ON CERTAIN SUMS GENERATING THE DEDEKIND SUMS AND THEIR RECIPROCITY LAWS

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1. Introduction. Let \( \{u\} = u - [u] \) denote the fractional part of \( u \) and let \( ((u)) = \{u\} - \frac{1}{2} \). Dedekind sums are defined for example, by

\[
S_i(h, k) = \sum_{\lambda=0}^{k-1} \left( \frac{\lambda}{k} \right) \left( \frac{ih}{k} \right)
\]

where \( h \) and \( k \) are relatively prime positive integers. These sums which were studied by Dedekind [7], and more recently by Rademacher and Whiteman [9], [12] in connection with the theory of the modular function \( \tau(\tau) \), occur also in the theory of partitions and in a great number of special papers. (Cf. for example [1]–[13].) The most important property of \( S_i(h, k) \) is the reciprocity law

\[
S_i(h, k) + S_i(k, h) = (h^2 + 3hk + k^2 + 1)/(12hk).
\]

A few years ago, Apostol [1] (for \( r = \nu \)) and Carlitz [3] introduced and investigated the so-called generalized Dedekind sums

\[
S^\nu(h, k) = \sum_{\lambda=0}^{k-1} P_{\nu+1-r}(\frac{\lambda}{k}) P_r(\frac{ih}{k}) \quad 0 \leq r \leq \nu + 1,
\]

where \( P_r \) denotes the well-known Bernoulli function defined by the expansion

\[
ze^{ux}/(e^x - 1) = \sum_{n=0}^{\infty} P_n(u)z^n/n! \quad |z| < 2\pi
\]

for \( 0 \leq u < 1 \) and by \( P_r(u) = P_r(\{u\}) \) for \( u \) arbitrary real. They found the corresponding extensions of (1.2) too.

Now, we shall continue to develop these results in two directions. Next we give a systematic treatment of certain exponential sums \((2.1), (2.3)\) generating

\[
S_{m,n}(a, b) = \sum_{s=0}^{c-1} P_m(\frac{sa}{c}) P_n(\frac{sb}{c}) \quad m, n = 0, 1, 2, \ldots
\]

with \((a, c) = (b, c) = 1, c > 0\). We obtain (among others) a three-term relation of new type (Theorem 1) which implies (in extended form) all the above reciprocity theorems (see \((5.1)-(5.10)\)). Let us remark that the sum function \((2.5)\) with other notations is also used in [6]. On the other hand, we get a functional equation for

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(1.5) \[ \mathcal{D}_{c}^{a,b}(w, z) = \sum_{\lambda=1}^{c-1} \zeta \left( w, \left\{ \frac{\lambda a}{c} \right\} \right) \zeta \left( z, \left\{ \frac{\lambda b}{c} \right\} \right) \]

where \( \zeta(s, u) \) is the Hurwitz zeta function (Theorem 2). By

\[ \zeta(1-n, u) = -P_{n}(u)/n \quad 0 < u \leq 1; \quad n = 1, 2, \cdots, \]

(1.5) can be regarded substantially as a (transcendental) generalization of (1.4).

2. Preliminaries on \( \mathcal{S}_{c}^{a,b}(x, y), \mathfrak{s}_{m,n} \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \). In what follows, \( x, y, w, z \) denote complex variables, \( a, b \) and \( c \) are integers and \( c > 0 \); for brevity we write, as usual, \( e(z) = e^{2\pi iz} \).

Let us put

(2.1) \[ S_{c}^{a,b}(x, y) = \sum_{(\lambda \mod c)} e \left( \left\{ \frac{\lambda a}{c} \right\} x + \left\{ \frac{\lambda b}{c} \right\} y \right) \]

with \( (a, c) = (b, c) = 1 \), the summation extending over a complete residue system modulo \( c \). It is obvious that (2.1) is independent of the choice of this residue system\(^1\) and for \( a = b \) or \( c = 1, 2 \) it is independent of \( a, b \). The function \( S_{c}^{a,b}(x, y) \) remains unaltered if we change \( a, b \) or \( x, y \) by multiplies of \( c \). By this periodicity, it is no restriction to suppose for example, that \( 0 \leq \Re(x) < c, \quad -c < \Re(y) \leq 0 \).

