. (Clearly $\mathcal{S}_q$ does not depend on the choice of $z_0$ in defining $Q$.) Then $\mathcal{S} \subset \mathcal{S}_q$.

On the other hand if $P$ being given, $\mathcal{S}_q$ contains a subset $\mathcal{S}^*$ satisfying conditions (CII.1), then a family $\{\omega_t\}$ satisfying (CII.2) exists (with $t \in \mathcal{S}^*$).

Define, for $t \in \mathcal{S}^*$, $z \in \Delta$,

$$\omega(z; t) = \begin{cases} Q^{-1}[Q(z) + t], & z \text{ not a singularity of } P(z) \\ z, & z \text{ a singularity of } P(z) \end{cases}$$

where $Q^{-1}$ denotes the function inverse to $Q$.

This definition defines $\omega$ uniquely. If $Q(z)$ refers to a particular branch of $Q$, then $\omega$ is uniquely determined (in $\Delta$) because of (CII.4); moreover, by the property of $Q$ mentioned on p. it is seen that the same point $\omega$ is defined no matter what branch of $Q$ is used in (II :12).

\[4\] $P \in \mathcal{P}$ and satisfying (CII.3) and (CII.4).
It is also clear that \( \omega(z; t) \) does not depend on the choice of \( z_0 \).

The function \( \omega(z; t) \) thus defined is analytic in \( z \) for each \( t \in \mathcal{T}^* \). This is clear if \( z \) is not a singularity of \( 1/P(z) \). If \( z_i \) is a singularity of \( 1/P(z) \) in \( \Delta \), it is necessary to show that \( \omega(z; t) \) is (for fixed \( t \)) continuous at \( z = z_i \); that is, (from II.12) \( \omega_i(z) \rightarrow z_i \) as \( z \rightarrow z_i \).

Since \( z_i \) is a pole of \( 1/P(z) \), one can say, by the definition of \( Q \), that there exist points \( \omega_i(z) \) approaching \( z_i \) as \( z \rightarrow z_i \), such that (II.12) is satisfied. But, by (CII.4), these points are the only ones in \( \Delta \) for which (II.12) is satisfied.

The other conditions of (CII.2) are readily verified for the functions \( \omega(z; t) \) as defined by (II.12).

The preceding results may be summed up as follows.

**Theorem** II.4. Let \( P(z) \) be in \( \mathcal{A} \), satisfying (CII.3) and (CII.4). Let \( Q(z) \) be defined by (II.9); let \( I_Q \) be the image of \( \Delta \) under \( Q \), let \( \mathcal{T}_Q \) be the set of translations of \( I_Q \) into or onto itself.

Then, there exists a set \( \mathcal{T} \) and a family \( \{ \omega_t \} \) satisfying (CII.1) and (CII.2), if and only if \( \mathcal{T}_Q \) contains a subset \( \mathcal{T}^* \) satisfying (CII.1). The maximum set \( \mathcal{T} \) is the "direct sum" of all subsets of \( \mathcal{T}_Q \) which satisfy (CII.1). Here "direct sum" is defined as follows: If \( \{ G_a \} \) is a collection of subsets of the complex plane, each containing the origin, the direct sum of the sets \( \{ G^a \} \) is defined to be the set consisting of all elements of the form \( t = t_1 + \cdots + t_n \) where \( n \) is a finite (positive) integer and where \( t_i \in \bigcup_a G_a \).

The last statement follows from the fact that the direct sum of subsets of \( \mathcal{T}_Q \) satisfying (C.II.1) is also a subset of \( \mathcal{T}_Q \) which satisfies (C.II.1).

One result of the previous theorem is the following.

**Theorem** II.5. If \( P(z) \in \mathcal{A} \), satisfying (CII.3) and (CII.4), and if there exists a set \( \mathcal{T} \) and a family \( \{ \omega_t \} \) satisfying (CII.1) and (CII.2), then \( 1/P(z) \) can have only a pole of first order in \( \Delta \).

