ON THE CASIMIR OPERATOR

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The Casimir operator is an important tool in the study of associative [4], Lie [4] and alternative algebras [7]. However its use has been for algebras of characteristic 0. We give a new definition of the Casimir operator for associative, Lie and alternative algebras, which keeps desirable properties of the usual Casimir operator and which is useful for arbitrary characteristic.

We show that under certain conditions our Casimir operator is the identity transformation and for non-degenerate alternative (or associative) algebras we show that it is the transformation into which the identity element of the algebra maps. We apply our results to obtain the first Whitehead lemma for non-degenerate alternative algebras of arbitrary characteristic. We also obtain a special case of the Levi theorem for Lie algebras of prime characteristic.

1. The Casimir Operator. Let $\mathfrak{g}$ be an associative, Lie or alternative algebra with basis $e_1, e_2, \ldots, e_n$ over an arbitrary field $\mathbb{F}$. For uniformity we use the notation $x \rightarrow S_x$ for a representation of $\mathfrak{g}$, where if $\mathfrak{g}$ is alternative we mean the $S_x$ part of a representation $x \rightarrow (S_x, T_x)$. If $\mathfrak{g}$ is a Lie or associative algebra, $f(x, y) = t(S_x S_y)$ where $t$ is the trace function, is an invariant symmetric bilinear form. In [7, p. 444] it is shown that if $\mathfrak{g}$ is alternative this form is invariant if $\mathfrak{g}$ is not of characteristic 2. For arbitrary characteristic we have

$$t(S_x S_y) = t(S_x S_y S_z + S_x T_y S_z - S_x S_z T_y)$$

$$= t(S_x S_y S_z + S_x T_y S_z - T_y S_x S_z) = t(S_x S_z).$$

Similarly $t(T_x T_y)$ is invariant.

We call $\mathfrak{g}$ non-degenerate if $t(R_x R_y)$ is non-degenerate where $R$ is the representation of right multiplications. It can be shown that this is equivalent to the non-degeneracy of the bilinear form $t(L_x L_y)$ of the left multiplications. It is well known that if $\mathfrak{g}$ is a non-degenerate alternative (or associative) algebra it is a direct sum of simple algebras. Dieudonne [3] has shown that this is also true for Lie algebras.

If $\mathfrak{g}$ is semi-simple and $\mathfrak{g}$ is of characteristic 0, the usual Casimir operator $I^*_\mathfrak{g}$ for the representation $S$ is defined as follows: Let $\mathfrak{R}$ be the set of all $x$ of $\mathfrak{g}$ such that $t(S_x S_y) = 0$ for all $y$ of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{R} \oplus \mathbb{C}$ where $\mathfrak{R}$ and $\mathbb{C}$ are semi-simple ideals of $\mathfrak{g}$. Let $e'_1, e'_2, \ldots, e'_k$ be the
complementary basis to a basis $e_1, e_2, \ldots, e_k$ of $\mathbb{C}$ such that $t(S_iS_j) = \delta_{ij}$ (Kroneker’s delta). (Note that the complementary basis depends on the representation.) Then $I_s^* = \sum_{i=1}^{k} S_iS_i'$.

For arbitrary $\mathfrak{g}$ we define a new Casimir operator $I_s$ for each non-degenerate $\mathfrak{g}$. This will include every semi-simple $\mathfrak{g}$ of characteristic 0, since $\mathfrak{g}$ is non-degenerate in this case. We use the same complementary basis $e_1', e_2', \ldots, e_n'$ such that $t(R_iR_j) = \delta_{ij}$ for every representation (or anti-representation) and define

\[ I_s = \sum_{i=1}^{n} S_iS_i'. \]

If $\mathfrak{g}$ is alternative we also define $I_t = \sum_{i=1}^{n} T_iT_i'$.

Unlike $I_s^*$, $I_s$ does not automatically reduce to zero when $t(S_xS_y) = 0$ for all $x, y$ of $\mathfrak{g}$. In fact it follows from Corollary 3.1 below that for alternative algebras $I_s \neq 0$ if $S \neq 0$. We note also that for the representation $x \rightarrow R_x$ we have $I_s^* = I_r$.

