A TOPOLOGICAL CHARACTERIZATION OF SETS OF REAL NUMBERS

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We will say that a space $E$ is of class $L$ if $E$ is a separable metric space which satisfies the following conditions:

1. Each component of $E$ is a point or an arc (closed, open, or half-open), and no interior point of an arc-component $A$ is a limit point of $E - A$.

2. Each point of $E$ has arbitrarily small neighborhoods whose boundaries are finite sets.

The purpose of this note is to show that a necessary and sufficient condition that a space be homeomorphic to a set of real numbers is that it be of class $L$.

This gives an affirmative answer to a question raised by de Groot in [1].

In [2] L. W. Cohen proved that a separable metric space is homeomorphic to a set of real numbers if and only if it satisfies (1) above and (3) and (4) below:

3. $E$ is zero-dimensional at each of its point-components.

4. If $p$ is an end point of an arc-component $A$, then the space $(E - A) \cup \{p\}$ is zero-dimensional at $p$.

Any set of real numbers is clearly of class $L$. To prove the converse it is sufficient to show that every space of class $L$ satisfies conditions (3) and (4). To this end it is clearly enough to show the following:

If $X$ is a component of the space $E$ of class $L$ and $\varepsilon$ is a positive number, there is a set $U(X, \varepsilon)$ which is both open and closed, contains $X$, and is contained in the union of $X$ with the $\varepsilon$-neighborhoods of its endpoints (if any).

Suppose $X$ is a component of a space $E$ of class $L$ and $\varepsilon$ is a positive number. There exists an open set $V$ which contains $X$ but contains no point whose distance from $X$ exceeds $\varepsilon$, such that the boundary $B$ of $V$ is finite; if $X$ is a point, we can apply (2) directly to obtain $V$; if $X$ is an arc, let $V$ consist of $X$ plus type (2) neighborhoods of the end points of $X$ (if any).

Let $G$ denote the sets of all points $p$ of $E$ such that $E$ is the union of two mutually separated sets $S_p$ and $T_p$, where $S_p$ contains $X$ and $T_p$ contains $p$. 

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Case I. $E - G = X$. Then $G$ contains $B$. Let $R$ be the union of all sets $T_p$ for $p$ in $B$. Since $B$ is finite, $R$ is both open and closed and $V - R$ is suitable for $U(X, \epsilon)$.

Case II. $E - G \not= X$. Since $X$ is a component, $E - G$ is the union of two mutually separated sets $Y$ and $Z$, where $Y$ contains $X$ and $Z$ is not empty. It will be shown that there is a set $K$ which is both open and closed and contains $Z$ but does not intersect $X$, thus contradicting the fact that $Z$ is not in $G$.

The definition of $G$, together with the fact that $E$ has a countable base, implies that $G = \bigcup_{n=1}^{\infty} G_n$, where each $G_n$ is both open and closed.

Let $p$ be a point of $Z$. If $q$ is a point of $G$, then $T_q$ contains $q$ and not $p$. The reasoning used in Case I shows that there is a neighborhood $N_p$ of $p$ which has no boundary point in $G$ and whose diameter is less than half the distance from $p$ to $Y$.

Let $\{H_n\} (n = 1, 2, 3, \cdots)$ be a countable base for $E$. If $H_n$ is not a subset of $N_p$ for any $p$ in $Z$, put $K_n = 0$. If, for some $p$ in $Z$, $H_n$ is a subset of $N_p$, let $N$ be one such $N_p$ and put $K_n = N - G_n$. Let $K = \bigcup_{n=1}^{\infty} K_n$. By the choice of $N_p$, $K$ has no limit point in $Y$. No $K_n$ has a boundary point in $G$ and only finitely many sets $K_n$ intersect any $G_i$. Consequently $K$ has no boundary points in $G$ and $K$ is both open and closed. Since $Z$ is a subset of $K$ and $X$ does not intersect $K$, the proof is complete.

References

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