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**INDUCED HOMOLOGY HOMOMORPHISMS FOR
SET-VALUED MAPS**

BARRETT O'NEILL

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§ 1. If X and Y are topological spaces, a set-valued function $F: X \rightarrow Y$ assigns to each point x of X a closed nonempty subset $F(x)$ of Y . Let H denote Čech homology theory with coefficients in a field. If X and Y are compact metric spaces, we shall define for each such function F a vector space of homomorphisms from $H(X)$ to $H(Y)$ which deserve to be called the induced homomorphisms of F . Using this notion we prove two fixed point theorems of the Lefschetz type.

All spaces we deal with are assumed to be compact metric. Thus the group $H(X)$ can be based on a group $C(X)$ of projective chains [4]. Define the support of a coordinate c_i of $c \in C(X)$ to be the union of the closures of the kernels of the simplexes appearing in c_i . Then the intersection of the supports of the coordinates of c is defined to be the support $|c|$ of c .

If $F: X \rightarrow Y$ is a set-valued function, let $F^{-1}: Y \rightarrow X$ be the function such that $x \in F^{-1}(y)$ if and only if $y \in F(x)$. Then F is *upper (lower) semi-continuous* provided F^{-1} is *closed (open)*. If both conditions hold, F is *continuous*. If $\varepsilon > 0$ is a real number, we shall also denote by $\varepsilon: X \rightarrow X$ the set-valued function such that $\varepsilon(x) = \{x' \mid d(x, x') \leq \varepsilon\}$ for each $x \in X$.

Let A and B be chain groups with supports in X and Y respectively, and let $\varepsilon > 0$ be a number. A chain map $\varphi: A \rightarrow B$ is *accurate* with respect to a set-valued function $F: X \rightarrow Y$ provided $|\varphi(a)| \subset F(|a|)$ for each $a \in A$. Further, φ is ε -*accurate* with respect to F provided φ is accurate with respect to the composite function $\varepsilon F \varepsilon$.

(1) DEFINITION. A homomorphism $h: H(X) \rightarrow H(Y)$ is an *induced homomorphism* of a set-valued function $F: X \rightarrow Y$ provided that given $\varepsilon > 0$ there is a chain map $\varphi: C(X) \rightarrow C(Y)$ such that φ is ε -accurate with respect to F and $\varphi_* = h$.

We shall say that a homology homomorphism h is *nontrivial* provided the 0-dimensional component $h_0: H_0(X) \rightarrow H_0(Y)$ is not the zero homomorphism. It will appear that a continuous set-valued function need not have a nontrivial induced homomorphism.

The set of all induced homomorphisms of an arbitrary set-valued function is, under the usual operations, a vector space. If h_F and h_G are induced homomorphisms of upper semi-continuous functions $F: X \rightarrow Y$ and $G: Y \rightarrow Z$, then $h_G h_F$ is an induced homomorphism of GF . If $F:$

$X \rightarrow Y$ is a point-valued map of a connected (compact metric) space into a compact polyhedron, then the induced homology homomorphisms of F are exactly the scalar multiples of the Čech homology homomorphism F_* . Corresponding to each function $F: X \rightarrow Y$, let $F': X \rightarrow X \times Y$ be such that $F'(x) = \{x\} \times F(x)$, for $x \in X$. If $h: H(X) \rightarrow H(X \times Y)$ is a non-trivial induced homomorphism of F' , then q_*h is a nontrivial induced homomorphism of F , where q is the projection of the productspace on Y .

If T is a triangulation of a compact polyhedron X , let $C(X, T)$ be the group of oriented simplicial chains based on T , with the given field of coefficients. We may assume that the sequence of coverings used to define $C(X)$ consists of the star-coverings associated with the successive barycentric subdivisions of a fixed triangulation of X .

(2) LEMMA. *Let X be a compact polyhedron, $F: X \rightarrow Y$ a set-valued function. Then $h: H(X) \rightarrow H(Y)$ is an induced homomorphism of F if and only if given $\varepsilon > 0$ there is an arbitrarily fine triangulation T of X and an ε -accurate chain map $\psi: C(X, T) \rightarrow C(Y)$ such that $\psi_* = h$.*

Proof. Given $\varepsilon > 0$, let T be one of the selected triangulations of X with mesh at most $\varepsilon/2$. Denote by p the projection of $C(X)$ onto the chains of the nerve of the star-covering associated with T , followed by the natural isomorphism onto $C(X, T)$. Denote by s the chain map which assigns to $c \in C(X, T)$ the element of $C(X)$ whose coordinates correspond to the successive subdivisions $Sd^i(c)$ of c . If h is an induced homomorphism, let $\varphi: C(X) \rightarrow C(Y)$ be ε -accurate and such that $\varphi_* = h$.

