ESTIMATES FOR THE EIGENVALUES OF INFINITE MATRICES

FULTON KOEHLER
ESTIMATES FOR THE EIGENVALUES OF INFINITE MATRICES

FULTON KOEHLER

1. Introduction. In most of the self-adjoint differential eigenvalue problems occurring in mathematical physics we are concerned with finding the extremal values of the quotient of two integro-differential quadratic forms in a certain space of admissible functions. By setting up a suitable basis in this space the problem can be reduced to that of finding the extremal values of a quotient of the form \((\alpha X, X)/(\beta X, X)\), where \(\alpha\) and \(\beta\) are infinite symmetric matrices and \(X\) is a vector. The ordinary Rayleigh-Ritz method of approximating the solutions of the latter problem is to replace the infinite matrices \(\alpha = (a_{ij})_n^\infty\) and \(\beta = (b_{ij})_n^\infty\) by their finite sections \(\alpha_n = (a_{ij})_n^n\) and \(\beta_n = (b_{ij})_n^n\). The extremal values of the quotient \((\alpha^n X^n, X^n)/(\beta^n X^n, X^n)\), where \(X^n\) is an \(n\) dimensional vector, are the roots \(\lambda\) of the equation

\[(1) \quad \det(\alpha^n - \lambda \beta^n) = 0,\]

and these are taken as approximations to the first \(n\) solutions of the original problem. If the roots of (1) are denoted by \(\lambda_k^n\) with \(\lambda_1^n \geq \lambda_2^n \geq \ldots \geq \lambda_n^n\), then for any fixed \(k\), \(\lambda_k^n\) increases monotonically with \(n\) and its limit as \(n \to \infty\) is the \(k\)th eigenvalue of the original problem. It should be stated here that the quotient of integro-differential quadratic forms in the original problem is taken as the reciprocal of the usual Rayleigh quotient so that the eigenvalues are all bounded.

If we let

\[(2) \quad \lambda_k = \lim_{n \to \infty} \lambda_k^n,\]

then the problem arises of estimating the difference \(\lambda_k - \lambda_k^n\).

We shall consider this problem under certain assumptions with regard to the matrices \(\alpha\) and \(\beta\). These assumptions are that \(\alpha\) and \(\beta\) are both positive definite, that the matrix \((b_{ij})_n^{n+1}\) has a positive lower bound independent of \(n\), that the matrix \((a_{ij})_n^{n+1}\) has an upper bound which tends towards zero as \(n \to \infty\), and that

\[\lim_{n \to \infty} \sum_{i=1}^n \sum_{j=n+1}^\infty a_{ij}^2 = 0, \quad \lim_{n \to \infty} \sum_{i=1}^n \sum_{j=n+1}^\infty b_{ij}^2 = 0.\]

2. The simplest case, which we take up first, is that in which \(\beta\)

Received September 22, 1955. Prepared under contract N onr 710 (16) (NR 044 004) between the University of Minnesota and the Office of Naval Research.

1391
is the unit matrix. Let \( X_n^{(n)} \) be the orthonormal eigenvectors corresponding to the eigenvalues \( \lambda_i \) as defined above. Let numbers \( \varepsilon_n \) and \( \rho_n \) be defined by

\[
\varepsilon_n \geq \left( \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} a_{ij}^2 \right)^{1/2},
\]

\[
\rho_n \geq \sup_{x_j} \frac{\sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij} x_i x_j}{\sum_{i=n+1}^{\infty} x_i^2}.
\]

In general the exact values of the right-hand members of (3) and (4) will not be available, and for this reason we define \( \varepsilon_n \) and \( \rho_n \) as merely upper bounds for these quantities. The more closely these upper bounds can be estimated, the better will be the subsequent estimates of the eigenvalues. For the effectiveness of the method it is necessary that the values of \( \varepsilon_n \) and \( \rho_n \) can be made arbitrarily small for \( n \) sufficiently large. One method of defining \( \rho_n \) is to take it as an upper bound for \( \left( \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij}^2 \right)^{1/2} \) in those cases where the latter series converges. A different method is given in the example of §6.

We shall adopt the convention that, if \( X \) is a vector, \( (x_j)^{n} \), then \( X^{n} \) stands for the \( n \)-dimensional vector \( (x_j)^{n} \). Let \( k \leq n < N \). By the minimax principle,

\[
\lambda_i^N = \min_{X_i} \max_{X} \left( \alpha^N X_i^N, X \right), \quad (X_i^N, U_i^N) = 0, \quad i = 1, 2, \ldots, k-1.
\]

Choose the vector \( U_i \) so that its first \( n \) components are equal respectively to those of \( X_i^{(n)} \) and its remaining components are zero. Let

\[
X = (x_i)^{n}, \quad y_i = (x_i^{2} + x_{i+1}^{2} + \cdots + x_N^{2})^{1/2}, \quad y_z = (x_{n+1}^{2} + x_{n+2}^{2} + \cdots + x_N^{2})^{1/2}.
\]

Then

\[
\lambda_i^N \leq \max_{X} \left( \alpha^N X_i^N, X \right), \quad (X_i^N, X^{(n)}) = 0, \quad i = 1, 2, \ldots, k-1
\]

\[
= \max_{X} \left[ (\alpha^N X_i^N, X^N) + 2 \sum_{j=n+1}^{N} \sum_{i=1}^{N} a_{ij} x_i x_j + \sum_{i=n+1}^{N} \sum_{j=n+1}^{N} a_{ij}^2 x_i x_j \right] / (y_i^2 + y_z^2)
\]

\[
(X_i^N, X^{(n)}) = 0, \quad i = 1, 2, \ldots, k-1
\]

\[
\leq \max_{y_z} \lambda_i^{(n)} y_i^2 + 2 \varepsilon_n y_i y_z + \rho_n y_z^2.
\]

The last step is justified by use of the maximum principle for the first term of the numerator and the Schwarz inequality for the second term.

