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In a paper [1] which appeared in 1936, J. A. Clarkson defined a property of Banach spaces known as uniform convexity. Let $\|f\|$ denote the norm of an element f of such a space and let $\{f'_n, f''_n\}$ be any sequence of pairs of elements such that $\|f'_n\|=\|f''_n\|=1$ and $\lim_{n\to\infty}\frac{1}{2}\|f'_n+f''_n\|=1$. The space is said to be *uniformly convex* if these conditions imply that $\lim_{n\to\infty}\|f'_n-f''_n\|=0$. It has been shown [2] that an equivalent definition is one in which the condition $\|f'_n\|=\|f''_n\|=1$ may be replaced with the weaker $\|f'_n\|\leqq 1$ and $\|f''_n\|\leqq 1$. Clarkson has been successful in showing that the Lebesgue spaces L_p are uniformly convex if $p\neq 1$ and that L_1 is not uniformly convex. The convexity properties of more general classes of Banach spaces have been investigated by M. M. Day [3], I. Halperin [4] and E. J. McShane [7].

A concept of convexity related to uniform convexity has been described and is termed *strict convexity*. It is defined in the following manner. Let f', f'' be any pair of elements in a Banach space such that $\|f'\|=\|f''\|=1$ and $\frac{1}{2}\|f'+f''_n\|=1$. The space is said to be strictly convex if these conditions imply that $\|f'-f''\|=0$. In a Euclidean space, strict convexity corresponds geometrically to the property that the unit sphere $\|f\|=1$ does not contain a segment. We remark that, if a space has the property of uniform convexity, then it possesses that of strict convexity as well; however, the converse implication is generally untrue.

The principal objective of this paper is to investigate the conditions which an Orlicz space [9] must satisfy to be uniformly convex. Also the related problem of determining the conditions for strict convexity is considered. A solution to both of these questions has been presented which may be regarded as complete in the sense that both the necessary and sufficient criteria are developed.

We begin by formulating the definitions of Orlicz spaces in accordance with the notations to be used subsequently. Except in minor details we shall adopt the standard conventions. Let $v=\varphi(u)$ be a monotonically nondecreasing function not identically zero, defined for all $0\leqq u$ such that $\varphi(u)=\varphi(u-)$ and $\varphi(0)=0$; also, let $\overline{\varphi}(u)$ denote the associated function $\overline{\varphi}(u)=\varphi(u+)$. Let $u=\psi(v)$ be the function inverse to $\varphi(u)$ which is defined by the relations:

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(i) $\psi(0)=0$,
(ii) $\psi(v)=u$ if $\varphi(u)=v$ and u is a point of continuity for $\varphi(u)$,
(iii) $\psi(v)=\psi(v-)$,
(iv) if $\varphi(u)\neq\bar{\varphi}(u)$, then $\psi(v)=u$ for all $\varphi(u)\leqq v\leqq\bar{\varphi}(u)$,
(v) if $\lim\limits_{u\to\infty}\varphi(u)=l<\infty$, then $\psi(v)=+\infty$ for all $l\leqq v$.

Also, let $\bar{\psi}(v)=\psi(v+)$. Since $\varphi(u)$ and $\psi(v)$ are monotonic functions they are Lebesgue measurable and their indefinite Lebesgue integrals define the functions:

$$\Phi(u)=\int_0^u\varphi(\bar{u})d\bar{u} \quad\text{and}\quad \Psi(v)=\int_0^v\psi(\bar{v})d\bar{v}\ .$$

Let Δ be a measure space with a σ-finite nonatomic measure μ and a σ-ring of measurable subsets. Let $f(x)$ be a μ-measurable function defined on Δ; then, the functions $\varphi(|f(x)|)$, $\bar{\varphi}(|f(x)|)$, $\Psi(|f(x)|)$, etc., are also μ-measurable on Δ. For each function $f(x)$, we define:

$$\|f\|_\Phi=\sup\int_\Delta|f(x)|g(x)\,d\mu$$

where the supremum is taken for all $g(x)\geqq0$ satisfying $\int_\Delta\Psi(g)\,d\mu\leqq1$. The Orlicz space $L_\Phi=L_\Phi(\Delta,\mu)$ is defined to be the collection of all functions $f(x)$, μ-measurable on Δ, for which $\|f\|_\Phi<\infty$. It may be shown (Zaanen, [10]) that the space L_Φ is a Banach space with the norm $\|f\|_\Phi$. If $\Phi(u)=u^p$, $1\leqq p<\infty$ then L_Φ is the classical Lebesgue space L_p.