Now s reduces supports, hence φs is the required chain map. Conversely, if $\psi: C(X, T) \rightarrow C(Y)$ is $\varepsilon/2$ -accurate and such that $\psi_* = h$, then ψp is the required chain map, since p is $\varepsilon/2$ -accurate with respect to the identity map of X .

(3) LEMMA. *Let X and Y be compact polyhedra. If, given $\varepsilon > 0$, there is an arbitrarily fine triangulation T of X and an ε -accurate $\psi: C(X, T) \rightarrow C(Y)$ such that ψ_* is nontrivial, then F has a nontrivial induced homomorphism.*

Proof. We may assume that X and Y are connected. Let L be the finite-dimensional vector space of homomorphisms from $H(X)$ to $H(Y)$. If $\varepsilon > 0$, let $A(\varepsilon)$ be the set of homomorphisms h in L such that h preserves Kronecker index and is induced by an ε -accurate chain map of $C(X, T)$, where T has mesh less than ε . By hypothesis $A(\varepsilon)$ is not empty. Furthermore one easily shows that $A(\varepsilon)$ is a coset modulo a subspace of L , and that if $\delta < \varepsilon$, then $A(\delta) \subset A(\varepsilon)$. Thus $\bigcap_{\varepsilon > 0} A(\varepsilon)$ is not empty, but an element of this intersection is, by the preceding

lemma, a nontrivial induced homomorphism of F .

§ 2. In order to establish the existence of nontrivial induced homomorphisms in certain cases, we need some general properties of set-valued functions. Note first that if $F: X \rightarrow Y$ is continuous and K is a component of the graph $\Gamma = \{(x, y) | y \in F(x)\}$ of F , then K projects onto a component of X . In fact, the continuity of F implies that the projection $p: \Gamma \rightarrow X$ is open and closed.

(4) LEMMA. *Let X be an arcwise connected, simply connected space, $F: X \rightarrow Y$ a set-valued map. If, for each $x \in X$, $F(x)$ has exactly n components, then the graph Γ of F has exactly n components.*

Proof. Let $A = \{(x, \alpha) | x \in X \text{ and } \alpha \text{ is a component of } F(x)\}$. Topologize A as follows. If $(x, \alpha) \in A$, select mutually disjoint neighborhoods V_1, \dots, V_n in Y for the components $\alpha = \alpha_1, \dots, \alpha_n$ of $F(x)$. Since F is continuous, there is a neighborhood of x such that if $x' \in U$, then $F(x') \subset V_1 \cup \dots \cup V_n$ and $F(x')$ meets each V_i . Since $F(x')$ has n components, there is one in each V_i . Let $U_i = \{(x', \alpha') | x' \in U \text{ and } \alpha' = F(x') \cup V_i\}$. The collection of all such subsets of A generates a topology on A for which the projection $\pi: A \rightarrow X$ is a covering map, where $\pi(x, \alpha) = x$ if $(x, \alpha) \in A$. Consequently π is a homomorphism on each component of A —of which there are thus exactly n . If K' is a component of A , then $K = \{(x, y) | (x, \alpha) \in K' \text{ and } y \in \alpha\}$ is a component of Γ . In fact, since K is open and closed in Γ , it suffices to show that K is connected, but this follows from the fact that $p|K$ is strongly continuous and that $p^{-1}(x) \cap K = \alpha$ is connected for each $x \in X$.

Replacing the last step above by an application of the Vietoris mapping theorem [1] we obtain:

(5) LEMMA. *Let F be a set-valued map of a simplex σ into an arbitrary (compact metric) space. If, for each $x \in \sigma$, $F(x)$ consists of exactly n homologically trivial components, then the graph Γ of F consists of exactly n homologically trivial components.*

(6) THEOREM. *Let n be an integer, $F: X \rightarrow Y$ a set-valued map of compact polyhedra such that if $x \in X$ then $F(x)$ is either homologically trivial or consists of n homologically trivial components. Then F has a nontrivial induced homomorphism.*

Proof. Let I be the (closed) set of points for which $F(x)$ is homologically trivial. Replace F by the associated map $F': X \rightarrow X \times Y$. Since $X \times Y$ is a compact polyhedron and H is (weakly) continuous [3], if $x \in X$ and $\epsilon > 0$ there is a δ_x such that the inclusion map $\delta(F'(x)) \subset$