The quantity on the right side of this inequality is the larger root \( \lambda \) of the equation
Hence,
\[
\lambda_k^N \leq \frac{\lambda_k^n + \rho_n + \sqrt{(\lambda_k^n - \rho_n)^2 + 4\varepsilon_n^2}}{2}.
\]
and, since the right side is independent of \( N \),
\[
(6)
\]
\[
\lambda_k^n \leq \lambda_k \leq \frac{\lambda_k^n + \rho_n + \sqrt{(\lambda_k^n - \rho_n)^2 + 4\varepsilon_n^2}}{2}.
\]
If \( \rho_n < \lambda_k^n \), this inequality gives the simpler, but less precise, one
\[
(6a)
\]
\[
\lambda_k^n \leq \lambda_k \leq \frac{\lambda_k^n + \rho_n}{\lambda_k^n - \rho_n}.
\]
The inequality (6) (or 6a) makes it possible to obtain arbitrarily close bounds for \( \lambda_k \) by taking \( N \) sufficiently large.

Better estimates for \( \lambda_k \) can be obtained if one makes full use of the available data, namely \( \lambda_k^n \) and \( X^{(n)} \). With these it is possible to transform \( \alpha \) into an equivalent matrix (one having the same eigenvalues) \( \overline{\alpha} = (\overline{a}_{ij}) \), where
\[
\overline{a}_{kk} = \lambda_k^n \quad (k=1, 2, \ldots, n),
\]
\[
\overline{a}_{ij} = 0 \quad (i, j=1, 2, \ldots, n; i \neq j),
\]
\[
\overline{a}_{ij} = a_{ij} \quad (i, j=n+1, n+2, \ldots),
\]
\[
\sum_{i=1}^{n} \overline{a}_{ij}^2 = \sum_{i=1}^{n} a_{ij}^2 \quad (j=n+1, n+2, \ldots).
\]
The actual formula for \( \overline{\alpha} \) is \( \overline{\alpha} = I'\alpha I' \) where \( I' = \begin{pmatrix} \Gamma^{(n)} & 0 \\ 0 & E \end{pmatrix}, \Gamma^{(n)} = (X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}) \) and the vectors \( X_k^{(n)} \) are orthonormal.

Let
\[
(7)
\]
\[
\varepsilon_{nk} \geq \left( \sum_{j=n+1}^{\infty} \overline{a}_{kj}^2 \right)^{1/2} \quad (k=1, 2, \ldots, n).
\]
If any one of the numbers \( \varepsilon_{nk} \) is equal to zero, then the corresponding eigenvalue \( \lambda_k^n \) of \( \overline{\alpha}^{(n)} \) is actually an eigenvalue of \( \overline{\alpha} \) and the \( k \)th row and column of \( \overline{\alpha} \) can be deleted before proceeding with any further calculations. We may therefore assume without loss of generality that all the numbers \( \varepsilon_{nk} \) appearing in subsequent formulas are different from zero.

Apply (5) with \( \alpha^N \) replaced by \( \overline{\alpha}^N \) and with \( U \), equal to the vector
whose \(i\)th component is 1 and whose remaining components are zero. This gives, with \(y=(x_{n+1}^2+\cdots+x_N^2)^{1/2}\)

\[
\lambda_k^N \leq \frac{\lambda_k^n x_k^2 + \lambda_{k+1}^n x_{k+1}^2 + \cdots + \lambda_n^n x_N^2 + 2 \sum_{i=k}^N \sum_{j=n+1}^N a_{ij} x_i x_j + \sum_{i=n+1}^N \sum_{j=n+1}^N a_{ij} x_i x_j}{x_k^2 + x_{k+1}^2 + \cdots + x_N^2}
\]

\[
\leq \frac{\lambda_k^n x_k^2 + \cdots + \lambda_n^n x_N^2 + 2 \sum_{i=k}^n \alpha_i |x_i| y + \rho_n y^2}{x_k^2 + \cdots + x_N^2 + y^2}
\]

The maximum value of the quotient

\[
\frac{\lambda_k^n x_k^2 + \cdots + \lambda_n^n x_N^2 + 2 \sum_{i=k}^n \alpha_i |x_i| y + \rho_n y^2}{x_k^2 + \cdots + x_N^2 + y^2}
\]

can be attained when the variables \(x_k, \cdots, x_n, y\) are restricted to non-negative values. Hence \(\lambda_k^n\) cannot exceed the largest root \(\lambda\) of the equation

\[
\begin{vmatrix}
\lambda_k^n - \lambda & 0 & \cdots & 0 & \varepsilon_{nk} \\
0 & \lambda_{k+1}^n - \lambda & \cdots & 0 & \varepsilon_{n,k+1} \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & \lambda_n^n - \lambda & \varepsilon_{nn} \\
\varepsilon_{nk} & \varepsilon_{n,k+1} & \cdots & \varepsilon_{nn} & \rho_n - \lambda
\end{vmatrix}
= (\rho_n - \lambda) \prod_{i=k}^n (\lambda_i^n - \lambda) - \sum_{j=k}^n \varepsilon_{nj}^2 \prod_{i=k}^n (\lambda_i^n - \lambda) = 0
\]