We have \( S_{c}^{a,b}(x, y) = S_{c}^{b,a}(y, x) \) and

(2.2) \[ S_{c}^{a,b}(x, y) = e(x)S_{c}^{a,-b}(-x, y) + 1 - e(x), \]

since \( \{-u\} = 0 \) or \( 1 - \{u\} \) according as \( u \) is an integer or not.

The function

(2.3) \[ \mathcal{S}_{c}^{a,b}(x, y) = [e(x) - 1]^{-1}[e(y) - 1]^{-1}S_{c}^{a,b}(x, y) \quad x, y \neq 0, \pm 1, \cdots \]

has corresponding trivial properties; in particular, (2.2) implies

(2.4) \[ \mathcal{S}_{c}^{a,b}(x, y) = -\mathcal{S}_{c}^{a,-b}(-x, y) - [e(y) - 1]^{-1}. \]

By the definition of Bernoulli functions and (1.4) we obtain

(2.5) \[ xy\mathcal{S}_{c}^{a,b}(x|2\pi i, y|2\pi i) = \sum_{m,n=0}^{\infty} \frac{x^{m}y^{n}}{m!n!} \mathfrak{s}_{m,n} \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \quad |x|, |y| < 2\pi. \]

Here

\(^1\) Hence we see that \( S_{c}^{a,b}(x, y) = S_{c}^{b,a}(x, y) \) for a suitable integer \( b' \); however, the above symmetric notation seems the most convenient.
\[ \delta_{0,n}(a/b) = \sum_{i=0}^{c-1} P_n\left(\frac{l}{c}\right) = c^{1-n}B_n, \quad n=0,1,\ldots, \]

\(B_n = P_n(0)\) denoting the Bernoullian numbers.

Note that \(\delta_{m,n}(a/b) = \delta_{n,m}(b/a)\) and \(\delta_{m,n}(a/a)\) does not depend on \(a\); especially we have \(\delta_{m,n}(1/b) = \delta_{n,m}^{(m+n-1)}(b,c)\), furthermore

\[ \delta_{m,n}(a/b) = B_mB_n, \quad \delta_{m,n}(1/b) = B_mB_n[1+(1-2^{1-m})(1-2^{1-n})] \]

\(m, n=0,1,\ldots\).

3. Representation by cotangents and Eulerian numbers respectively. Let \(c>1\). The identity

\[ \sum_{\mu=1}^{c-1} e\left(\frac{\mu x}{c}\right) e\left(\frac{\mu v}{c}\right) = [e(x) - 1]\left[ e\left(\frac{x+v}{c}\right) - 1 \right]^{-1} \]

yields after multiplication by \(e\left(-\frac{\mu v}{c}\right) (\nu=0,1,\ldots,c-1)\) and summation

\[ e\left(\frac{\mu x}{c}\right) = \frac{1}{c} [e(x) - 1] \sum_{\nu=0}^{c-1} \left[ e\left(\frac{x+v}{c}\right) - 1 \right]^{-1} e\left(-\frac{\mu v}{c}\right) \]

\(\mu=0,1,\ldots,\nu-1;\)

(3.1) and (3.2) hold clearly provided that \((x+\nu)/c\) is not an integer \((\nu=0,1,\ldots,c-1)\). Hence by putting \(\mu = c\{a\lambda/c\}\), \(a\) and \(c\) being coprime we get

\[ e\left(\frac{a\lambda}{c}\right) = \frac{1}{c} [e(x) - 1] \sum_{\nu=0}^{c-1} \left[ e\left(\frac{x+v}{c}\right) - 1 \right]^{-1} e\left(-\nu\frac{a\lambda}{c}\right) \]

Furthermore, by using the corresponding expression for \(e(y\{b\lambda/c\})\), \((b,c)=1,

\[ S_{x,y}^{ab}(x,y) = \frac{1}{c^a} [e(x) - 1][e(y) - 1] \sum_{p,q\pmod{c}} \left[ e\left(\frac{x+p}{c}\right) - 1 \right]^{-1} \left[ e\left(\frac{y+q}{c}\right) - 1 \right]^{-1} \]

\[ \times \sum_{\lambda=0}^{c-1} e\left(-\lambda\left(\frac{ap+bq}{c}\right)\right). \]

