NOTE ON NORMAL NUMBERS

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Introduction. Let \( \alpha \) be a real number with fractional part \( .a_1a_2a_3\ldots \) when written to base \( r \). Let \( Y_n \) denote the block of the first \( n \) digits in this representation and let \( N(d, Y_n) \) denote the number of occurrences of the digit \( d \) in \( Y_n \). The number \( \alpha \) is said to be simply normal to base \( r \) if

\[
\lim_{n \to \infty} \frac{N(d, Y_n)}{n} = \frac{1}{r}
\]

for each of the \( r \) distinct choices of \( d \). \( \alpha \) is said to be normal to base \( r \) if each of the numbers \( \alpha, r\alpha, r^2\alpha, \ldots \) are simply normal to each of the bases \( r, r^2, r^3, \ldots \). These definitions, due to Emile Borel [1], were introduced in 1909. In 1940 S. S. Pillai [3] showed that a necessary and sufficient condition that \( \alpha \) be normal to base \( r \) is that it be simply normal to each of the bases \( r, r^2, r^3, \ldots \), thus considerably reducing the number of conditions needed to imply normality. The purpose of the present note is to show that \( \alpha \) is normal to base \( r \) if and only if there exists a set of positive integers \( m_1 < m_2 < m_3 < \cdots \) such that \( \alpha \) is simply normal to base \( r^{m_i} \) for each \( i \geq 1 \), and also to show that no finite set of \( m \)'s will suffice.

Notation. We make use of the following additional conventions.

If \( B_k \) is any block of \( k \) digits to base \( r \), \( N(B_k, Y_n) \) will denote the number of occurrences of \( B_k \) in \( Y_n \) and \( N_i(B_k, Y_n) \) will denote the number of occurrences of \( B_k \) starting in positions congruent to \( i \) modulo \( k \) in \( Y_n \).

The term "relative frequency" will denote the asymptotic frequency with which an event occurs. For example, \( B_k \) occurs in \( (\alpha) \), the fractional part of \( \alpha \), with relative frequency \( r \) if

\[
\lim_{n \to \infty} \frac{N(B_k, Y_n)}{n} = r
\]

Proof of the theorems. The following lemmas are easily proved.

**Lemma 1.** If \( \lim_{n \to \infty} \sum_{i=1}^{m} f_i(n) = 1 \) and \( \lim \inf_{n \to \infty} f_i(n) \geq 1/m \) for \( i = 1, 2, \ldots, m \); then \( \lim f_i(n) = 1/m \) for each \( i \).

**Lemma 2.** The real number \( \alpha \) is simply normal to base \( r \) if and
only if \( \lim_{n \to \infty} N_{i}(B_{k}, Y_{n})/n = 1/kr^{k} \) for every block \( B_{k} \) of \( k \) digits to base \( r \).

**Theorem 1.** The real number \( \alpha \) is normal to base \( r \) if and only if there exist positive integers \( m_{1} < m_{2} < m_{3} < \cdots \) such that \( \alpha \) is simply normal to each of the bases \( r^{m_{1}}, r^{m_{2}}, r^{m_{3}}, \cdots \).

**Proof.** The necessity of the condition follows immediately from the definition of normality.

Now suppose the condition of the theorem prevails. Let \( \nu \) be an arbitrary positive integer and let \( B \) be an arbitrary block of \( \nu \) digits to base \( r \). Choose \( k \) so large that \( m_{k} \geq \nu \). It follows from Lemma 2 that

\[
\lim_{n \to \infty} \frac{N_{i}(A_{m_{k}}, Y_{n})}{n} = \frac{1}{m_{k}r^{m_{k}}}
\]

for each block \( A_{m_{k}} \) of \( m_{k} \) digits to base \( r \). If \( B \) occurs exactly \( t \) times in each \( A_{m_{k}} \), then it follows that

\[
\liminf_{n \to \infty} \frac{N(B, Y_{n})}{n} \geq \frac{T}{m_{k}r^{m_{k}}}
\]

where \( T = \sum_{i=1}^{\infty} t(A_{m_{k}}) \) with the sum running over all blocks of \( m_{k} \) digits to base \( r \). Now there are \( r^{m_{k} - \nu} \) distinct blocks \( A_{m_{k}} \) which contain \( B \) starting in position \( i \) for \( i = 1, 2, \cdots, m_{k} - \nu + 1 \) so that \( T = (m_{k} - \nu + 1)r^{m_{k} - \nu} \). Thus it follows that

\[
\liminf_{n \to \infty} \frac{N(B, Y_{n})}{n} \geq (m_{k} - \nu + 1)r^{m_{k} - \nu} = \frac{1}{m_{k}r^{m_{k}}}.
\]

But, since this argument can be made for arbitrarily large values of \( k \) and \( m_{k} \geq k \), this implies that

\[
\liminf_{n \to \infty} \frac{N(B, Y_{n})}{n} \geq \frac{1}{r^\nu}.
\]

With Lemma 1 this implies that

\[
\lim_{n \to \infty} \frac{N(B, Y_{n})}{n} = \frac{1}{r^\nu}
\]

so that \( \alpha \) is normal to base \( r \) by a result of Niven and Zuckerman [2].

The next theorem implies that no finite set of \( m \)'s will suffice in Theorem 1.

**Theorem 2.** If \( m_{1}, m_{2}, \cdots, m_{s} \) is an arbitrary collection of distinct
positive integers, then there exists at least one real number $\alpha$ simply normal to each of the bases $r^{m_1}, r^{m_2}, \ldots, r^{m_s}$ but not normal to base $r$.

Proof. Writing to base $r^m$ form the periodic decimal

$$\alpha = .012\ldots(r^m-1)$$

where $m$ is the least common multiple of $m_1, m_2, \ldots, m_s$. It is clear that $\alpha$ is simply normal to base $r^m$ and that it is not normal to base $r$. To show that it is simply normal to base $r^m_i$ for $i=1, 2, \ldots, s$ we prove that if $d$ divides $m$ then $\alpha$ is simply normal to base $r^d$.

Let $m=qd$ and let $B_d$ be an arbitrary but fixed block of $d$ digits to base $r$. In view of Lemma 2 it suffices to show that

$$\lim_{n\to\infty} \frac{N_r(A_d, Y_n)}{n} = \frac{1}{dr^d}.$$ 

A simple counting process shows that there are precisely $\left(\frac{q}{d}\right)(r^d-1)^{q-1}$ distinct blocks $A_m$ of $m$ digits to base $r$ which contain $B_d$ exactly $i$ times starting in a position congruent to one modulo $d$. Therefore, since

$$\lim_{n\to\infty} \frac{N_r(A_m, Y_n)}{n} = \frac{1}{mr^m}$$

for each $A_m$, it follows that

$$\lim_{n\to\infty} \frac{N_r(B_d, Y_n)}{n} = \frac{1}{mr^m} \sum_{i=1}^{q} i \left(\frac{q}{d}\right)(r^d-1)^{q-i} = \frac{1}{dr^d}$$

as required.

References

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