If a number \(r\) appears \(m+1\) times in the set \(\lambda_k^n, \lambda_{k+1}^n, \cdots, \lambda_n^n\), then this number is an \(m\)-fold root of (9). If \(\mu_1 > \mu_2 > \cdots > \mu_r\) are the distinct values in the set \(\lambda_k^n, \lambda_{k+1}^n, \cdots, \lambda_n^n\), then (9) also has roots \(r_1, r_2, \cdots, r_{l+1}\), where \(r_1 < \mu_1 < r_2 < \mu_2 < \cdots < \mu_r < r_{l+1}\). The latter roots are all the roots of the equation

\[
(9a) \quad \lambda - \rho_n = \sum_{j=k}^n \frac{\varepsilon_{nj}^2}{\lambda_j^n - \lambda}
\]

3. As a simple example illustrating the estimates of the last section, let us take the problem of finding the eigenvalues \(\lambda\) defined by

\[
y'' = -\lambda (1 + x) y, \quad (0 < x < 1),
y(0) = y(1) = 0.
\]
The reciprocals of these will be the extremal values \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \) of the quotient

\[
Q(y) = \frac{\int_0^1 (1 + x) y^2 \, dx}{\int_0^1 y^2 \, dx}
\]

in the space \( \mathcal{F} \) consisting of all functions \( y(x) \) with sectionally continuous first derivatives and with \( y(0) = y(1) = 0 \). As a basis for this space we take

\[
\varphi_n(x) = \sqrt{2} \sin \frac{n\pi x}{\pi} \quad (n = 1, 2, \ldots)
\]

and let

\[
a_{ij} = \int_0^1 (1 + x) \varphi_i \varphi_j \, dx = \begin{cases} 3 & \text{if } i = j, \\ \frac{2i^2}{\pi} & \text{if } i \neq j, \\ \frac{4[-1]^{i+j} - 1}{\pi' (i^2 - j^2)} & \text{if } i \neq j, 
\end{cases}
\]

\[
b_{ij} = \int_0^1 \varphi'_i \varphi'_j \, dx = \delta_{ij}.
\]

If \( y = \sum_{i=1}^{\infty} x_i \varphi_i \), then

\[
Q(y) = \begin{pmatrix} \alpha X, X \end{pmatrix} \begin{pmatrix} \beta X, X \end{pmatrix},
\]

where \( \alpha = (a_{ij})^\sigma, \beta = (b_{ij})^\sigma, X = (x_i)^\sigma \), so the problem is reduced to one of the type for which the estimates of the last section apply.

Let \( n = 3 \). The equation for \( \lambda_1, \lambda_2, \lambda_3 \) is

\[
\begin{vmatrix} 3 - \lambda & -8 & 0 \\ -8 & 9\pi^2 - \lambda & -8 \\ 0 & -8 & 8\pi^2 - \lambda \\ 25\pi^2 & 18\pi^2 - \lambda \\
\end{vmatrix} = 0.
\]

The eigenvalues and eigenvectors are:

\[
\lambda_1 = .1527 0819, \quad X_1^{(3)} = (.99684, -.07935, .00192),
\]

\[
\lambda_2 = .0377 8273, \quad X_2^{(3)} = (.07869, .98480, -.15482),
\]

\[
\lambda_3 = .0163 7316, \quad X_3^{(3)} = (.01040, .15449, .98794).
\]
We make the following estimates

\[ \sum_{j=1}^{\infty} a_{ij} = \frac{64}{\pi^3} \sum_{\sigma=2}^{\infty} \frac{1}{(4\sigma^2 - 1)} \]

\[ \leq \frac{64}{\pi^3} \left[ \frac{1}{15^1} + \frac{1}{35^1} + \frac{1}{63^1} + \sum_{\sigma=5}^{\infty} \frac{1}{(3\sigma^1)} \right] \]

\[ < \frac{64}{\pi^3} \left[ \frac{1}{15^1} + \frac{1}{35^1} + \frac{1}{63^1} + \frac{1}{81} \int_{1}^{\infty} \frac{dx}{x^3} \right] = 1.389 \times 10^{-7} \]

\[ \sum_{j=1}^{\infty} a_{ij}^2 = \frac{64}{\pi^3} \sum_{\sigma=2}^{\infty} \frac{1}{[(2\sigma+1)^2 - 4]} \]

\[ < \frac{64}{\pi^3} \left[ \frac{1}{21^1} + \frac{1}{45^1} + \frac{1}{77^1} + \frac{1}{256} \int_{1}^{\infty} \frac{dx}{x^3} \right] = 0.368 \times 10^{-7} \]

\[ \sum_{j=1}^{\infty} a_{ij}^3 = \frac{64}{\pi^3} \sum_{\sigma=2}^{\infty} \frac{1}{(4\sigma^2 - 9)} \]

