BIORTHOGONAL SYSTEMS IN BANACH SPACES

SHAUL FOGUEL
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1. Introduction. We shall be interested, in this paper, in the following question: Given a biorthogonal system \((x_n, f_n)\) in a separable Banach space \(B\), under what conditions can one assert that the sequence \(\{x_n\}\) constitutes a basis? The system \((x_n, f_n)\) is called a biorthogonal system if

\[ x_n \in B, \quad f_n \in B^* \quad \text{and} \quad f_n(x_m) = \delta_{nm}. \]

We shall assume throughout the paper that \(\|x_n\|=1\) and the sequence \(\{x_n\}\) is fundamental. When the sequence \(\{x_n\}\) constitutes a basis it will be called regular otherwise irregular.

2. Irregular systems. Let \(\{x_n\}\) be an irregular sequence. (For example the trigonometric functions for \(C(-\pi, \pi)\)). The following definitions will be used.

\[ \varphi_n(x) = \sum_{i=1}^{n} f_i(x)x_i \]

\[ \|\|x\|| = \sup \{ \|\varphi_n(x)\|, n=1, 2, 3, \cdots \} \]

Compare [4]

\[ E_{0} = \{x| \lim_{n \to \infty} \varphi_n(x) = x\} \]

\[ E_{1} = \{x| \|\|x\|| < \infty\} \]

\[ E_{2} = \{x| \lim_{n \to \infty} \|\varphi_n(x)\| = \infty\} \]

\[ E_{3} = \{x| \|\|x\|| = \infty\} \]

We have \(E_{0} \subset E_{1}\) and \(E_{2} \subset E_{3}\). For regular systems \(E_{0}=E_{1}=B\) and \(E_{2}=E_{3}=\phi\) where \(\phi\) is the null set. The system is regular if and only if the sequence \(\{\|\varphi_n\|\}\) is bounded [2], and if the sequence \(\{\|\varphi_n\|\}\) is not bounded the set

\[ \bigcap_{n=1}^{\infty} \{x| \|\varphi_n(x)\| \leq K\} \]

is nowhere dense [2], hence for irregular systems the set

\[ E_{1} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \{x| \|\varphi_n(x)\| \leq K\} \]

is of the first category. Also \(E_{3}=B-E_{1}\) is dense and of the second

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category. In the case of regular systems there exists a number \( K \geq 1 \) such that if \( \| x \| = 1 \) then \( 1 \leq \| x \| \leq K \). The existence of such a bound, \( K \), is equivalent to the equiboundedness of \( \{ \| \varphi_n(x) \| \} \| x \| = 1 \) and therefore for irregular systems for any number \( a \), there exists a point \( x \) such that \( \| x \| = 1 \) and \( \| x \| > a \), moreover such a point might be found in the linear manifold generated by \( \{ x_n \} \). (Equiboundedness of \( \{ \| \varphi_n(x) \| \} \) on a dense subset of the unit sphere would imply equiboundedness on the unit sphere.) It is interesting to note that for every number \( a \geq 1 \) there exists a point \( x \) such that \( \| x \| = 1 \) and \( \| x \| = a \). There exists a point \( y_n \) satisfying

\[
y_n = \sum_{i=1}^{n} \alpha_i x_i, \quad \| y_n \| = 1, \quad \| y_n \| > a.
\]

On the other hand \( \| x_1 \| = 1 \) and \( \| x_1 \| = 1 \). Let \( 0 \leq t \leq 1 \), then \( (1-t)x_1 + ty_n \neq 0 \). Define

\[
g(t) = \left\| \varphi \left( \frac{(1-t)x_1 + ty_n}{\| (1-t)x_1 + ty_n \|} \right) \right\|, \quad \nu = 12 \cdots n
\]

The functions \( g(t) \) are continuous in \( t \), and so is \( g(t) \) where

\[
g(t) = \sup \{ g(t) \mid 1 \leq \nu \leq n \}.
\]

\[
g(0) = 1
\]

and

\[
g(1) = \sup \{ \| \varphi_n(y_n) \| \mid 1 \leq \nu \leq n \} = \| y_n \| > a.
\]

There exists a number \( t_0 \) such that

\[
0 \leq t_0 < 1 \quad \text{and} \quad g(t_0) = a.
\]

This following generalization of Baire’s theorem [1] will be used:

Let \( \{ u_n(x) \} \) be a sequence of real valued continuous functions defined on a metric space \( c \), and \( \lim u_n(x) = u(x) \), \( | u_n(x) | \leq M \), then the set of points of discontinuity of \( u \) is of the first category.