If we consider the complete residue systems \((\pmod{c})\): \(p=-br, q=ar\)

\((r, \rho=0,1,\ldots,c-1)\) and take into account that \(\sum_{\lambda=0}^{c-1} e\left(-\lambda\left(\frac{ab(\rho-r)}{c}\right)\right)\)
vanishes except for $p=r$ when it has the value $c$, it follows simply that

$$\mathcal{S}_{c}^{a,b}(x, y) = \frac{1}{c} \sum_{r \text{(mod } c)} \left[ e^{\left(\frac{x-br}{c}\right)} - 1 \right]^{-1} \left[ e^{\left(\frac{y+ar}{c}\right)} - 1 \right]^{-1},$$

holds for all $x, y \neq 0, \pm 1, \cdots$ and, because of the definition (2.3), in the case $c=1$ too. By $[1-e(z)]^{-1} = \frac{1}{2}(1+i \cot \pi z)$ and

$$\sum_{n=0}^{c-1} \cot \pi \left(z + \frac{\mu}{c}\right) = c \cdot \cot c \pi z,$$

we have the equivalent formula:

$$\mathcal{S}_{c}^{a,b}(x, y) = \frac{1}{4} [1 + i(\cot \pi x + \cot \pi y)]
- \frac{1}{4c} \sum_{r \text{(mod } c)} \cot \pi \frac{x-br}{c} \cot \pi \frac{y+ar}{c};$$

(3.4) or (3.5) expresses the sum (2.3) by means of periodic elementary functions, without using the arithmetical function \{w\}.

(3.4) leads immediately to corresponding representations of $s_{m,n}(\frac{a}{c}, \frac{b}{c})$ by means of the so-called Eulerian numbers $H_{n}(\gamma^{k})$, defined for a root of unity $\gamma^{k} = e^{\left(k/c\right)}$, $c > 1$, $c \nmid k$ by

$$\frac{1-\gamma^{k}}{(e^{\gamma^{k}} - 1)} = \sum_{n=0}^{\infty} H_{n}(\gamma^{k}) \frac{x^{n}}{n!} \quad |z| < 2\pi \{k/c\}.$$

In fact, after expanding the right-hand members of

$$xy \mathcal{S}_{c}^{a,b}(x/2\pi i, y/2\pi i) = (xy/c) (e^{\gamma^{k}} - 1)^{-1} (e^{\gamma^{k}} - 1)^{-1}
+ (xy/c) \sum_{r=1}^{c-1} (e^{\gamma^{k}r} \gamma^{-br} - 1)^{-1} (e^{\gamma^{k}r} \gamma^{-ar} - 1)^{-1},$$

we find

$$xy \mathcal{S}_{c}^{a,b}(x/2\pi i, y/2\pi i) = c + \sum_{n=1}^{\infty} \frac{B_{n}}{n!} (x^{n} + y^{n})
+ \sum_{m,n=1}^{\infty} \frac{1}{m!n!} \left[ B_{m}B_{n} + mn \sum_{r=1}^{c-1} \frac{H_{m-1}(\gamma^{br})H_{n-1}(\gamma^{-ar})}{(\gamma^{ar} - 1)(\gamma^{-br} - 1)} \right] \quad |x|, |y| < 2\pi/c,$$

so that comparison with (2.5) gives in addition to (2.6)

$$s_{m,n}(\frac{a}{c}, \frac{b}{c}) = \frac{1}{c^{m+n-1}} \left[ B_{m}B_{n} + mn \sum_{r=1}^{c-1} \frac{H_{m-1}(\gamma^{br})H_{n-1}(\gamma^{-ar})}{(\gamma^{ar} - 1)(\gamma^{-br} - 1)} \right]$$

$$m, n=1, 2, \cdots,$$
a formula implying a result of Carlitz [3, (6.5)]. In particular, for
\( m=n-1 \) (3.8) becomes
\[
\varphi_{\mp}(a, b, c) = \frac{1}{4c} \sum_{r=1}^{c-1} \left( \eta^ar - 1 \right) \left( \eta^{-br} - 1 \right)^{-1}
\]
\[
= \frac{1}{4c} \sum_{r=1}^{c-1} \frac{\pi ar}{c} \frac{\pi br}{c},
\]
which contains two equivalent representations due to Rademacher and Rédei (for \( a=1 \); cf. for example, [4], (2.2) and [2], (5) respectively).