\[ < \frac{64}{\pi^3} \left[ \frac{1}{7^1} + \frac{1}{27^1} + \frac{1}{55^1} + \frac{1}{81} \int_{1}^{\infty} \frac{dx}{x^3} \right] = 28.234 \times 10^{-7} \]

\[ \sum_{j=1}^{\infty} a_{ij} a_{kj} = \frac{64}{\pi^3} \sum_{\sigma=2}^{\infty} \frac{1}{(4\sigma^2 - 1)(4\sigma^2 - 9)} \]

\[ < \frac{64}{\pi^3} \left[ \frac{1}{15^2 \cdot 7^1} + \frac{1}{35^2 \cdot 27^1} + \frac{1}{63^2 \cdot 55^1} + \frac{1}{81} \int_{1}^{\infty} \frac{dx}{x^3} \right] = 6.206 \times 10^{-7} \]

\[ \sum_{j=1}^{\infty} a_{ij} a_{kj} = \sum_{j=1}^{\infty} a_{ij} a_{kj} = 0 \]

\[ \sum_{j=1}^{\infty} (a_{ij}^2 + a_{ij}^3 + a_{ij}^3) < 29.991 \times 10^{-7} = \epsilon_3^2 \]

\[ \sum_{i,j=1}^{\infty} a_{ij} = \frac{9}{4\pi^1} \sum_{\sigma=1}^{\infty} \frac{1}{\sigma^1} + \frac{128}{\pi^3} \sum_{n=2}^{\infty} \sum_{\sigma=0}^{\infty} \frac{1}{[(2n+2\sigma+1)^2 - 4n^2]} \]

\[ + \frac{128}{\pi^3} \sum_{n=2}^{\infty} \sum_{\sigma=1}^{\infty} \frac{1}{[(2n+2\sigma)^2 - (2n+1)^2]} \]

\[ < \frac{9}{4\pi^1} \sum_{\sigma=1}^{\infty} \frac{1}{\sigma^1} + \frac{128}{\pi^3} \left\{ \sum_{n=2}^{\infty} \sum_{\sigma=0}^{\infty} \frac{1}{[4n(1+2\sigma)]} \right\} \]

\[ + \sum_{n=2}^{\infty} \sum_{\sigma=1}^{\infty} \frac{1}{[2(2n+1)(2\sigma-1)]} \]

\[ = \frac{9}{4\pi^1} \sum_{\sigma=1}^{\infty} \frac{1}{\sigma^1} + \frac{8}{\pi^3} \sum_{\sigma=0}^{\infty} \frac{1}{(1+2\sigma)^1} \left\{ \sum_{n=1}^{\infty} \frac{1}{(2n)^1} + \sum_{n=2}^{\infty} \frac{1}{(2n+1)^1} \right\} \]
If the matrix $a$ is transformed into the equivalent matrix $\bar{a}$ in which the upper left hand $3 \times 3$ matrix is diagonalized, the formulas for the elements $\bar{a}_{ij}$ are (for $j \geq 4$):

$$
\bar{a}_{1j} = 0.99684 \ a_{1j} - 0.07935 \ a_{2j} + 0.00192 \ a_{3j},
$$

$$
\bar{a}_{2j} = 0.07869 \ a_{1j} + 0.98480 \ a_{2j} - 0.15482 \ a_{3j},
$$

$$
\bar{a}_{3j} = 0.01040 \ a_{1j} + 0.15449 \ a_{2j} + 0.98794 \ a_{3j}.
$$

Hence,

$$
\sum_{j=1}^{\infty} \bar{a}_{1j}^2 < 1.395 \times 10^{-7} = \varepsilon_{01}^2,
$$

$$
\sum_{j=1}^{\infty} \bar{a}_{2j}^2 < 1.042 \times 10^{-7} = \varepsilon_{02}^2,
$$

$$
\sum_{j=1}^{\infty} \bar{a}_{3j}^2 < 27.630 \times 10^{-7} = \varepsilon_{03}^2.
$$

The first three extremal values of the quotient $Q(y)$ can now be estimated by either (6), (6a), or (9a). From (6) we get

$$
0.152 \ 708 \leq \lambda_1 \leq 0.152 \ 730,
$$

$$
0.037 \ 782 \leq \lambda_2 \leq 0.037 \ 905,
$$

$$
0.016 \ 373 \leq \lambda_3 \leq 0.017 \ 167;
$$

whereas (9a) yields the following more precise estimates:

$$
0.152 \ 7081 \leq \lambda_1 \leq 0.152 \ 7092,
$$

$$
0.037 \ 7827 \leq \lambda_2 \leq 0.037 \ 7871,
$$

$$
0.016 \ 3731 \leq \lambda_3 \leq 0.017 \ 1139.
$$

4. Returning to the general problem, let us assume that, by a preliminary transformation, the matrices $\alpha$ and $\beta$ are already diagonalized in the $n \times n$ upper left-hand corner; that is, that

$$
a_{ii} = \lambda_i^n, \quad b_{ii} = 1 \quad (i = 1, 2, \ldots, n),
$$

$$
a_{ij} = b_{ij} = 0 \quad (i, j = 1, 2, \ldots, n; \ i \neq j).
$$

Let the bounds $\nu_n$ and $\varepsilon_{nk}$ be defined by (4) and (7) (with $\bar{a}_{kj}$ replaced
by $a_{k,l}$): In addition let bounds $\delta_{nk}$ and $r_n$ be defined by

\begin{equation}
\delta_{nk} \geq \left( \sum_{j=n+1}^{\infty} b_{kj}^2 \right)^{1/2} \quad (k=1, 2, \ldots, n),
\end{equation}

\begin{equation}
r_n \leq \inf \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} b_{ij}x_i x_j / \sum_{i=n+1}^{\infty} x_i^2.
\end{equation}