$\epsilon(F'(x))$ induces the zero homology homomorphism in positive dimensions. Using the compactness of I and the continuity of F one can show that a δ may be selected which is independent of x . Thus, if the dimension of X is $d \geq 1$, there exist numbers $0 < \delta < \epsilon_1 < \dots < \epsilon_d = \epsilon$ such that if $x \in X$ and $1 \leq i \leq d$ then :

(1) $\epsilon_i + \epsilon_1 < \epsilon_{i+1}$,

(2) each positive-dimensional cycle of $(\epsilon_i + \epsilon_1)(F'(x))$ bounds in $\epsilon_{i+1}(F'(x))$, and

(3) $F'(\delta(x)) \subset \epsilon_1(F'(x))$. Let T be a triangulation of X with mesh at most δ , and for each simplex σ of T which meets I select a point $x_\sigma \in \sigma \cap I$. By Lemma 3 it suffices to find an ϵ -accurate chain map $\varphi : C(X, T) \rightarrow C(X \times Y)$ such that φ_* is nontrivial. We may associate with each point p of $X \times Y$ a 0-chain, $\bar{p} \in C(X \times Y)$ with support $\{p\}$ and such that $\bar{p} \sim 0$ and $\bar{p} \sim \bar{p}'$ if and only if p and p' are in the same component of $X \times Y$. Define a homomorphism $\varphi_0 : C(X, T) \rightarrow C(X \times Y)$ as follows. If the vertex v is in I , let $\varphi_0(v) = n\bar{p}$, where $p \in F'(v)$. Otherwise, let $\varphi_0(v) = \bar{p}_1 + \dots + \bar{p}_n$, where there is one point p_i in each component of $F'(v)$. Now φ_0 is extendable to a chain map φ_1 accurate with respect to F' provided that if vw is a 1-simplex of T then $\varphi_0(v) \sim \varphi_0(w)$ in $F'(vw)$. Using Lemma 4 and the preceding remark, one checks this homology in case vw does, or does not, meet I . Clearly φ_1 may be accurately extended to those chains in $C(X, T)$ whose supports avoid I . We complete the definition of φ by an induction on dimension, defining the chain map φ_q on chains of dimension at most q so that $|\varphi_q(c)| \subset \subset \epsilon_q(F'(|c|)$. The homomorphism φ_1 is correctly defined. If φ_{q-1} has been defined ($q \geq 1$), it suffices to define $\varphi_q(\sigma)$, where σ is an oriented q -simplex of T which meets I . If τ is an oriented $(q-1)$ -face of σ , then

$$\varphi_{q-1}(\tau) \subset \epsilon_{q-1}(F'(\tau)) \subset \epsilon_{q-1}(F'(\delta(x_\sigma))) \subset (\epsilon_{q-1} + \epsilon_1)(F'(x_\sigma)).$$

Thus $\varphi_{q-1}(\partial\sigma)$ bounds in $\epsilon_q(F'(\sigma))$ and $\varphi_q(\sigma)$ may be correctly defined.

(7) THEOREM. *Let X be a compact 1-dimensional polyhedron with first Betti number $R_1 \leq 1$. Then any set-valued map F of X into a compact polyhedron has a nontrivial induced homomorphism.*

Proof. Given $\epsilon > 0$ and a triangulation T of X we must find a chain map ϵ -accurate with respect to F' and such that φ_* is nontrivial. If vw is a 1-simplex of T and φ is defined accurately on v , then φ can be accurately defined on w and vw so that $\partial(\varphi(vw)) = \varphi(w) - \varphi(v)$, because each component of $F'(v)$ is contained in a component of $F'(vw)$ which meets $F'(w)$. Thus if $R_1 = 0$, the required chain map exists. If $R_1 = 1$, then X may be expressed as the union of a circle C and a finite num-

ber of trees, each meeting C in at most one point. If v^0, v^1, \dots, v^n are the vertices of C , we may suppose that the 1-simplexes of C are $v^0v^1, v^1v^2, \dots, v^{n-1}v^n$. Pick $y \in F(v^0)$ and let $\varphi^0(v^0) = \bar{y}$. There is a point $z \in F(v^1)$ such that if $\varphi^0(v^1) = \bar{z}$, then $\varphi^0(v^0v^1)$ can be correctly defined. Repeating this step we reach $\varphi^0(v^n)$, then $\varphi^1(v^0)$, and so on. Since Y is a compact polyhedron there exist integers $j < k$ such that $\varphi^k(v^0) - \varphi^j(v^0) = \partial c$, with $|c| < \epsilon(F(v^0))$. Then on C let $\varphi = \sum_j^k \varphi^j$, except that

$$\varphi(v^nv^0) = \sum_j^k \varphi^j(v^nv^0) + c.$$

So far φ is ϵ -accurate and accurate on vertices. But then as in the case $R_1=0$ this homomorphism may be extended correctly to $C(X, T)$.