4. **The main property of** \( \varphi_{\mp}(x, y) \). Our next purpose is to deduce a peculiar symmetry relation relating to the sums in question, by applying the calculus of residues.

**Theorem 1.** We have for \( a, b, c \) positive, mutually coprime, and for \( 0 \leq \Re(x) < 1, -1 < \Re(y) \leq 0 \) the relation
\[
\varphi_{\mp}(ax+by, -cx) + \varphi_{\mp}(cx, cy) + \varphi_{\mp}(-cy, ax+by)
\]
\[
=[1-e(ax+by)]^{-1},
\]
provided that \( ax+by, cx \) and \( cy \) are not integers.

**Proof.** We consider the integral
\[
\frac{1}{2\pi i} \int_{Q} \left[ e(z) - 1 \right]^{-1} \left[ e\left( x - \frac{b}{c} z \right) - 1 \right]^{-1} \left[ e\left( y + \frac{a}{c} z \right) - 1 \right]^{-1} dz
\]
the path of integration being a rectangle whose vertices are the points \( -\epsilon \pm ti, c - \epsilon \pm ti \) with
\[
t > \max \left\{ \frac{c}{b} |\Im(x)|, \frac{c}{a} |\Im(y)| \right\}
\]
and
\[
0 < \epsilon < \min \left\{ \frac{c}{b} (1 - \Re(x)), \frac{c}{a} (1 + \Re(y)) \right\},
\]
taken in positive direction. A straight-forward calculation shows that only singularities of the integrand inside \( Q \) are at the points:
\[
z = \lambda \quad \lambda = 0, 1, \ldots, c-1;
\]
\[
z = \frac{c}{b} (\mu + x) \quad \mu = 0, 1, \ldots, b-1;
\]
\[
z = \frac{c}{a} (\nu - y) \quad \nu = 0, 1, \ldots, a-1;
\]
by our assumptions, these are all distinct and poles of order 1 only of
the first, second, and third factor respectively. Since
\[ \text{res}_{z=\lambda} [e(z)-1]^{-1} = \frac{1}{2\pi i} \]
\[ \text{res}_{z=(c/b)(\mu\pm\sigma)} [e(x-bz/c)-1]^{-1} = -c/2\pi ib \]
\[ \text{res}_{z=(c/a)(\nu-y)} [e(y+az/c)-1]^{-1} = c/2\pi ia \]
the residue theorem yields
\[ 2\pi i \cdot \mathcal{F} = \sum_{\lambda=0}^{c-1} \left[ e\left(x - \frac{\lambda b}{c}\right) - 1 \right]^{-1} \left[ e\left(y + \frac{\lambda a}{c}\right) - 1 \right]^{-1} \]
\[ -\frac{c}{b} \sum_{\mu=0}^{b-1} \left[ e\left(\frac{a}{b} x + y + \frac{\mu a}{b}\right) - 1 \right]^{-1} \left[ e\left(\frac{c}{b} x + \frac{\mu c}{b}\right) - 1 \right]^{-1} \]
\[ + \frac{c}{a} \sum_{\nu=0}^{a-1} \left[ e\left(\frac{b}{a} y + \frac{\nu b}{a}\right) - 1 \right]^{-1} \left[ e\left(-\frac{c}{a} y + \frac{\nu c}{a}\right) - 1 \right]^{-1} \]
and therefore, by (3.4), we obtain
\[ (4.3) \quad \mathcal{E}_{a,b}^{\alpha,\beta}(cx, cy) - \mathcal{E}_{b,-a}^{\alpha,\beta}(ax+by, cx) + \mathcal{E}_{a}^{\alpha,\beta}(ax+by, -cy) = (2\pi i/c) \mathcal{F} \] .