Analogous to the corresponding result for $I_s^*$ for Lie and associative algebras [4, p. 682] and for alternative algebras [7, p. 445] we have the following theorem.

**Theorem 1.** Let $I_s$ be the Casimir operator (1) for a representation $x \rightarrow S_x(x \rightarrow (S_x, T_x))$ of a non-degenerate Lie or associative (alternative) algebra $\mathfrak{g}$ over an arbitrary field. Then $I_s$ commutes with $S_x$ (and $T_x$) for all $x$ of $\mathfrak{g}$.

Except for the commutativity of $I_s$ and $T_x$ which will be proved along with Lemma 3.2, the proof is similar to those in the references.

We also have the following result which follows from the properties of the complementary basis.

**Theorem 2.** Let $\mathfrak{g}$ be a non-degenerate associative, Lie or alternative algebra over an arbitrary field. Then the Casimir operators $I_r$ and $I_l$ of the right and left multiplications of $\mathfrak{g}$ are both the identity transformation.

2. Application to alternative (and associative) algebras. Since every associative algebra is an alternative algebra, the results of this section hold for associative algebras.

In place of the identities (4) of [6] used in the definition of a representation $x \rightarrow (S_x, T_x)$ of an alternative algebra $\mathfrak{g}$, we will use the

For simplification we write $S_t$ as $S_i$ and $S_t'$ as $S_i'$. 

\[ 2 \text{ For simplification we write } S_t \text{ as } S_i \text{ and } S_t' \text{ as } S_i'. \]
equivalent (except for characteristic 2) identities

\[(2) \quad S_x^e = S_x, \quad T_x^e = T_x^e \quad \text{for all } x \text{ of } \mathfrak{A},\]

in order to insure that the semi-direct sum [6, p. 3] or split null extension \( \mathfrak{E} = \mathfrak{A} + \mathfrak{M} \) of \( \mathfrak{A} \) and the representation space \( \mathfrak{M} \) is an alternative algebra for arbitrary characteristic.

Theorem 3. For every representation \( S \) of a non-degenerate alternative algebra \( \mathfrak{A} \), \( I'_S = S_e \) where \( e = \sum e_i e'_i \) is the identity element of \( \mathfrak{A} \).

The proof follows from Theorem 2 and the properties of the complementary basis.

Corollary 3.1. If \( S \neq 0 \) the matrix of \( I'_S \) can be taken to have the form \( \text{diag} (I, 0) \). Hence if in addition the representation is irreducible, \( I'_S \) is the identity transformation.

Proof. By (2), \( S_x^e = S_e \) and the result follows.

Corollary 3.2. \( I'_S S_x = S_x \) for all \( x \) of \( \mathfrak{A} \).

Proof. Assume \( S \neq 0 \) and take \( I'_S \) to have the form \( \text{diag} (I, 0) \). Then the matrix of \( S_x \) must have the form \( \text{diag} (S'_x, S'_x) \) where \( I \) and \( S'_x \) have the same order. By identity (4) of [6] we have \( T_x I'_S - I'_S T_x = S_x - S_x I'_S \). Hence \( S_x'' = 0 \) and \( T_x = \text{diag} (T'_x, T''_x) \) and so \( S_x I'_S = S_x \). This completes the proof of Theorem 1, for we also have \( T_x I'_S = I'_S T_x \).

Evidently all of the above results also hold when \( S \) is replaced by \( T \).

Now for a non-degenerate alternative algebra \( \mathfrak{A} \) with neither \( S \) nor \( T = 0 \) we may apply Corollary 3.1 and Theorem 1 to take

\[(3) \quad I'_S = \text{diag} (I^{(1)}, I^{(2)}, 0^{(3)}, 0^{(4)}), \quad I'_T = \text{diag} (I^{(1)}, 0^{(2)}, I^{(3)}, 0^{(4)}) \]

where the superscript \( (i) \) indicates the matrix has order \( k_i \) and each \( I \) is an identity matrix and \( S_x^{(3)} = 0^{(3)}, \ T_x^{(2)} = 0^{(2)} \). Also \( x \to (S_x^{(1)}, T_x^{(2)}), \ (i=1, 2, 3) \) are representations of \( \mathfrak{A} \) with respective Casimir operators

\[(4) \quad I'_S^{(1)} = I^{(1)}; \quad I'_S^{(2)} = I^{(2)}; \quad I'_T^{(2)} = 0^{(2)}; \quad I'_S^{(3)} = 0^{(3)}; \quad I'_T^{(3)} = I^{(3)}.\]