We assume that all these bounds exist, that

\begin{equation}
r_n > \sum_{k=1}^{n} \delta_{nk}^2,
\end{equation}

and that $\varepsilon_{nk} + \delta_{nk} \neq 0 \quad (k=1, 2, \ldots, n)$ (see remark following (7)).

By the minimax principle with $k \leq n < N$,

$$
\lambda_k^N = \min_{U_i} \max_{x} \left( \alpha^N X^N, X^N \right), \\
(\beta^N X^N, U_i) = 0, \quad i = 1, 2, \ldots, k-1.
$$

Proceeding as before, let $U_i$ be the vector whose $i$th component is 1 and whose remaining components are zero. Then

$$
\lambda_k^N \leq \max_{x_i} \frac{\lambda_k^N x_k^2 + \cdots + \lambda_n^N x_n^2 + 2 \sum_{i=k}^{n} \sum_{j=n+1}^{N} a_{ij} x_i x_j + \sum_{i=n+1}^{N} \sum_{j=i+1}^{N} a_{ij} x_i x_j}{x_k^2 + \cdots + x_n^2 + 2 \sum_{i=k}^{n} \sum_{j=n+1}^{N} b_{ij} x_i x_j + \sum_{i=n+1}^{N} \sum_{j=i+1}^{N} b_{ij} x_i x_j},
$$

$$
\leq \max_{x_i} \frac{\lambda_k^N x_k^2 + \cdots + \lambda_n^N x_n^2 + 2 \sum_{i=k}^{n} \varepsilon_{ni} |x_i| y + \rho_n y^2}{x_k^2 + \cdots + x_n^2 - 2 \sum_{i=k}^{n} \delta_{ni} |x_i| y + r_n y^2},
$$

where $y = (x_{n+1} + x_{n+2} + \cdots + x_N)^{1/2}$. The condition (12) is equivalent to the positive definiteness of the denominator of the last expression. Hence, $\lambda_k^N$ and therefore $\lambda_k$, cannot exceed the largest root $\hat{\lambda}$ of the equation

\begin{equation}
\left| \begin{array}{cccccc}
\lambda_k^N - \lambda & \cdots & 0 & \varepsilon_{nk} + \lambda \delta_{nk} \\
0 & \cdots & \lambda_n^N - \lambda & \varepsilon_{nn} + \lambda \delta_{nn}
\end{array} \right|
\end{equation}

$$
= (\rho_n - \lambda r_n) \prod_{i=k}^{n} (\lambda_i^N - \lambda) - \sum_{j=k}^{n} (\varepsilon_{nj} + \lambda \delta_{nj})^2 \prod_{i=k}^{n} (\lambda_i^N - \lambda) = 0,
$$

which is the same thing as the largest root of the equation

\begin{equation}
\lambda r_n - \rho_n = \sum_{j=k}^{n} \frac{(\varepsilon_{nj} + \lambda \delta_{nj})^2}{\lambda - \lambda_j^n}.
\end{equation}
To analyze the location of the largest root of (13a), let
\[
\psi(\lambda) = \sum_{j=k}^{n} \left( \frac{\varepsilon_{nj} + \lambda \delta_{nj}}{\lambda - \lambda_j^n} \right)^2.
\]
Then
\[
\psi'(\lambda) = \sum_{j=k}^{n} \left[ \frac{2\delta_{nj}(\varepsilon_{nj} + \lambda \delta_{nj}) - (\varepsilon_{nj} + \lambda \delta_{nj})^2}{(\lambda - \lambda_j^n)^3} \right],
\]
\[
\psi''(\lambda) = 2 \sum_{j=k}^{n} \left( \frac{\varepsilon_{nj} + \lambda_j^n \delta_{nj}}{\lambda - \lambda_j^n} \right)^2,
\]
\[
\lim_{\lambda \to \infty} \psi'(\lambda) = \sum_{j=k}^{n} \delta_{nj}^2.
\]
For \( \lambda > \lambda_k^n \), \( \psi''(\lambda) > 0 \), and therefore in this range the graph of \( \psi(\lambda) \) can intersect that of the function \( r_n \lambda - \rho_n \) in at most two points. Since \( \lim_{\lambda \to \infty} \psi(\lambda) = +\infty \) and since, by (12), \( r_n \lambda - \rho_n > \psi(\lambda) \) for all \( \lambda \) sufficiently large, there must be exactly one point of intersection, that is, one root of (13) or (13a), in the range \( \lambda > \lambda_k^n \). This root is the upper bound which we obtain for \( \lambda_k \).

