

# Pacific Journal of Mathematics

**A TOPOLOGICAL CHARACTERIZATION OF SETS OF REAL  
NUMBERS**

MARY ELLEN RUDIN

# A TOPOLOGICAL CHARACTERIZATION OF SETS OF REAL NUMBERS

MARY ELLEN RUDIN

We will say that a space  $E$  is of class  $L$  if  $E$  is a separable metric space which satisfies the following conditions :

(1) *Each component of  $E$  is a point or an arc (closed, open, or half-open), and no interior point of an arc-component  $A$  is a limit point of  $E - A$ .*

(2) *Each point of  $E$  has arbitrarily small neighborhoods whose boundaries are finite sets.*

The purpose of this note is to show that a necessary and sufficient condition that a space be homeomorphic to a set of real numbers is that it be of class  $L$ .

This gives an affirmative answer to a question raised by de Groot in [1].

In [2] L. W. Cohen proved that a separable metric space is homeomorphic to a set of real numbers if and only if it satisfies (1) above and (3) and (4) below :

(3)  *$E$  is zero-dimensional at each of its point-components.*

(4) *If  $p$  is an end point of an arc-component  $A$ , then the space  $(E - A) \cup \{p\}$  is zero-dimensional at  $p$ .*

Any set of real numbers is clearly of class  $L$ . To prove the converse it is sufficient to show that every space of class  $L$  satisfies conditions (3) and (4). To this end it is clearly enough to show the following :

*If  $X$  is a component of the space  $E$  of class  $L$  and  $\epsilon$  is a positive number. there is a set  $U(X, \epsilon)$  which is both open and closed, contains  $X$ , and is contained in the union of  $X$  with the  $\epsilon$ -neighborhoods of its endpoints (if any).*

Suppose  $X$  is a component of a space  $E$  of class  $L$  and  $\epsilon$  is a positive number. There exists an open set  $V$  which contains  $X$  but contains no point whose distance from  $X$  exceeds  $\epsilon$ , such that the boundary  $B$  of  $V$  is finite; if  $X$  is a point, we can apply (2) directly to obtain  $V$ ; if  $X$  is an arc, let  $V$  consist of  $X$  plus type (2) neighborhoods of the end points of  $X$  (if any).

Let  $G$  denote the sets of all points  $p$  of  $E$  such that  $E$  is the union of two mutually separated sets  $S_p$  and  $T_p$ , where  $S_p$  contains  $X$  and  $T_p$  contains  $p$ .

*Case I.*  $E-G=X$ . Then  $G$  contains  $B$ . Let  $R$  be the union of all sets  $T_p$  for  $p$  in  $B$ . Since  $B$  is finite,  $R$  is both open and closed and  $V-R$  is suitable for  $U(X, \varepsilon)$ .

*Case II.*  $E-G \neq X$ . Since  $X$  is a component,  $E-G$  is the union of two mutually separated sets  $Y$  and  $Z$ , where  $Y$  contains  $X$  and  $Z$  is not empty. It will be shown that there is a set  $K$  which is both open and closed and contains  $Z$  but does not intersect  $X$ , thus contradicting the fact that  $Z$  is not in  $G$ .

The definition of  $G$ , together with the fact that  $E$  has a countable base, implies that  $G = \bigcup_{n=1}^{\infty} G_n$ , where each  $G_n$  is both open and closed.

Let  $p$  be a point of  $Z$ . If  $q$  is a point of  $G$ , then  $T_q$  contains  $q$  and not  $p$ . The reasoning used in Case I shows that there is a neighborhood  $N_p$  of  $p$  which has no boundary point in  $G$  and whose diameter is less than half the distance from  $p$  to  $Y$ .

Let  $\{H_n\}$  ( $n=1, 2, 3, \dots$ ) be a countable base for  $E$ . If  $H_n$  is not a subset of  $N_p$  for any  $p$  in  $Z$ , put  $K_n=0$ . If, for some  $p$  in  $Z$ ,  $H_n$  is a subset of  $N_p$ , let  $N$  be one such  $N_p$  and put  $K_n=N-G_n$ . Let  $K = \bigcup_{n=1}^{\infty} K_n$ . By the choice of  $N_p$ ,  $K$  has no limit point in  $Y$ . No  $K_n$  has a boundary point in  $G$  and only finitely many sets  $K_n$  intersect any  $G_i$ . Consequently  $K$  has no boundary points in  $G$  and  $K$  is both open and closed. Since  $Z$  is a subset of  $K$  and  $X$  does not intersect  $K$ , the proof is complete.

#### REFERENCES

1. J. de Groot, *On Cohen's topological characterization of sets of real numbers*, Nederl. Akad. Wetensch. Proc. Ser. A, **58** (1955), 33-35.
2. L. W. Cohen, *A characterization of those subsets of metric separable space which are homeomorphic with subsets of the linear continuum*, Fund. Math. **14** (1929), 281-303.

UNIVERSITY OF ROCHESTER

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. L. ROYDEN  
Stanford University  
Stanford, California

R. A. BEAUMONT  
University of Washington  
Seattle 5, Washington

A. L. WHITEMAN  
University of Southern California  
Los Angeles 7, California

E. G. STRAUS  
University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH  
C. E. BURGESS  
M. HALL  
E. HEWITT

A. HORN  
V. GANAPATHY IYER  
R. D. JAMES  
M. S. KNEBELMAN

L. NACHBIN  
I. NIVEN  
T. G. OSTROM  
M. M. SCHIFFER

G. SZEKERES  
F. WOLF  
K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
OREGON STATE COLLEGE  
UNIVERSITY OF OREGON  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF UTAH  
WASHINGTON STATE COLLEGE  
UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
HUGHES AIRCRAFT COMPANY  
THE RAMO-WOOLDRIDGE CORPORATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

# Pacific Journal of Mathematics

Vol. 7, No. 2

February, 1957

William F. Donoghue, Jr., <i>The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation</i> .....	1031
Michael (Mihály) Fekete and J. L. Walsh, <i>Asymptotic behavior of restricted extremal polynomials and of their zeros</i> .....	1037
Shaul Foguel, <i>Biorthogonal systems in Banach spaces</i> .....	1065
David Gale, <i>A theorem on flows in networks</i> .....	1073
Ioan M. James, <i>On spaces with a multiplication</i> .....	1083
Richard Vincent Kadison and Isadore Manual Singer, <i>Three test problems in operator theory</i> .....	1101
Maurice Kennedy, <i>A convergence theorem for a certain class of Markoff processes</i> .....	1107
G. Kurepa, <i>On a new reciprocity, distribution and duality law</i> .....	1125
Richard Kenneth Lashof, <i>Lie algebras of locally compact groups</i> .....	1145
Calvin T. Long, <i>Note on normal numbers</i> .....	1163
M. Mikolás, <i>On certain sums generating the Dedekind sums and their reciprocity laws</i> .....	1167
Barrett O'Neill, <i>Induced homology homomorphisms for set-valued maps</i> .....	1179
Mary Ellen Rudin, <i>A topological characterization of sets of real numbers</i> .....	1185
M. Schiffer, <i>The Fredholm eigen values of plane domains</i> .....	1187
F. A. Valentine, <i>A three point convexity property</i> .....	1227
Alexander Doniphan Wallace, <i>The center of a compact lattice is totally disconnected</i> .....	1237
Alexander Doniphan Wallace, <i>Two theorems on topological lattices</i> .....	1239
G. T. Whyburn, <i>Dimension and non-density preservation of mappings</i> .....	1243
John Hunter Williamson, <i>On the functional representation of certain algebraic systems</i> .....	1251