**Theorem 1.** The set \( E_2 \) is of the first category.

**Proof.** Define the functions \( u_n(x) \) by

\[
u_n(x) = \frac{\| \varphi_n(x) \|}{1 + \| \varphi_n(x) \|}.
\]

We have \( 0 \leq u_n(x) \leq 1 \) and if \( x \in E_0 \cup E_2 \) then

\[
\lim u_n(x) = u(x)
\]
where \( u(x) = 1 \) for \( x \in E_2 \) and

\[
u(x) = \frac{\|x\|}{1 + \|x\|} \quad \text{for} \quad x \in E_0.
\]

If \( E_2 \) is a set of the second category then there exists at least one point of continuity of \( u \). Let us denote such a point by \( x_0 \).

The set \( E_0 \) is dense in \( B \). Let \( \{y_n\} \) by a sequence of points in \( E_0 \) with \( \lim y_n = x_0 \), then

\[
u(x_0) = \lim \nu(y_n) = \lim \frac{\|y_n\|}{1 + \|y_n\|} = \frac{\|x_0\|}{1 + \|x_0\|}
\]

The set \( E_2 \) is dense in \( B \). If \( x \in E_2 \) and \( y \in E_0 \) then \( x + y \in E_2 \). Let \( \{z_n\} \) be a sequence of points in \( E_2 \) with \( \lim z_n = x_0 \), then \( u(z_n) = 1 \) and

\[
\frac{\|x_0\|}{1 + \|x_0\|} = u(x_0) = \lim u(z_n) = 1
\]

which is absurd.

**Theorem 2.** Let \( S \) be a subset of \( B \) such that each point \( x \in S \) is the limit of some sequence \( \{y_n\} \), \( y_n \in B \), and the sequence \( \{\|y_n\|\} \) is bounded, then \( S \) is of the first category.

**Proof.** Define the functions \( v_n(x) \) by

\[
v_n(x) = \frac{\|\varphi_n(x)\|}{1 + \|\varphi_n(x)\|}
\]

then \( 0 \leq v_1(x) \leq v_2(x) \leq \cdots < 1 \). If \( x \in E_2 \) let \( \lim v_n(x) = v(x) \). \( v(x) = 1 \) for \( x \in E_3 \) and the set \( E_3 \) is dense, hence \( v(x) = 1 \) at every point of continuity of \( v \). Let \( x \) be a point of continuity of \( v \) and \( \{z_n\} \) a sequence with \( \lim z_n = x \), then

\[
\lim v(z_n) = v(x) = 1
\]

therefore the sequence \( \{\|z_n\|\} \) is unbounded. Thus the set \( S \) is contained in the set of points of discontinuity of \( v \) which is a set of the first category by Baire’s theorem.

3. General criteria for regularity. From Theorems 1 and 2 we derive the following criteria.

**Theorem 3.** A necessary and sufficient condition for the regularity of the system \( (x_n, f_n) \) is.
sup \{||\varphi_n(x)||, n=1, 2, \cdots \} = \infty

implies

\lim ||\varphi_n(x)|| = \infty. \quad (or \ E_2=E_3).

Proof. If the system is regular, then \ E^2=E_2=\phi. On the other hand, if the system is irregular \ E_2 is of the first category and \ E_3 of the second category.

Let

\[ \varphi_n(x) = \sum_{i=1}^{n} a_i^n x_i \]

denote the point nearest to \( x \) on the subspace spanned by

\[ \{x_1, x_2, x_3, \cdots, x_n\} \]

\textbf{THEOREM 4.} The system \((x_n, f_n)\) is regular if and only if the sequence \{||\varphi_n(x)||\} is bounded for each \( x \).

Proof. If the system is regular, then there exists a positive number \( K \), such that \( ||x|| \leq K ||x|| \). Then

\[ ||\varphi_n(x)|| \leq K ||\varphi_n(x)|| \leq K(||x|| + ||x-\varphi_n(x)||) \leq K2||x||, \]

hence the condition is necessary. Sufficiency is clear by Theorem 2.

4. Biorthogonal systems in Hilbert spaces. In this section we assume that \( B \) is a Hilbert space. In order to use Theorem 4 let us compute \(||\varphi_n(x)||\). \( \varphi_n(x) = \sum_{i=1}^{n} a_i^n x_i \) and the coefficient \( a_i^n \) may be computed from the equation

\[ (x-\sum_{i=1}^{n} a_i^n x_i, x_k) = 0 \quad k=1, 2, \cdots, n \]

or

\[ (x, x_k) = \sum_{i=1}^{n} a_i^n (x_i, x_k) \quad \text{see [5]} \]

We introduce the following notation

\[ (x_i, x_k) = c_{ik} \]

\[ C_n = (c_{ik}) \quad 1 \leq i \leq n \quad 1 \leq k \leq n \]

\[ ((x, x_i), (x, x_2), \cdots, (x, x_n)) = (\gamma_n) \]

\[ (a_1^n, a_2^n, \cdots, a_n^n) = (a_n) \]