We shall see in the next section that this theorem does not hold if the condition on either the dimension or the first Betti number is omitted.

§ 3. The Lefschetz theorem holds for set-valued functions in this form :

(8) LEMMA. *Let X be a compact polyhedron, $F: X \rightarrow X$ an upper semi-continuous set-valued function. If h is an induced homology homomorphism of F and the Lefschetz number $\Lambda(h) = \sum (-1)^q \text{trace } h_q$ is not zero, then F has a fixed point.*

The usefulness of this fact, of course, depends on our knowledge of the induced homomorphisms of a given set-valued map. From § 2 we get :

(9) THEOREM. *Let F be a set-valued self-map of a compact polyhedron X such that if $x \in X$, $F(x)$ is homologically trivial or consists of n homologically trivial components. Then F has a nontrivial homomorphism h such that if $\Lambda(h) \neq 0$, then F has a fixed point. If, further, X is homologically trivial, then F has a fixed point.*

The case $n=1$ is the polyhedral form of the Eilenberg-Montgomery theorem [2], except that the requirement that F be lower semi-continuous is then superfluous. However, if $n > 1$ upper semi-continuity alone is not sufficient. For example, consider the self-map F of the Euclidean interval $[-1, 1]$ for which $F(0) = \{-1, 1\}$ and $F(x)$ is 1 for $x > 0$, -1 for $x < 0$. Also, if $n > 1$ the space of induced homomorphisms need not be 1-dimensional as in the case $n=1$.

It does not appear that this result can be generalized by altering the number of components $F(x)$ is permitted to have. Mr. Richard Dunn has shown by a series of examples (unpublished) that if S is any finite set of positive integers—except certain sets of the form $\{2, n\}$ and, necessarily, $\{1, n\}$ —there is a self-map F of the 2-cell without fixed points

and such that for each point x the number of points in $F(x)$ occurs in S .

(10) THEOREM. *Let X be a compact 1-dimensional polyhedron with first Betti number $R_1 \leq 1$. Every set-valued self-map F of X has a non-trivial induced homomorphism such that if $\Delta(h) \neq 0$ then F has a fixed point.*

In particular, as is known, every set-valued self-map of a tree has a fixed point.

The last remark of § 2 may be justified by exhibiting suitable fixed-point-free maps. As for the restriction on the Betti number for example, let X be a compact connected 1-dimensional polyhedron without end points and such that $R_1 > 1$. If $\varepsilon > 0$ is sufficiently small, the function $F: X \rightarrow X$ for which $F(x) = \{x' \mid d(x, x') = \varepsilon\}$ will be continuous if d is a suitable metric and any induced homomorphism of F will be a scalar multiple of the identity homomorphism of $H(X)$. Thus a nontrivial induced homomorphism of such a function would have nonzero Lefschetz number, contradicting (8).

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Vol. 7, No. 2

February, 1957

William F. Donoghue, Jr., <i>The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation</i>	1031
Michael (Mihály) Fekete and J. L. Walsh, <i>Asymptotic behavior of restricted extremal polynomials and of their zeros</i>	1037
Shaul Foguel, <i>Biorthogonal systems in Banach spaces</i>	1065
David Gale, <i>A theorem on flows in networks</i>	1073
Ioan M. James, <i>On spaces with a multiplication</i>	1083
Richard Vincent Kadison and Isadore Manual Singer, <i>Three test problems in operator theory</i>	1101
Maurice Kennedy, <i>A convergence theorem for a certain class of Markoff processes</i>	1107
G. Kurepa, <i>On a new reciprocity, distribution and duality law</i>	1125
Richard Kenneth Lashof, <i>Lie algebras of locally compact groups</i>	1145
Calvin T. Long, <i>Note on normal numbers</i>	1163
M. Mikolás, <i>On certain sums generating the Dedekind sums and their reciprocity laws</i>	1167
Barrett O'Neill, <i>Induced homology homomorphisms for set-valued maps</i>	1179
Mary Ellen Rudin, <i>A topological characterization of sets of real numbers</i>	1185
M. Schiffer, <i>The Fredholm eigen values of plane domains</i>	1187
F. A. Valentine, <i>A three point convexity property</i>	1227
Alexander Doniphan Wallace, <i>The center of a compact lattice is totally disconnected</i>	1237
Alexander Doniphan Wallace, <i>Two theorems on topological lattices</i>	1239
G. T. Whyburn, <i>Dimension and non-density preservation of mappings</i>	1243
John Hunter Williamson, <i>On the functional representation of certain algebraic systems</i>	1251