Let us now assume that
\[
(14) \quad r_n \lambda_k^n - \rho_n \geq \alpha > 0
\]
for all \( n \) sufficiently large, and that
\[
(15) \quad \lim_{n \to \infty} \sum_{j=1}^{n} (\varepsilon_{nj}^2 + \delta_{nj}^2) = 0.
\]
Then, for any \( \varepsilon > 0 \), and for \( n \) sufficiently large, \( \psi(\lambda_k^n + \varepsilon) < r_n (\lambda_k^n + \varepsilon) - \rho_n \) and so the largest root of (13) or (13a) is less than \( \lambda_k^n + \varepsilon \). Therefore, (14) and (15) are sufficient to ensure that the method gives arbitrarily close bounds on \( \lambda_k \), for any \( k \), by taking \( n \) sufficiently large.

5. To illustrate the method of the last section let us consider the problem:
\[
\frac{d}{dx} \left( (1+x) \frac{dy}{dx} \right) = -\lambda y \quad (0 < x < 1),
\]
\[
y(0) = y(1) = 0.
\]
The reciprocals of the eigenvalues \( \lambda \) of this problem are the extremal values of the quotient
\[ Q(y) = \int_0^1 y^2 \, dx \left/ \int_0^1 (1 + x)y'^2 \, dx \right. \]

on the space of functions \( y(x) \) with sectionally continuous first derivatives and with \( y(0) = y(1) = 0 \). If \( \{ \varphi_n(x) \}_{i=1}^\infty \) is a basis in this space and

\[
a_{i,j} = \int_0^1 \varphi_i \varphi_j \, dx, \quad b_{i,j} = \int_0^1 (1 + x) \varphi'_i \varphi'_j \, dx,
\]

then the problem is reduced to that of finding the extremal values of the quotient \( (\alpha X, X)/(\beta X, X) \), where \( \alpha = (a_{i,j})^\infty_{i,j=1} \), \( \beta = (b_{i,j})^\infty_{i,j=1} \).

Let the sequence \( \{ \varphi_i \} \) be defined as follows:

\[
\varphi_i = \sum_{j=1}^3 c_{ij} \sin j\pi x \quad (i = 1, 2, 3),
\]

\[
\varphi_i = \sqrt{2} \sin \frac{i\pi x}{i\pi} \quad (i \geq 3),
\]

where the constants \( c_{ij} \) are chosen in such a way that

\[
(b_{i,i})^i_i = E,
\]

\[
(a_{i,i})^i_i = \begin{pmatrix}
0.0696 & 820 & 0 & 0 \\
0 & 0.0173 & 553 & 0 \\
0 & 0 & 0.0073 & 9145
\end{pmatrix}.
\]

The values of the constants \( c_{ij} \) are given by the table:

<table>
<thead>
<tr>
<th>( i ) ( \backslash ) ( j )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.3713655</td>
<td>.0378935</td>
<td>.0039777</td>
</tr>
<tr>
<td>2</td>
<td>-.0189824</td>
<td>.1828646</td>
<td>.0301791</td>
</tr>
<tr>
<td>3</td>
<td>.0007276</td>
<td>-.0197241</td>
<td>.1199722</td>
</tr>
</tbody>
</table>

We now apply the method of the last section with \( n = 2 \). Since the matrix \( \alpha \) is of diagonal form, \( \epsilon_{11} \) and \( \epsilon_{22} \) may be taken as zero and \( \mu_2 \) may be taken as the maximum of the elements \( a_{ii} \) (\( i \geq 3 \)), namely \( a_{33} = .0073 \) 9145.

For \( i = 1, 2 \) we have

\[
\sum_{j=3}^{\infty} b_{i,j}^2 = \sum_{j=4}^{\infty} b_{i,j}^2
\]

\[
= 2\pi^2 \sum_{j=4}^{\infty} \left( \int_0^1 (1 + x)(c_{11} \cos \pi x + 2c_{12} \cos 2\pi x + 3c_{13} \cos 3\pi x) \cos j\pi x \, dx \right)^2
\]

\[
= 2\pi^2 \sum_{j=1}^{\infty} c_{11}^2 \left( \int_0^1 (1 + x) \cos \pi x \cos j\pi x \, dx \right)^2 + 4c_{12}^2 \left( \int_0^1 (1 + x) \cos 2\pi x \cos j\pi x \, dx \right)^2
\]
\[ + 9c_{i3}^2 \left( \int_0^1 (1 + x) \cos 3\pi x \cos j\pi x \, dx \right)^2 + 6c_{i4}c_{i3} \left( \int_0^1 (1 + x) \cos \pi x \cos j\pi x \, dx \right) \]
\[ \times \left( \int_0^1 (1 + x) \cos 3\pi x \cos j\pi x \, dx \right) \]
\[ = \frac{8}{\pi^2} \left[ c_{i1}^2 \sum_{\sigma=2}^{\infty} \frac{(1 + 4\sigma^2)^2}{(4\sigma^2 - 1)^t} + 4c_{i2}^2 \sum_{\sigma=2}^{\infty} \frac{(4 + (2\sigma + 1)^2)^2}{((2\sigma + 1)^2 - 4)^t} \right. \]
\[ + 9c_{i3}^2 \sum_{\sigma=2}^{\infty} \frac{(9 + 4\sigma^2)^2}{(4\sigma^2 - 9)^t} + 6c_{i4}c_{i3} \sum_{\sigma=2}^{\infty} \frac{(1 + 4\sigma^2)(9 + 4\sigma^2)}{(4\sigma^2 - 1)^t(4\sigma^2 - 9)^t} \].