Thus the representation space \( \mathfrak{M} \) can be expressed as \( \mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 \).
where $\mathfrak{M}_i$ is an invariant subspace of dimension $k_i$ and hence is an ideal of the split-null extension $\mathcal{S} = \mathfrak{A} + \mathfrak{M}$. It also follows that $\mathfrak{M}_2$ and $\mathfrak{M}_3$ are in the nucleus [2] of $\mathcal{S}$.

We are now able to obtain the following generalization of the first Whitehead lemma (see [8]) for alternative algebras of characteristic zero [6, Theorem 3].

**Theorem 4.** Let $\mathfrak{A}$ be a non-degenerate alternative algebra over an arbitrary field and let $x \rightarrow (S_x, T_x)$ be a representation of $\mathfrak{A}$ acting in a space $M$. Let $\mathcal{S}$ be the split null extension $\mathcal{S} = \mathfrak{A} + \mathfrak{M}$ and let $h(x)$ be a linear mapping of $\mathfrak{A}$ into $\mathfrak{M}$ such that

$$h(xy) = xh(y) + h(x)y = h(x)S_y + h(y)T_x$$

for all $x, y$ of $\mathfrak{A}$. Then $h(x)$ is an inner derivation of $\mathcal{S}$. If $\mathfrak{A}$ is not of characteristic 2 then

$$h(x) = [x, g] + \frac{x}{2} \sum_{i=1}^{n} \{[R_i, R_{h(e_i)}] + [L_i, L_{h(e_i)}]\}$$

where $g$ is in the nucleus of $\mathcal{S}$; $R, L$ are right and left multiplications in $\mathcal{S}$ and $e_1, e_2, \ldots, e_n$ are a complementary basis to a basis $e_1, e_2, \ldots, e_n$ of $\mathfrak{A}$.

**Proof.** If either $S$ or $T$ is zero the theorem follows similarly to the associative characteristic zero case, so assume neither is. Since $\mathfrak{M}_i$ is invariant,

$$h(x) = h_0(x) + h_1(x) + h_2(x)$$

where $h_j(x)$ is a linear mapping of $\mathfrak{A}$ into $\mathfrak{M}_j$ ($\mathfrak{M}_i = \mathfrak{M}$) such that

$$h_j(xy) = xh_j(y) + h_j(x)y = h_j(x)S_y + h_j(y)T_x.$$

Then we have

$$h_j(x)I'_s = \sum_{i=1}^{n} \{h_j(xe_i)e_i' - xh_j(e_i)e_i'\} = \sum_{i=1}^{n} \{h_j(e_i)(e'_ix) - xh_j(e_i)\cdot e_i'\}.$$

Consequently for $j = 0, 1, 2, 3$

$$h_j(x)I'_s = x \sum_{i=1}^{n} \{L_iL_{h_j(e_i)} - R_{h_j(e_i)}R_i\}.$$

Similarly

$$h_j(x)I'_r = x \sum_{i=1}^{n} \{R_iR_{h_j(e_i)} - L_{h_j(e_i)}L_i\}.$$
By (3) and (4) we have
\[ h(x) = h_1(x)\Gamma_S + h_2(x)\Gamma_S + h_3(x)\Gamma_T. \]
Hence by (7) and (8) \( h(x) = xD \) where
\[
D = \sum \{L'_i L_{h_1(e_i)} - R_{h_1(e_i)} R_i\} + \sum \{L'_i L_{h_2(e_i)} - R_{h_2(e_i)} R_i\}
+ \sum \{R'_i R_{h_3(e_i)} - L_{h_3(e_i)} L_i\}. \]

To show that \( D \) is inner it suffices to show that for \( x, y \) in \( \mathfrak{g} \), \( L_x L_y - R_y R_x \) is in the Lie algebra \( \mathcal{L}(\mathfrak{g}) \) of linear transformations generated by the right and left multiplications of \( \mathfrak{g} \). This is true since \( L_x L_y - R_y R_x = 2[R_y, L_x] + L_{yx} - R_{yx} \).