**Proof.** If \( 1/P(z) \) had a pole of order higher than the first, then \( I_Q \) would have a bounded (and non-null) complement; therefore \( \mathcal{T}_Q \) would consist only of the point \( t = 0 \).

Thus, if \( \zeta_0 \) is the singularity of \( 1/P(z) \), then \( Q(z) \) can be written

\[
Q(z) = q_0 \ln (z - \zeta_0) + Q_1(z)
\]

where \( Q(z) \) is analytic in \( \mathcal{T} \).

Theorems II.6 and II.7 refer to families of transformations generated by \( P(z) \) satisfying (CII.3) and (CII.4).
THEOREM II.6. If \( \omega(z_1; t) = z_1, \) \( z_1 \in \mathcal{J}, \) for \( t \neq 2\pi ikq_0, \) \( k=0, \pm 1, \pm 2, \ldots, \) then \( z_1 = \zeta_0. \)

Proof. \( Q[\omega(z; t)] = Q(z) + t \) for \( z \neq \zeta_0. \)
Therefore \( Q[z_1] = Q[z_1] + t \) if \( z \neq \zeta_0. \)
Therefore \( t = 2\pi ikq_0, \) \( k=0, \pm 1, \ldots. \)

THEOREM II.7. If \( \omega(z_1; t) = \omega(z_2; t), \) \( t \in \mathcal{J}, \) then \( z_1 = z_2. \)

Proof. Suppose first that \( z_1, z_2 \neq \zeta_0. \) Then \( \omega(z_1; t) = \omega(z_2; t) \) would imply \( Q(z_1) = Q(z_2) \) or, by (CII.4), \( z_1 = z_2. \) On the other hand, if, say, \( z_1 = \zeta_0, \) then \( \omega(z_1, t) = z_1 = \omega(z_2; t) \) and so \( z_2 = z_1 \) by Theorem II.6.
Thus, conditions (CII.3) and (CII.4) when imposed on the function \( P(z) \) imply that the family \( \{\omega_t\} \) is a family of schlicht functions.

It is clear that the functions \( \omega_t \) as well as the set \( \mathcal{J} \) are unaltered if the definition of \( Q \) is altered by the addition of an arbitrary constant.

It is also easy to see that multiplying \( Q \) (that is, multiplying \( 1/P \)) by a constant \( c \neq 0 \) yields essentially the same family of transformations:
Let \( \mathcal{J}, \{\omega_t\} \) correspond to \( P(z) \) and let \( \mathcal{J}', \{\omega'_t\} \) correspond to \( \frac{1}{c} P(z). \) (Here the primes do not, of course, imply differentiation.) Then clearly, \( \mathcal{J}' = c \mathcal{J}. \) Also, for \( t' \in \mathcal{J}', \)
\[
cQ[\omega'(z; t')] = cQ(z) + t',
\]
or
\[
Q[\omega'(z; t')] = Q(z) + \frac{t'}{c},
\]
so that
\[
(\text{II.14}) \quad \omega'(z; t') = \omega\left(Z; \frac{t'}{c}\right); \quad t' \in \mathcal{J}', \quad \frac{t'}{c} \in \mathcal{J}.
\]

In other words, there is a one-to-one correspondence between the transformations corresponding to \( P(z) \) and those corresponding to \( \frac{1}{c} P(z); \) the correspondence is given by (II.14).

Now consider, for \( t \in \mathcal{J} \cap I_q, \) the parameter defined by

(II.15) \[ \beta = Q^{-1}(t) \quad t \in \mathcal{J} \cap I_q \]
Then \( \beta \in \mathcal{J} \) and (II.12) becomes, writing \( \omega[z; t(\beta)] \) simply as \( \omega(z; \beta), \)

(II.16) \[ \omega(z; \beta) = Q^{-1}[Q(z) + Q(\beta)], \]
\[ z, \beta \in \mathcal{J}. \]
Here $\beta$ is defined on $Q^{-1}[\mathcal{S} \cap I_0]$.