Then
\[(r)_n = (a)_n C_n \quad \text{or} \quad (a)_n = (r)_n C_n^{-1}\]

since \(C_n^{-1}\) exists. Now

\[
|| \sum_{i=1}^{j} a_i^n x_i ||^2 = \sum_{i,k=1}^{j} a_i^n a_k^n e_{i,k} = (a)_n E_j^n C_n E_j^n (a)_n^* \]

where \(E_j^n\) is the matrix \((e_{l,m})\) with

\[
e_{l,m} = \begin{cases} 
1 & l = m \leq j \\
0 & \text{otherwise}
\end{cases}
\]

\(C_n^* = C_n\) and \((a)_n = (r)_n C_n^{-1}\) hence

\[
|| \sum_{j=1}^{j} a_i^n x_i ||^2 = (r)_n C_n^{-1} E_j^n C_n E_j^n C_n^{-1} (r)_n^* \]

Orthogonalizing the sequence \(\{x_n\}\) by Schmidt’s process we get the sequence \(\{y_n\}\) with

\[
x_1 = y_1 \quad x_k = \sum_{a=1}^{k} d_{k,a} y_a
\]

where

\[
d_{k,a} = \begin{cases} 
(x_k, y_a) & \alpha \leq k \\
0 & \alpha > k
\end{cases}
\]

and \(d_{11} = 1\) see [3].

Let \(D_n\) denote the triangular matrix

\[(d_{k,a}) \quad 1 \leq \alpha \leq n \quad 1 \leq k \leq n.
\]

\[
(x_i, x_j) = \sum_{a} d_{i,a} d_{j,a} \quad \text{or} \quad C_n = D_n D_n^*.
\]

The matrix \(D_n\) can be computed from this relation.

Let

\[
(\delta)_n = ((x, y_1), (x, y_2), \cdots, (x, y_n))
\]

\[
(x, x_n) = \sum_{a=1}^{k} (x, y_a) d_{k,a} \quad \text{or} \quad (r)_n = (\delta)_n D_n^*
\]

and hence

\[
|| \sum_{i=1}^{j} a_i^n x_i ||^2 = (r)_n C_n^{-1} E_j^n C_n E_j^n C_n^{-1} (r)_n^* = (\delta)_n D_n^* C_n^{-1} E_j^n C_n E_j^n C_n^{-1} D_n (\delta)_n^* \]

\[
= (\delta)_n (D_n^{-1} E_j^n D_n) (D_n^* E_j^n (D_n^*)^{-1}) (\delta)_n^* \]

Let \(A_j^n = D_n^{-1} E_j^n D_n\) then
\[ \| \varphi_n(x) \| = \max \{ (\delta)_n A_j^n (A_j^n)^* (\delta)_j^n | 1 \leq j \leq n \} \]

The triangular matrix \( A_j^n \) is an operator defined on the Hilbert space. If

\[ x = \sum_{i=1}^{\infty} \delta_i y_i \]

then

\[ A_j^n (x) = (\delta_1, \cdots, \delta_n) A_j^n. \]

By Theorem 4 and the above computation the system is regular if and only if for each \( x \)

\[ \sup \{ \| A_j^n (x) \| | 1 \leq j \leq n \quad n = 12 \cdots \} < \infty \]

or by the uniform boundedness theorem.

**THEOREM 5.** The system is regular if and only if the double sequence \( \{ \| A_j^n \| \} \) is bounded, or, in other words, if and only if the set of characteristic roots of \( A_j^n (A_j^n)^* \) is bounded.

We shall use Theorem 3 to derive the following theorems.

**THEOREM 6.** The system \((x_t, f_t)\) is regular and \( \sum_{i=1}^{\infty} |f'_i(x)|^2 < \infty \) if and only if for every \( x \in \mathbb{B} \) there exists a real number \( \alpha = \alpha(x) \) such that

\[ (1) \quad 2 \Re \{ \sum_{i=1}^{n} \sum_{j=1}^{i-1} f_i(x) f_j(x) c_{ij} \} > \alpha \]

**Proof.** If the system is regular and

\[ \sum_{i=1}^{\infty} |f'_i(x)|^2 < \infty \]

then

\[ \| x \|^2 - \sum_{j \neq j} |f'_j(x)|^2 = \sum_{j \neq j} f'_i(x) f_j(x) c_{ij} \]

\[ = 2 \Re \{ \sum_{j \neq j} f'_i(x) f_j(x) c_{ij} \} \]