Now let \( \mathfrak{g} \) have characteristic \( \neq 2 \) and use (7) and (8) to get
\[
h(x)(\Gamma_S + \Gamma_T) = x\left\{ \sum \left[ R_i, R_{h(e_i)} \right] + \sum \left[ L'_i, L_{h(e_i)} \right] \right\}.
\]
Then by (7) and the nucleus property of \( \mathfrak{m} \), we have \( h_2(x) \Gamma_S = [x, v_2] \) where \( v_2 = \sum h_2(e_i)e_i \) is in \( \mathfrak{m} \). Similarly \( h_3(x) \Gamma_T = [x, v_3] \) where \( v_3 \) is in \( \mathfrak{m} \). But
\[
h(x)(\Gamma_S + \Gamma_T) = h_2(x) \Gamma_S + h_3(x) \Gamma_T = 2h(x)
\]
hence
\[
h(x) = [x, g] + x \left\{ \frac{1}{2} \sum \left[ R_i, R_{h(e_i)} \right] + \frac{1}{2} \sum \left[ L'_i, L_{h(e_i)} \right] \right\}
\]
where \( g = \frac{1}{2} (v_2 + v_3) \) is in the nucleus of \( \mathfrak{g} \).

As is the case for similar theorems, the first part of Theorem 4 can be stated in the following form.

**Theorem 5.** Let \( \mathfrak{A} \) be a non-degenerate subalgebra of an alternative algebra \( \mathfrak{B} \) over an arbitrary field. Then any derivation of \( \mathfrak{A} \) into \( \mathfrak{B} \) can be extended to an inner derivation of \( \mathfrak{B} \).

3. **Application to Lie algebras.** We obtain the following special case of the generalization of the Levi theorem to algebras of prime characteristic.

**Theorem 6.** Let \( \mathfrak{L} \) be a Lie algebra over an arbitrary field with radical \( \mathfrak{R} \neq \mathfrak{L} \) such that \( 2\mathfrak{R} = 0 \) and \( \mathfrak{L}/\mathfrak{R} \) is non-degenerate. Then there is an algebra \( \mathfrak{G} \) (which is isomorphic to \( \mathfrak{L}/\mathfrak{R} \) and is a direct sum of

\[
\text{This actually } = -v_2x \text{ since } xv_2 = 0.
\]
simple algebras) such that $\mathfrak{L}$ is the direct sum $\mathfrak{L}=\mathfrak{S} \oplus \mathfrak{R}$.

Proof. Let $e_1, e_2, \ldots, e_n$ be a basis for $\mathfrak{L}$ such that $e_1, e_2, \ldots, e_k$ are a basis for a subspace $\mathfrak{B}$ and $e_{k+1}, \ldots, e_n$ are a basis for $\mathfrak{R}$. Then the right multiplication of each $x$ of $\mathfrak{L}$ has the form

$$R_x=\begin{bmatrix} P_x & Q_x \\ 0 & 0 \end{bmatrix}$$

where $P_x=Q_x=0$ if $x$ is in $\mathfrak{R}$ and $P_x$ is the right multiplication of the image $\bar{x}$ of $x$ in $\mathfrak{L}/\mathfrak{R}$. Now if $I'=\sum_{i=1}^k P_i P_i'$ is the Casimir operator (1) for the representation $P$ of $\mathfrak{L}/\mathfrak{R}$, then by Theorem 2, $I'$ is the identity $I$ and hence

$$I'=\sum_{i=1}^k R_i R_i' = \begin{bmatrix} I & Q \\ 0 & 0 \end{bmatrix}.$$

By using the properties of the complementary basis of $\mathfrak{L}/\mathfrak{R}$ and the fact that the Lie algebra of right multiplications of the elements of $\mathfrak{B}$ is isomorphic to $\mathfrak{L}/\mathfrak{R}$ it can be shown that $I'$ commutes with $R_x$ for all $x$ of $\mathfrak{L}$.