Now, if we write
\[ \int_{Q} = \int_{c-t+i} + \int_{c-t-i} + \int_{c+t-i} + \int_{c+t+i} \]
with the integrand of (4.2) and straight-line paths, the sum of the first
and third member on the right vanishes because of the periodicity (with
period c) of
\[ [e(z)-1]^{-1} [e(x-bz/c)-1]^{-1} [e(y+az/c)-1]^{-1} . \]
On the other hand, using the estimate \(|e(u+iv)-1| \geq |e^{-2\pi v}-1|\) (u, v
arbitrary real), we find at once that the integrals along the horizontal
segments tend to zero as \( t \to \infty \). Hence (4.3) implies for \( t \to \infty \)
\[ (4.4) \quad \mathcal{E}_{a,b}^{\alpha,\beta}(ax+by, -cy) - \mathcal{E}_{b,-a}^{\alpha,\beta}(ax+by, cx) + \mathcal{E}_{c}^{\alpha,\beta}(cx, cy) = 0 \]
which is, by (2.4), equivalent to (4.1).

5. Applications: extension of the well-known reciprocity theorems.

(1) If we write
\[ (5.1) \quad \mathcal{E}_{c}^{\alpha,\beta}(x, y) = \frac{1}{c} \sum_{r(\mod c)} \cotg \frac{x-br}{c} \cotg \frac{y+ar}{c} \]
and use (3.5), then (4.1) becomes
By (3.9), this may be regarded as a generalization of the reciprocity theorem of Dedekind sums. For, by putting \( y = -x \) in (5.2) and making \( x \to 0 \), we obtain on the basis of the Laurent expansion \( \cot z = z^{-1} - \frac{1}{2} z - \cdots \)

\[
(5.3) \quad \hat{s}_{11}(b, c, a) + \hat{s}_{11}(c, a, b) + \hat{s}_{11}(a, b, c) = \frac{1}{2} + \frac{1}{12} bc + ca + ab,
\]

a remarkably symmetric three-term relation which for \( a = 1 \) reduces to (1.2) with \( h = b, k = c \). (Cf. also a result of Rademacher in \([11]\).)

(2) Let us replace in (4.1) \( x, y \) by \( x/2\pi i \) and \( y/2\pi i \) respectively, multiply both sides by \( e^{xy}(ax + by) \) and expand every member by applying (2.5), (2.6) and the power series of \( z/(e^z - 1) \). We obtain

\[
cy \sum_{m,n=1}^\infty \frac{(ax + by)^m(-cx)^n}{m!n!} \hat{s}_{m,n}(c, a, b) - \frac{x}{m!n!} \hat{s}_{m,n}(a, b, c) \]
\[
+ cx \sum_{m,n=1}^\infty \frac{(-cy)^{(ax + by)^n}}{m!n!} \hat{s}_{m,n}(b, c, a) = c^2 xy \left[ 1 + \sum_{\nu=1}^\infty B_\nu(ax + by)^\nu \right]
\]
\[
- cy \sum_{\nu=1}^\infty \frac{B_\nu(x+y)^\nu}{\nu!} - \frac{(ax + by)^\nu}{\nu!} + (cx)^\nu + c(ax + by) \sum_{\nu=1}^\infty \frac{B_\nu(x + y)^\nu}{\nu!}
\]
\[
- cx \sum_{\nu=1}^\infty \frac{B_\nu(x - y)^\nu}{\nu!} = c^2 xy \left[ 1 - \sum_{\nu=1}^\infty B_\nu(ax + by)^\nu \right]
\]

this holding identically for \( |x|, |y| < 2\pi \). If one uses still the binomial theorem and arranges our absolutely convergent series in terms of \( x^v, y^v \) \((v=1, 2, \cdots)\), then comparison of the corresponding coefficients leads without difficulty to the following system of relations:

