A TOPOLOGICAL CHARACTERIZATION OF SETS OF REAL NUMBERS

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We will say that a space $E$ is of class $L$ if $E$ is a separable metric space which satisfies the following conditions:

1. Each component of $E$ is a point or an arc (closed, open, or half-open), and no interior point of an arc-component $A$ is a limit point of $E-A$.

2. Each point of $E$ has arbitrarily small neighborhoods whose boundaries are finite sets.

The purpose of this note is to show that a necessary and sufficient condition that a space be homeomorphic to a set of real numbers is that it be of class $L$.

This gives an affirmative answer to a question raised by de Groot in [1].

In [2] L. W. Cohen proved that a separable metric space is homeomorphic to a set of real numbers if and only if it satisfies (1) above and (3) and (4) below:

3. $E$ is zero-dimensional at each of its point-components.

4. If $p$ is an end point of an arc-component $A$, then the space $(E-A) \cup \{p\}$ is zero-dimensional at $p$.

Any set of real numbers is clearly of class $L$. To prove the converse it is sufficient to show that every space of class $L$ satisfies conditions (3) and (4). To this end it is clearly enough to show the following:

If $X$ is a component of the space $E$ of class $L$ and $\epsilon$ is a positive number, there is a set $U(X, \epsilon)$ which is both open and closed, contains $X$, and is contained in the union of $X$ with the $\epsilon$-neighborhoods of its endpoints (if any).

Suppose $X$ is a component of a space $E$ of class $L$ and $\epsilon$ is a positive number. There exists an open set $V$ which contains $X$ but contains no point whose distance from $X$ exceeds $\epsilon$, such that the boundary $B$ of $V$ is finite; if $X$ is a point, we can apply (2) directly to obtain $V$; if $X$ is an arc, let $V$ consist of $X$ plus type (2) neighborhoods of the end points of $X$ (if any).

Let $G$ denote the sets of all points $p$ of $E$ such that $E$ is the union of two mutually separated sets $S_p$ and $T_p$, where $S_p$ contains $X$ and $T_p$ contains $p$. 

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Case I. \( E - G = X \). Then \( G \) contains \( B \). Let \( R \) be the union of all sets \( T_p \) for \( p \) in \( B \). Since \( B \) is finite, \( R \) is both open and closed and \( V - R \) is suitable for \( U(X, \varepsilon) \).

Case II. \( E - G \neq X \). Since \( X \) is a component, \( E - G \) is the union of two mutually separated sets \( Y \) and \( Z \), where \( Y \) contains \( X \) and \( Z \) is not empty. It will be shown that there is a set \( K \) which is both open and closed and contains \( Z \) but does not intersect \( X \), thus contradicting the fact that \( Z \) is not in \( G \).

The definition of \( G \), together with the fact that \( E \) has a countable base, implies that \( G = \bigcup_{n=1}^{\infty} G_n \), where each \( G_n \) is both open and closed.

Let \( p \) be a point of \( Z \). If \( q \) is a point of \( G \), then \( T_q \) contains \( q \) and not \( p \). The reasoning used in Case I shows that there is a neighborhood \( N_p \) of \( p \) which has no boundary point in \( G \) and whose diameter is less than half the distance from \( p \) to \( Y \).

Let \( \{H_n\} (n=1, 2, 3, \cdots) \) be a countable base for \( E \). If \( H_n \) is not a subset of \( N_p \) for any \( p \) in \( Z \), put \( K_n = 0 \). If, for some \( p \) in \( Z \), \( H_n \) is a subset of \( N_p \), let \( N \) be one such \( N_p \) and put \( K_n = N - G_n \). Let \( K = \bigcup_{n=1}^{\infty} K_n \). By the choice of \( N_p \), \( K \) has no limit point in \( Y \). No \( K_n \) has a boundary point in \( G \) and only finitely many sets \( K_n \) intersect any \( G_i \). Consequently \( K \) has no boundary points in \( G \) and \( K \) is both open and closed. Since \( Z \) is a subset of \( K \) and \( X \) does not intersect \( K \), the proof is complete.

**References**


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