It is always possible to define $Q$ in such a way that $\mathcal{S} \subset I_0$ and therefore $\mathcal{S} \cap I_0 = \mathcal{S}$. In such a case, (II.15) and (II.16) hold for all $t \in \mathcal{S}$. For example, in defining $Q$ by (II.9), it is clear that $Q(z_0) = 0$ for $z_0 \in A$. Thus, for $Q$ defined as in (II.9) with $z_0 \in A$, we have $\beta = Q^{-1}(t) = \omega(z_0; t)$.

It is, however, often possible and more convenient to define $Q$ such that $\mathcal{S}$ is the closure of $I_Q$. It is also often possible to extend the definition of $Q$ to the boundary of $A$ in such a way that the boundary of $A$ goes (under $Q$) into the boundary of $I_Q$. (An example of this is given by the family of transformations studied in the next section.) In such cases, (II.15) holds for all $t \in \mathcal{S}$ and, in (II.16), $\beta$ may be a point on the boundary of $A$.

The law of composition of the transformations $T_{\omega_2} = T_{\beta_2}$ in terms of the parameter $\beta$ is

\begin{equation}
\begin{cases}
T_{\beta_1}T_{\beta_2} = T_{\beta_3} \\
\beta_3 = \omega(\beta_1; \beta_2)
\end{cases}
\tag{II.17}
\end{equation}

This can be shown as follows.

\[
\omega[\omega(z; t_1); t_2] = \omega(z; t_1 + t_2),
\]

so

\[
\omega[\omega(z; \beta_1); \beta_2] = \omega[z; t = Q(\beta_1) + Q(\beta_2)] = \omega[z; \beta = \omega(\beta_1; \beta_2)].
\]

By simply looking at the set $I_q$, one is usually able to determine many of the properties of the family $\{T_t\}$. For example, one may determine (a) whether or not such a family exists for the given $P(z)$; (b) what the maximum parameter domain $\mathcal{S}$ is; (c) whether $\{T_t\}$ is a group or a semigroup; (d) which of the functions $\omega_i$ transform $A$ onto itself and which transform $A$ into but not onto itself;

D. Possible applications. The above results provide the basis for obtaining a variety of theorems by rephrasing known results in the theory of transformations in Banach space in terms of transformations in the function spaces $H^p$ of the kind studied above. Three possible categories of results are:

(a) Representations of the transformations $T_t$ in terms of the generator $A$ or the resolvent of $A$ ([2] contains many such formulas).

(b) Application of results in the theory of analytic Banach-space-

\footnote{The addition of a constant to $Q$ changes $I_q$ but leaves $\mathcal{S}$ unaltered.}
valued functions of a complex variable ([2], [7], [9])
(c) Other theorems concerning properties of semigroups or groups of transformations in Banach space.

III. Two Special Cases

A. The family \( \{ T_w \} \) defined by \( T_w f(z) = f(wz) \), \( |w| \leq 1 \).

Let

\[
\begin{align*}
P(z) &= -z \\
Q(z) &= \int_1^z -d\zeta = -\ln z.
\end{align*}
\]

Then \( I_q \) is the open right half plane: \( \Re(z) > 0 \). \( \mathcal{T}_q \) is the closed right half plane: \( \Re(z) \geq 0 \). Clearly, \( \mathcal{T}_q \) itself satisfies conditions (CII.1) and is therefore the maximum domain \( \mathcal{T} \) of the parameter \( t \). We have

\[
\begin{align*}
&\omega(z; t) = ze^{-t} \\
&\text{or, if we let} \\
&w = e^{-t}
\end{align*}
\]

then, writing \( \omega[z; t(w)] \) simply as \( \omega(z; w) \),

\[
\begin{align*}
&\omega(z; w) = wz \\
&z \in \mathcal{D}, \ |w| \leq 1
\end{align*}
\]