Therefore the necessity of condition (1) is verified. Assume that condition (1) is satisfied then

\[ \| \varphi_{n+p}(x) \|^2 = \sum_{i=1}^{n+p} f'_i(x) f_j(x) c_{ij} \]

\[ = \| \varphi_n(x) \|^2 + \sum_{i=n+1}^{n+p} |f'_i(x)|^2 + 2 \Re \{ \sum_{i=n+1}^{n+p} \sum_{j=1}^{i-1} f'_i(x) f_j(x) c_{ij} \} \]

\[ \geq \| \varphi_n(x) \|^2 + 2 \alpha \]
Therefore $\sup \|\varphi_n(x)\| = \infty$ implies $\lim \|\varphi_n(x)\| = \infty$. According to Theorem 3 the system is regular. Moreover

$$\sum_{i=1}^{n} |f_i(x)|^2 = |\varphi_n(x)|^2 - 2\Re \left\{ \sum_{j, i \leq n} c_{ij} f_i(x) f_j(x) \right\} \leq \|x\|^2 - \alpha < \infty.$$  

An immediate consequence is the following. The system is regular if $\sum_{i \neq j} |c_{ij}| < \infty$ and the sequence $\{\|f_i\|\}$ is bounded.

Professor R. C. James called my attention to the fact that this may be proved directly and without the assumption of boundedness of the sequence $\{\|f_i\|\}$ as follows. We may assume without loss of generality that $\sum_{i \neq j} |c_{ij}| = r < 1$

$$\left| \sum_{i \neq j} a_i a_j c_{ij} \right| \leq \max |a_i a_j| \cdot r \leq \sum_{i=1}^{n} |a_i|^2 r$$

$$\|\sum_{i=1}^{n} a_i x_i \|^2 = \sum_{i=1}^{n} |a_i|^2 + \sum_{i \neq j} a_i a_j c_{ij} \leq 2 \sum_{i=1}^{n} |a_i|^2$$

Hence

$$\|\sum_{i=1}^{n+p} a_i x_i \|^2 = \sum_{i=1}^{n+p} |a_i|^2 \geq \sum_{i=1}^{n+p} |a_i|^2 (1-r) \geq \sum_{i=1}^{n} |a_i|^2 \frac{1-r}{2} \left\| \sum_{i=1}^{n} a_i x_i \right\|^2$$

and by [4] the system is regular.

Using the same method as in Theorem 6 we arrive at the following.

**Theorem 7.** The system is regular if and only if for each $x$

$$\text{(2)} \quad \inf_{n, p} \Re \left\{ \sum_{i=1}^{n} \sum_{j=n+1}^{p} f_i(x) f_j(x) c_{ij} \right\} > -\infty$$

**Proof.**

$$\|\sum_{i=1}^{n+p} f_i(x) x_i \|^2 = \|\sum_{i=1}^{n} f_i(x) x_i \|^2 + \|\sum_{i=n+1}^{n+p} f_i(x) x_i \|^2$$

$$+ 2\Re \left\{ \left( \sum_{i=n+1}^{n} f_i(x) x_i, \sum_{j=n+1}^{p} f_j(x) x_j \right) \right\}.$$ 

If condition (2) is satisfied then according to Theorem 3 the system is regular. If the system is regular then
\[ |\sum_{i=1}^{n} f_i(x)x_i - \sum_{j=n+1}^{n+p} f_j(x)x_j| \leq 2\|x\|^2. \]

As a simple application we note the following.

If \( e_{ij} = 0 \) when \( |i-j| > N \) then the system is regular if and only if the sequence \( \{|f_i|\} \) is bounded.

**References**


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William F. Donoghue, Jr., The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation ........................................... 1031
Michael (Mihály) Fekete and J. L. Walsh, Asymptotic behavior of restricted extremal polynomials and of their zeros ........................................... 1037
Shaul Foguel, Biorthogonal systems in Banach spaces ............................. 1065
David Gale, A theorem on flows in networks ............................................. 1073
Ioan M. James, On spaces with a multiplication ......................................... 1083
Richard Vincent Kadison and Isadore Manual Singer, Three test problems in operator theory .............................................................. 1101
Maurice Kennedy, A convergence theorem for a certain class of Markoff processes ...................................................................................... 1107
G. Kurepa, On a new reciprocity, distribution and duality law ...................... 1125
Richard Kenneth Lashof, Lie algebras of locally compact groups ................. 1145
Calvin T. Long, Note on normal numbers .................................................. 1163
M. Mikolás, On certain sums generating the Dedekind sums and their reciprocity laws ................................................................. 1167
Barrett O’Neill, Induced homology homomorphisms for set-valued maps ........ 1179
Mary Ellen Rudin, A topological characterization of sets of real numbers ......... 1185
M. Schiffer, The Fredholm eigen values of plane domains ......................... 1187
F. A. Valentine, A three point convexity property ....................................... 1227
Alexander Doniphan Wallace, The center of a compact lattice is totally disconnected ................................................................. 1237
Alexander Doniphan Wallace, Two theorems on topological lattices ............. 1239
G. T. Whyburn, Dimension and non-density preservation of mappings .......... 1243
John Hunter Williamson, On the functional representation of certain algebraic systems .............................................................. 1251