\[
(5.4) \quad a^v \cdot (v+1)b^v c \hat{s}_{v,1}(b, c, a) + b^v \sum_{\mu=1}^\infty (-1)^{\mu+1} \binom{v+1}{\mu} e^{\mu a^{v+1-\mu}} \hat{s}_{v+1-\mu, \mu}(c, a, b)
\]
\[
\quad + c^v \cdot (v+1)ab^v \hat{s}_{v,1}(a, b, c) = B_{v+1}(a^{v+1} + b^{v+1} + (c)^{v+1}) - (v+1)B_v(ab)^v c
\]
\[
v = 1, 2, \cdots ,
\]

furthermore, by \( \binom{\alpha}{\beta} \binom{\gamma}{\delta} = \binom{\gamma - \delta}{\alpha - \beta} \binom{\gamma}{\delta} \binom{\gamma}{\alpha} \),

\[
(5.5) \quad a^v \cdot \binom{v+1}{p+1} \sum_{\mu=1}^p (-1)^{\mu+1} \binom{p+1}{\mu} b^{v+1-\mu} e^{\mu a^{v+1-\mu}} \hat{s}_{v+1-\mu, \mu}(b, c, a)
\]
\[
\quad + b^v \cdot \binom{v+1}{p} \sum_{\mu=1}^{p+1} (-1)^{\mu+1} \binom{v+1}{\mu} e^{\mu a^{v+1-\mu}} \hat{s}_{v+1-\mu, \mu}(c, a, b)
\]
The results can be written briefly in symbolic form as follows

\[ (5.6) \quad \sum_{p=1}^{\nu+1} c^p \left[ \begin{array}{c} \nu+1 \\ p+1 \end{array} \right] a^{p+1} b^{\nu-p} \delta_{\nu-p, p+1} \left( \begin{array}{c} a \\ b \end{array} \right) + \sum_{p=1}^{\nu+1} \left( \begin{array}{c} \nu+1 \\ p \end{array} \right) a^p b^{\nu+1-p} \delta_{\nu+1-p, p} \left( \begin{array}{c} a \\ c \end{array} \right) \]

\[ = B_{\nu+1} \left[ \begin{array}{c} \nu+1 \\ p \end{array} \right] a^{\nu+1} + \left( \begin{array}{c} \nu+1 \\ p+1 \end{array} \right) b^{\nu+1} - (\nu+1) B_{\nu} \left( \begin{array}{c} \nu \\ p \end{array} \right) (ab)^{\nu+1} \]

\[ 1 \leq p \leq \nu-1. \]

The factor \((-1)^{\mu}\) may plainly be suppressed in the last summand, that is,

\[ (Bc-Ba)^{\nu+1} = (Bc+Ba)^{\nu+1}. \]
therefore (5.4), (5.6) generalize (5.3) and Apostol's reciprocity theorem [1, Theorem 1].

On the other hand, putting \( \nu = 3, 5, 7, \ldots \) in (5.7), we get for \( e = 1 \)
\[
(\nu + 1) a^{\nu - \psi(s^{(\nu)} - b)p + 1} - \frac{b^{p(s^{(\nu)} - a)\nu - p + 1}}{p} 
\]
while the case \( b = 1 \) yields
\[
(\nu + 1) a^{\nu - \psi} - \frac{b^{p} c^{(\nu)}(a, c)}{p} + \frac{\psi^{\nu + 1} - \psi(\nu)}{b^{p} c^{(\nu)}(a, c)}
\]
the symbolic notations being understood in similar sense as above. (5.9) and (5.10) express the first and second reciprocity law of Carlitz respectively [3, Theorems 1, 2]³, so that we have in (5.5), (5.7) a common extension of them.

