

# Pacific Journal of Mathematics

**THE CENTER OF A COMPACT LATTICE IS TOTALLY  
DISCONNECTED**

ALEXANDER DONIPHAN WALLACE

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The purpose of this note is to prove the theorem of the title. A topological lattice is a Hausdorff space together with a pair of continuous functions  $\wedge: L \times L \rightarrow L$ ,  $\vee: L \times L \rightarrow L$  satisfying the usual conditions for lattice operations. As is customary we may write  $x \wedge y$  in place of  $\wedge(x, y)$ . All references are to Chapter II of [1]. We assume the reader to be familiar with the elementary facts concerning topological algebras (groups, lattices, semigroups) and set-theoretic topology.

**THEOREM.** *The center of a compact lattice is totally disconnected.*

*Proof.* Let  $L$  be a compact lattice. As is wellknown  $L$  has a zero and a unit, 0 and 1. If  $A$  is the set of pairs  $(x, y) \in L \times L$  such that  $x \wedge y = 0$  and  $x \vee y = 1$  then  $A = \wedge^{-1}(0) \cap \vee^{-1}(1)$  so that  $A$  is closed. The projection  $(x, y) \rightarrow x$  takes  $A$  onto the closed set  $B$  and  $B$  is the set of all  $x \in L$  which admit a complement.

Now  $N$ , the set of neutral elements of  $L$ , is the intersection of the maximal distributive sublattices by Theorem 11. But if  $D$  is a distributive sublattice of  $L$  its closure is also a distributive sublattice. It follows that  $N$  is closed. By the corollary to Theorem 10 the center  $C$  of  $L$  is  $N \cap B$  so that  $C$  is closed.

By the lemma on page 27 each element  $x \in C$  has a *unique* complement  $k(x) \in C$ . We will show that  $k: C \rightarrow C$  is continuous. If  $G$  is the subset of  $C \times C$  consisting of all  $(x, k(x))$  with  $x \in C$  it is enough to show that  $G$  is closed since  $C$  is compact. But by the remarks above we have  $G = (C \times C) \cap \wedge^{-1}(0) \cap \vee^{-1}(1)$ .

Now  $C$  is a distributive lattice (Theorem 9 and Corollary p. 29) with unique complements. Thus  $C$  is a commutative *topological* group under the operations

$$x + y = (x \wedge k(y)) \vee (k(x) \wedge y), \quad -x = x$$

all of whose elements are of order 2, that is,  $x + x = 0$  for all  $x$ . If  $Q$  is the component of  $C$  containing 0 and if  $q \in Q$ ,  $q \neq 0$ , then there is a continuous homomorphism  $f$  taking  $Q$  into  $Z$ , the reals mod 1, such that  $f(q) \neq f(0)$ . Since  $f(Q)$  is connected it contains an interval of  $Z$  and therefore contains an element not of finite order. Since the order of each element of  $Q$  is two this is a contradiction. Hence  $Q$  contains

only 0 and therefore is totally disconnected. The proof of the Theorem is complete.

#### REFERENCE

1. G. Birkhoff, *Lattice theory*, New York, 1948.

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