We now show that the associative algebra $\mathfrak{L}^*$ generated by the $R_x$ for all $x$ of $\mathfrak{L}$ is isomorphic to the associative algebra $\mathfrak{B}^*$ generated by the $P_x$. Certainly by (9) there is a homomorphism of $\mathfrak{L}^*$ onto $\mathfrak{B}^*$ which maps any polynomial $p(R_x, R_y, \ldots)$ into $p(P_x, P_y, \ldots)$. Now if $p(R_x, R_y, \ldots)=0$ then $p(P_x, P_y, \ldots)=0$ since $I'$ commutes with $p(R_x, R_y, \ldots)$. Hence $\mathfrak{L}^* \simeq \mathfrak{B}^*$.

Now $\mathfrak{L}/\mathfrak{R}$ is a direct sum of simple algebras and therefore [1, Lemma 2], $\mathfrak{B}^*$ (and hence $\mathfrak{L}^*$) is semi-simple. Consequently [1, Lemma 2] $\mathfrak{L}$ is a direct sum of an algebra $\mathfrak{S}$, which is a direct sum of simple algebras, and an abelian algebra $\mathfrak{R}_1$. But we must have $\mathfrak{R}_1=\mathfrak{R}$ completing the proof.

It is to be noted that it is easy to give examples of prime characteristic where all but the non-degeneracy of $\mathfrak{L}/\mathfrak{R}$ of the hypothesis is satisfied but for which the conclusion is false.

References


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<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silvio Aurora</td>
<td>Multiplicative norms for metric rings</td>
<td>1279</td>
</tr>
<tr>
<td>Ross A. Beaumont and John Richard Byrne</td>
<td>On the construction of R-modules and rings with polynomial multiplication</td>
<td>1305</td>
</tr>
<tr>
<td>Fred Brafman</td>
<td>An ultraspherical generating function</td>
<td>1319</td>
</tr>
<tr>
<td>Howard Ernest Campbell</td>
<td>On the Casimir operator</td>
<td>1325</td>
</tr>
<tr>
<td>Robert E. Edwards</td>
<td>Representation theorems for certain functional operators</td>
<td>1333</td>
</tr>
<tr>
<td>Tomlinson Fort</td>
<td>The five-point difference equation with periodic coefficients</td>
<td>1341</td>
</tr>
<tr>
<td>Isidore Heller</td>
<td>On linear systems with integral valued solutions</td>
<td>1351</td>
</tr>
<tr>
<td>Harry Hochstadt</td>
<td>Addition theorems for solutions of the wave equation in parabolic coordinates</td>
<td>1365</td>
</tr>
<tr>
<td>James A. Hummel</td>
<td>The coefficient regions of starlike functions</td>
<td>1381</td>
</tr>
<tr>
<td>Fulton Koehler</td>
<td>Estimates for the eigenvalues of infinite matrices</td>
<td>1391</td>
</tr>
<tr>
<td>Henry Paul Kramer</td>
<td>Perturbation of differential operators</td>
<td>1405</td>
</tr>
<tr>
<td>R. Sherman Lehman</td>
<td>Development of the mapping function at an analytic corner</td>
<td>1437</td>
</tr>
<tr>
<td>Harold Willis Milnes</td>
<td>Convexity of Orlicz spaces</td>
<td>1451</td>
</tr>
<tr>
<td>Vikramaditya Singh</td>
<td>Interior variations and some extremal problems for certain classes of univalent functions</td>
<td>1485</td>
</tr>
<tr>
<td>William Lee Stamey</td>
<td>On generalized euclidean and non-euclidean spaces</td>
<td>1505</td>
</tr>
<tr>
<td>Alexander Doniphan Wallace</td>
<td>Retractions in semigroups</td>
<td>1513</td>
</tr>
<tr>
<td>R. L. Wilder</td>
<td>Monotone mappings of manifolds</td>
<td>1519</td>
</tr>
</tbody>
</table>