6. The sum \( \mathcal{D}_{c}(w, z) \). We now use the generalized zeta function, defined by
\[
\zeta(z, u) = \sum_{n=0}^{\infty} (u + n)^{-z}
\]
for \( \Re(z) > 1 \) and by analytic continuation for other values of \( z, u \) denoting a fixed number with \( 0 < u \leq 1 \). There holds the well-known formula of Hurwitz :
\[
\zeta(z, u) = 2(2\pi)^{z-1} \Gamma(1 - z) \times \left( \sin \frac{\pi z}{2} \sum_{n=1}^{\infty} n^{z-1} \cos 2n\pi u + \cos \frac{\pi z}{2} \sum_{n=1}^{\infty} n^{z-1} \sin 2n\pi u \right) \quad \Re(z) < 0
\]
Next we establish a functional equation for the sum
\[
\mathcal{D}_{c}^{a,b}(w, z) = \sum_{k=1}^{c-1} \zeta \left( w, \left\{ \frac{ka}{c} \right\} \right) \zeta \left( z, \left\{ \frac{kb}{c} \right\} \right)
\]
with \( (a, c) = (b, c) = 1, c > 1 \), in observing that [cf. (1.4)]
\[
\mathcal{D}_{c}^{a,b}(1-m, 1-n) = \frac{1}{mn} \left[ \mathcal{D}_{m,n}(a \ b) - B_{m}B_{n} \right] \quad m, n = 1, 2, \ldots
\]
³ In formula (3.2) of [3], the lack of the corresponding binomial coefficients before the Bernoullian numbers appears to be a typographical error.
and, by \( \zeta(z, \frac{1}{2}) = (2^s - 1) \zeta(z) \) where \( \zeta(z) = \zeta(z, 1) \) is Riemann's zeta function,

(6.4) \[ \mathcal{D}_c^{a,b}(w, z) = (2^w - 1)(2^z - 1) \cdot \zeta(w) \zeta(z). \]

**Theorem 2.** For \((a, c) = (b, c) = 1, c > 2\) and for any \(w, z\) distinct from 0 and 1 we have the relation

(6.5) \[ \mathcal{D}_c^{a,b}(w, z) = (e^{w+z} - 1) \zeta(w) \zeta(z) + \pi^{-1} (2\pi)^{w+z-1} \Gamma(1-w) \Gamma(1-z) \]
\[ \times \left\{ \cos \frac{\pi}{2} (w-z) \mathcal{D}_c^{a,b}(1-w, 1-z) - \cos \frac{\pi}{2} (w+z) \mathcal{D}_c^{a,b}(1-w, 1-z) \right\}. \]

**Proof.** 1° First let \( \Re(w) < 0, \Re(z) < 0 \). We transform

(6.6) \[ \mathcal{D}_c^{a,b}(w, z) = \sum_{\lambda=1}^{c} \zeta \left( w, \left\{ \frac{\lambda a}{c} \right\} \right) \zeta \left( z, \left\{ \frac{\lambda b}{c} \right\} \right) \]

by means of (6.1). Since the series involved in Hurwitz's formula are absolutely convergent, one obtains after substitution into (6.6)

(6.7) \[ \mathcal{D}_c^{a,b}(w, z) = 4(2\pi)^{w+z-2} \Gamma(1-w) \Gamma(1-z) \]
\[ \times \sum_{m,n=1}^{c} m^{w-1} n^{z-1} \left( \phi_{m,n} \sin \frac{\pi w}{2} \sin \frac{\pi z}{2} + \psi_{m,n} \cos \frac{\pi w}{2} \cos \frac{\pi z}{2} \right), \]

where

(6.8) \[ \phi_{m,n} = \sum_{\mu=1}^{c} \cos 2\pi \frac{\mu a}{c} \cos 2\pi \frac{\mu b}{c} = \begin{cases} c, & \text{if } c \mid am \pm bn, \\ 0 & \text{for } c \nmid am \pm bn, \\ c/2 & \text{otherwise}, \end{cases} \]

(6.9) \[ \psi_{m,n} = \sum_{\mu=1}^{c} \sin 2\pi \frac{\mu a}{c} \sin 2\pi \frac{\mu b}{c} = \begin{cases} c/2, & \text{if } c \mid am - bn \text{ but } c \nmid am + bn, \\ -c/2, & \text{if } c \mid am + bn \text{ and } c \nmid am - bn, \\ 0 & \text{otherwise}. \end{cases} \]