The corresponding family of transformations \( \{ T_w \} \) is then given by

\[
T_w f = g
\]

where

\[
g(z) = f(wz)
\]

The generator \( A \) is defined for those \( f \in H^p \) for which the limit

\[
Af(z) = \lim_{w \to 1} \frac{T_w f - f}{1 - w} \quad |w| \leq 1
\]

exists in the \( H^p \) norm. Thus,

\[
Af(z) = -zf'(z) \quad \text{for } f \in \mathcal{D} \ (A).
\]

For \( 1 \leq p < \infty \), \( \mathcal{D} \ (A) \) is the set of functions \( f \in H^p \) for which \( f'(z) \) defines an element of \( H^p \). This follows from Theorem II.2. The crucial point in applying Theorem II.2 is in verifying condition (ii) of

\[6 \text{ Here } z_0 = 1 \text{ is not in } \mathcal{D}, \text{ but in this case this is immaterial.} \]
that theorem. This amounts to the following. Let \( h \) be in \( H^p \) \( (1 \leq p < \infty) \), and let \( T_w h(z) = h(wz) \) for \( |w| \leq 1 \). Then \( T_w h \rightarrow h \) in the norm of \( H^p \) as \( w \rightarrow 1 \) in the closure of \( \Delta \). It is not difficult to prove this.

Also, for \( 1 \leq p < \infty \), \( A \) is a closed operator with domain dense in \( H^p \).

For \( p = \infty \), (III.7) still holds, but one cannot verify condition (ii) of Theorem II.2 and it is easily seen that \( \mathcal{D}(A) \) is not dense in \( H^\infty \).

**B. The family \( \{L_a\} \) defined by**

Let

(III.8) \[ P(z) = (1 - z^2) \]

and

(III.9) \[ Q(z) = \int_0^z \frac{d\zeta}{1 - \zeta^2} = \tanh^{-1} z \]

Then \( I_0 \) is the strip \( |\Im(z)| < \pi/4 \). \( \mathcal{R}_0 \) is the real axis. Clearly \( \mathcal{R}_0 \) satisfies conditions (CII.1) and is therefore the maximum domain \( \mathcal{J} \) of the parameter \( t \). We have

(III.10) \[ \omega(z; t) = \frac{z + \tanh t}{1 + z \tanh t}, \quad t \in \mathcal{R}_0, \ z \in \Delta. \]

If we let

(III.11) \[ \alpha = \tanh t, \quad t \in \mathcal{R}_0 \]

then, writing \( \omega[z; t(\alpha)] \) simply as \( \omega(z; \alpha) \),

(III.12) \[ \omega(z; \alpha) = \frac{z + \alpha}{1 + \alpha z}, \quad z \in \Delta, \ -1 < \alpha < 1. \]

The family of transformations \( \{L_a\} \) is given by

(III.13) \[ L_a f = g \]

where

\[ g(z) = f\left(\frac{z + \alpha}{1 + \alpha z}\right) \]

The norm of \( L_a \) is

(III.14) \[ \| L_a \|_{H^p} = \left[ \frac{1 + |\alpha|}{1 - |\alpha|} \right]^{1/p} \]

\[ ^8 \text{The path of integration lying entirely in } \Delta. \]
The generator $A$ is defined for those $f \in H^p$ for which the limit

$$Af = \lim_{\alpha \to 0} L_\alpha f = f'$$

exists in the $H^p$ norm. Hence

$$Af(z) = (1 - z^\alpha)f'(z) \quad \text{for } f \in \mathcal{D}(A).$$

For $1 \leq p < \infty$, $\mathcal{D}(A)$ is the set of functions $f \in P^p$ for which $(1 - z^\alpha)f'(z)$ defines an element of $H^p$: also, $A$ is a closed operator with domain dense in $H^p$. As with the previous example, these statements do not hold for $H^\infty$.

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