We make the following estimates:

\[ \sum_{\sigma=2}^{\infty} \frac{(1 + 4\sigma^2)^2}{(4\sigma^2 - 1)^t} < \frac{17^2}{15^t} + \frac{37^2}{35^t} + \frac{65^2}{63^t} + \frac{1}{15} \sum_{\sigma=6}^{\infty} \frac{1}{\sigma^t} = .00712722 , \]
\[ \sum_{\sigma=2}^{\infty} \frac{(4 + (2\sigma + 1)^2)^2}{((2\sigma + 1)^2 - 4)^t} < \frac{29^2}{21^t} + \frac{53^2}{45^t} + \frac{85^2}{77^t} + \frac{5}{4} \sum_{\sigma=5}^{\infty} \frac{1}{(2\sigma + 1)^t} = .00541918 , \]
\[ \sum_{\sigma=2}^{\infty} \frac{(9 + 4\sigma^2)^2}{(4\sigma^2 - 9)^t} < \frac{25^2}{7^t} + \frac{45^2}{27^t} + \frac{73^2}{55^t} + \frac{1}{8} \sum_{\sigma=5}^{\infty} \frac{1}{\sigma^t} = .26514737 , \]
\[ \sum_{\sigma=2}^{\infty} \frac{(1 + 4\sigma^2)(9 + 4\sigma^2)}{(4\sigma^2 - 1)^t(4\sigma^2 - 9)^t} < \frac{17 \cdot 25}{15^t \cdot 7^t} + \frac{37 \cdot 45}{35^t \cdot 27^t} + \frac{65 \cdot 73}{63^t \cdot 55^t} + \frac{1}{8} \sum_{\sigma=5}^{\infty} \frac{1}{\sigma^t} \]
\[ = .04125482 . \]

This gives

\[ \sum_{j=3}^{\infty} b_{ij}^2 < .0011490 = \delta_{21}^3 , \]
\[ \sum_{j=3}^{\infty} b_{ij}^2 < .0023514 = \delta_{22}^3 . \]

To obtain a value for \( r_2 \) we let \( F(x) = \sum_{i=3}^{N} x_i \varphi_i(x) \), where \( (x_i)^y \) is any given vector. Then

\[ \int_0^1 F''(x) \, dx = \sum_{i=4}^{N} x_i \int_0^1 \varphi_i''(x) \, dx + \sum_{i=4}^{N} x_i \]
\[ = 0.646936x_3^2 + \sum_{i=4}^{N} x_i \geq 0.646936 \sum_{i=3}^{N} x_i^2 , \]
\[ \int_0^1 (1 + x) F''(x) \, dx = \sum_{i=3}^{N} \sum_{j=3}^{N} b_{ij} x_i x_j . \]

Hence,
\[
\sum_{i=3}^{N} \sum_{j=3}^{N} b_{i,j} x_i x_j \geq \int_0^1 \frac{(1 + x)F''(x) \, dx}{\int_0^1 F''(x) \, dx} (.646936) \geq .646936.
\]

Since the bound on the right side is independent of \( N \) we may take
\[
r_2 = .646936.
\]

The use of equation (13a) now gives the following results, where \( \lambda_1 \) and \( \lambda_2 \) are the reciprocals of the first two eigenvalues of the original problem:
\[
.06968 \leq \lambda_1 \leq .06984,
\]
\[
.01735 \leq \lambda_2 \leq .01754.
\]

6. In conclusion we shall show how the method would work on the two dimensional problem of an oscillating square membrane of variable density; namely,
\[
u_{xx} + u_{yy} = -\lambda u \quad \text{in } R,
\]
\[
u = 0 \quad \text{on } C,
\]

where \( R \) is the region \( 0 < x < 1, \quad 0 < y < 1 \), \( C \) is the boundary \( \partial R \) and \( g \) is a nonnegative function with the derivative \( g_{xy} \) sectionally continuous in \( R + C \). The reciprocals \( \sigma \) the eigenvalues \( \lambda \) are the extremal values of the quotient
\[
Q(u) = \frac{\int_0^1 \int_0^1 \nu u^2 \, dx \, dy}{\int_0^1 \int_0^1 \nu (u_x^2 + u_y^2) \, dx \, dy}
\]
in the space of functions \( u(x, y) \) with sectionally continuous first derivatives in \( R + C \) and vanishing on \( C \).

As a basis for this problem we take the functions
\[
\frac{2 \sin m\pi x \sin n\pi y}{\pi (m^2 + n^2)^{1/2}}, \quad m, n = 1, 2, 3, \ldots,
\]
and arrange them in a sequence \( \varphi_1, \varphi_2, \varphi_3, \ldots \) ordered according to the value of \( m^2 + n^2 \); that is,
\[
\varphi_i = \frac{2 \sin m_i \pi x \sin n_i \pi y}{\pi \sigma_i}, \quad \sigma_i = (m_i^2 + n_i^2)^{1/2},
\]
\[
\sigma_1 \leq \sigma_2 \leq \sigma_3 \ldots.
\]