Hence it follows easily that

(6.10) \[ \mathcal{D}_c^{a,b}(w, z) = 2c(2\pi)^{w+z-2} \Gamma(1-w) \Gamma(1-z) \cdot \left\{ 2 \sin \frac{\pi w}{2} \sin \frac{\pi z}{2} \sum_{c \mid m, c \nmid n} m^{w-1} n^{z-1} \right\} \]
\[ + \cos \frac{\pi}{2} (w-z) \sum_{c \mid m, c \nmid n} m^{w-1} n^{z-1} - \cos \frac{\pi}{2} (w+z) \sum_{c \mid m, c \nmid n} m^{w-1} n^{z-1} \}

Now, by the functional equation of \( \zeta(s) \) we have
(6.11) \[ 4e(2\pi)^{w+z-2} \Gamma(1-w)\Gamma(1-z) \sin \frac{\pi w}{2} \sin \frac{\pi z}{2} \sum_{c|n, c \neq n} m^{w-1}n^{z-1} \]
\[ = e^{w+z-1} \zeta(w)\zeta(z). \]

Furthermore, \( ar \ (r=0, 1, \cdots, c-1) \) and \( br \ (r=0, 1, \cdots, c-1) \) being complete systems of residues mod \( c \), we can write

(6.12) \[ \sum_{\substack{am+bn \mod c \in \mathbb{N} \colon c \nmid m, c \nmid n}} m^{w-1}n^{z-1} = c^w+z-2 \sum_{r=1}^{c-1} \left( \sum_{M=0}^{\infty} \left( \left\{ \frac{rb}{c} \right\} + M \right)^{w-1} \right) \left( \sum_{N=1}^{\infty} \left( \left\{ \frac{ra}{c} \right\} + N \right)^{z-1} \right) \]
\[ = c^{w+z-2} \sum_{r=1}^{c-1} \zeta(1-w, \left\{ \frac{rb}{c} \right\}) \zeta(1-z, \left\{ \frac{ra}{c} \right\}) \]

and similarly

(6.13) \[ \sum_{\substack{am+bn \mod c \in \mathbb{N} \colon c \nmid m, c \nmid n}} m^{w-1}n^{z-1} = c^w+z-2 \sum_{r=1}^{c-1} \left( \sum_{M=0}^{\infty} \left( \left\{ \frac{rb}{c} \right\} + M \right)^{w-1} \right) \left( \sum_{N=1}^{\infty} \left( \left\{ \frac{ra}{c} \right\} + N \right)^{z-1} \right) \]
\[ = c^{w+z-2} \sum_{r=1}^{c-1} \zeta(1-w, \left\{ \frac{rb}{c} \right\}) \zeta(1-z, \left\{ \frac{ra}{c} \right\}). \]

(6.10) – (6.13) yield together

(6.14) \[ \mathfrak{D}_{c}^{a,b}(w, z) = e^{w+z-1} \zeta(w)\zeta(z) + \pi^{-1}(2\pi)^{w+z-1} \Gamma(1-w)\Gamma(1-z) \]
\[ \times \left\{ \cos \frac{\pi}{2} (w-z) \mathfrak{D}_{c}^{a}(1-w, 1-z) - \cos \frac{\pi}{2} (w+z) \mathfrak{D}_{c}^{a}(1-w, 1-z) \right\}. \]

2° Finally, (6.5) follows immediately from (6.14), in view of

\[ \mathfrak{D}_{c}^{a,b}(w, z) = \overline{\mathfrak{D}_{c}^{a,b}(w, z)} - \zeta(w)\zeta(z) \quad \Re(w) < 0, \Re(z) < 0 \]

and by analytic continuation.

7. Some remarks. In [2], Apostol finds certain finite sum representations for \( s_{c}^{h,k}(u, v) \), involving cotangents, \( \zeta(z, u) = \Gamma'(z)/\Gamma(z) \) and he uses these expressions to give a short analytic proof of (5.8) [Theorems 1, 2]. It may be noted that the above Theorem 2 implies the results in question, arising as limiting cases for \( w \to 0 \), and \( z \to 0 \), \( z=-1, -2, \cdots \).

The form of \( \mathfrak{S}_{c}^{a,b}(x, y) \), \( \mathfrak{D}_{c}^{a,b}(w, z) \) suggests applications in connection with certain Lambert series, generalizing those investigated by Rademacher, Apostol and Carlitz. I hope to return on this problem in another paper.
REFERENCES


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