As \( N \to \infty \), \( \sigma_N = O(\sqrt{N}) \). Let
If \( u = \sum_{i=1}^{\infty} x_i \varphi_i \), then

\[
Q(u) = (\alpha X, X)/(\beta X, X)
\]

where

\[
\alpha = (a_{ij})_{i}^\infty, \quad \beta = (\delta_{ij})_{i}^\infty, \quad X = (x_i)_{i}^\infty.
\]

In order to show that the method will give arbitrarily close estimates of the eigenvalues, we must show that the quantity defined in (4) can be determined and made arbitrarily small, and that \( \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} a_{ij}^2 \) can be made arbitrarily small by taking \( n \) sufficiently large. The estimate \( \rho_n \) can be managed by noting that (4) is equivalent, in the present case, to

\[
\frac{\int_0^1 \int_0^1 g^2 \varphi^2 \, dx \, dy}{\int_0^1 \int_0^1 (v^2 + v_y^2) \, dx \, dy},
\]

where \( \varphi \) is the set of admissible functions which are orthogonal to \( \varphi_1, \varphi_2, \ldots, \varphi_n \). Let \( g \leq M \) in \( R \). Then we may define \( \rho_n \) by

\[
\rho_n = \sup_{\varphi \in \varphi_n} M \frac{\int_0^1 \int_0^1 \varphi^2 \, dx \, dy}{\int_0^1 \int_0^1 (v^2 + v_y^2) \, dx \, dy},
\]

and this gives

\[
\rho_n = \frac{M}{\pi^2 \sigma^2_{n+1}} = O\left( \frac{1}{n} \right)
\]

since the functions \( \{\varphi_i\} \) are the extremal functions for the quotient in (16).

Next, the numbers \( a_{ij} \) satisfy

\[
|a_{ij}| \leq \frac{C}{\sigma_i \sigma_j} A_i \bar{A}_{ij}
\]

where \( C \) is an absolute constant, and

\[
A_i = \begin{cases} \frac{1}{|m_i - m_j|} & \text{if } m_i \neq m_j, \\ 1 & \text{if } m_i = m_j. \end{cases}
\]
\[
\Delta_{ij} = \begin{cases} 
\frac{1}{|n_i - n_j|} & \text{if } n_i \neq n_j, \\
1 & \text{if } n_i = n_j.
\end{cases}
\]

Hence, for \(1 \leq i \leq n\),
\[
\sum_{j=n+1}^{\infty} a_{ij}^2 \leq -\frac{C^2}{\sigma_i^2 \sigma_{n+1}^2} \sum_{j=n+1}^{\infty} \Delta_{ij} \Delta_{ij},
\]
and
\[
\sum_{j=n+1}^{\infty} \Delta_{ij} \Delta_{ij} \leq \left(1 + 2 \sum_{s=1}^{\infty} \frac{1}{s^2}\right)^2,
\]
so
\[
\sum_{j=n+1}^{\infty} a_{ij}^2 \leq \frac{C_1}{i(n+1)}.
\]

Therefore,
\[
\sum_{i=1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij}^2 \leq C_2 \log \frac{n}{n} \quad (n > 1),
\]
where \(C_1\) and \(C_2\) are absolute constants.

References


University of Minnesota
PACIFIC JOURNAL OF MATHEMATICS

EDITORS
H. L. ROYDEN
Stanford University
Stanford, California

R. A. BEAUMONT
University of Washington
Seattle 5, Washington

ASSOCIATE EDITORS
E. F. BECKENBACH
C. E. BURGESS
M. HALL
E. HEWITT
A. HORN
V. GANAPATHY IYER
R. D. JAMES
M. S. KNEBELMAN
L. NACHBIN
I. NIVEN
T. G. OSTROM
M. M. SCHIFFER
G. SZERKES
F. WOLF
K. YOSIDA

SUPPORTING INSTITUTIONS
UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
THE RAMO-WOOLDRIDGE CORPORATION

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.
Silvio Aurora, *Multiplicative norms for metric rings* .................. 1279
Ross A. Beaumont and John Richard Byrne, *On the construction of
R-modules and rings with polynomial multiplication* .................. 1305
Fred Brafman, *An ultraspherical generating function* .................. 1319
Howard Ernest Campbell, *On the Casimir operator* .................. 1325
Robert E. Edwards, *Representation theorems for certain functional
operators* ........................................................................... 1333
Tomlinson Fort, *The five-point difference equation with periodic
coefficients* ......................................................................... 1341
Isidore Heller, *On linear systems with integral valued solutions* .... 1351
Harry Hochstadt, *Addition theorems for solutions of the wave equation in
parabolic coordinates* .......................................................... 1365
James A. Hummel, *The coefficient regions of starlike functions* .... 1381
Fulton Koehler, *Estimates for the eigenvalues of infinite matrices* ... 1391
Henry Paul Kramer, *Perturbation of differential operators* ........... 1405
R. Sherman Lehman, *Development of the mapping function at an analytic
corner* .................................................................................. 1437
Harold Willis Milnes, *Convexity of Orlicz spaces* .................... 1451
Vikramaditya Singh, *Interior variations and some extremal problems for
certain classes of univalent functions* ..................................... 1485
William Lee Stamey, *On generalized euclidean and non-euclidean
spaces* .................................................................................... 1505
Alexander Doniphan Wallace, *Retractions in semigroups* ............ 1513
R. L. Wilder, *Monotone mappings of manifolds* ...................... 1519