Necessary and sufficient conditions for both types of convexity will be expressed directly in terms of the functions φ, ψ, etc. For strict convexity of L_Φ these conditions are simply that $\psi(v)$ and $\Psi(v)$ should be continuous in the extended sense. By this we mean that if V_0 is defined by $V_0=\sup\limits_{\psi(v)<\infty}v$ then $\psi(v)$ and $\Psi(v)$ are continuous for $v<V_0$ and $\lim\limits_{v\to V_0-}\psi(v)=\infty$ and $\lim\limits_{v\to V_0-}\Psi(v)=\infty$. Of course, requirements additional to those for strict convexity must be satisfied to imply uniform convexity. It is found that these conditions are alternative according as Δ is assumed to have finite or infinite measure. If Δ is of infinite measure it is necessary and sufficient that the space satisfy the following requirements: not only must the functions $\psi(v)$ and $\Psi(v)$ be continuous in the extended sense but the function $\varphi(u)$ may neither increase too rapidly nor too slowly. Precisely stated, there must be a constant $0<N<\infty$ such that $\Phi(2u)/\Phi(u)\leqq N$ for all $0<u$ (or what is readily shown to be an equivalent statement, that there exist a constant $0<N<\infty$ such that $\varphi(2u)/\varphi(u)\leqq N$ for all $0<u$), and also that for each constant $0<\varepsilon<1/4$ there is a corresponding constant $1<R_\varepsilon<\infty$ such that if $0<u$ then $R_\varepsilon<\varphi(u)/\varphi((1-\varepsilon)u)$. When Δ is of finite measure, the functions $\psi(v)$ and $\Psi(v)$, as before, must be continuous in the extended sense; however, slightly less stringent conditions apply to the functions $\varphi(u)$ and $\Phi(u)$. It is required merely that the conditions stated for Δ of infinite measure apply only in the limiting sense; namely, that there exist a constant $0<N<\infty$ such that $\limsup_{u\to\infty} \Phi(2u)/\Phi(u)\leqq N$ and that for each constant $0<\varepsilon<1/4$ there is a corresponding constant $1<R_\varepsilon<\infty$ such that $R_\varepsilon<\liminf_{u\to\infty} \varphi(u)/\varphi((1-\varepsilon)u)$.

We begin the demonstration by establishing first the statements relative to strict convexity.

LEMMA 1. *If $f(x)\in L_\Phi$ is a step function, then*

$$\|f\|_\Phi=\sup_{g(x)\geqq 0}\int_\Delta |f|g\,d\mu$$

where $\int_\Delta \Psi(g)\,d\mu\leqq 1$ and where $g(x)$ is also a step function with the same regions of constancy as $f(x)$ and $g(x)=0$ whenever $f(x)=0$.

Proof. Let $|f(x)|=f_i$ on sets e_i of measure $\mu(e_i)=\lambda_i>0$ $i=1, 2, \cdots, n$. Let $h(x)\geqq 0$ be any function such that $\int_\Delta \Psi(h)\,d\mu\leqq 1$. Define:

$$g(x)=\lambda_i^{-1}\int_{e_i} h(x)\,d\mu=g_i \quad \text{on} \quad e_i\ .$$

Since $\Psi(v)$ is the integral of a monotone nondecreasing function, it is a convex function so that by Jensen's inequality [10]:

$$\lambda_i^{-1}\int_{e_i}\Psi(h)\,d\mu\geqq\Psi\Big(\lambda_i^{-1}\int_{e_i}h(x)\,d\mu\Big)=\Psi(g_i)$$

and therefore:

$$\begin{aligned}\int_\Delta \Psi(g)\,d\mu=\sum_{i=1}^n \Psi(g_i)\lambda_i &\leqq \sum_{i=1}^n\int_{e_i}\Psi(h)\,d\mu\\ &=\int_\Delta \Psi(h)\,d\mu\leqq 1\ .\end{aligned}$$

On the other hand:

$$\begin{aligned}\int_\Delta |f|g\,d\mu=\sum_{i=1}^n f_ig_i\lambda_i &= \sum_{i=1}^n f_i\Big(\lambda_i^{-1}\int_{e_i}h\,d\mu\Big)\lambda_i\\ &=\int_\Delta |f|h\,d\mu\ .\end{aligned}$$

It is clear that we may take $g(x)=0$ where $f(x)=0$ since the integral $\int_\Delta |f|g\,d\mu$ will remain unaltered in value while $\int_\Delta \Psi(g)\,d\mu$ can only become smaller.

Theorem 1. *If $\Psi(v)$ is discontinuous, then L_Φ is not strictly convex.*

Proof. Since $\Psi(v)$ is defined as the integral of a positive function, the only type of discontinuity which can arise is of the form $\Psi(V_0)<\infty$ while $\Psi(V_0+)=\infty$ where $0<V_0<\infty$. (It is to be remarked that the definition of the space L_Φ excludes the case $V_0=0$ as trivial). Let $\lambda\leqq\min[\mu(\Delta)/2, 1/2\Psi(V_0)]$ be a finite number, $A\subseteqq\Delta$, $B\subseteqq\Delta$ be two sets such that $A\cap B=0$ and $\mu(A)=\mu(B)=\lambda$. Define $f'(x)=1/\lambda V_0$ on A and 0 elsewhere, $f''(x)=1/\lambda V_0$ on B and 0 elsewhere. By Lemma 1, if c', c'' represent positive numbers:

$$\|f'\|_\Phi=\sup_{c'\geqq 0}\frac{1}{\lambda V_0}\cdot\lambda c' \quad \text{where} \quad \Psi(c')\leqq\frac{1}{\lambda},$$

$$\|f''\|_\Phi=\sup_{c''\geqq 0}\frac{1}{\lambda V_0}\cdot\lambda c'' \quad \text{where} \quad \Psi(c'')\leqq\frac{1}{\lambda}.$$

Since $\Psi(V_0)\leqq 1/2\lambda$ and $\Psi(v)=\infty$ for $V_0<v$ the largest value of c' or c'' that may be chosen is $c'=c''=V_0$. Thus $\|f'\|_\Phi=\|f''\|_\Phi=1$.

But $\frac{1}{2}|f'(x)+f''(x)|=\frac{1}{2}|f'(x)-f''(x)|=1/2\lambda V_0$ on $A\cup B$ and 0 elsewhere, so that by Lemma 1:

$$\left\|\frac{f'+f''}{2}\right\|_\Phi=\left\|\frac{f'-f''}{2}\right\|_\Phi=\sup_{c\geqq 0}\frac{1}{2\lambda V_0}\cdot 2\lambda c$$

where c represents any positive number with $\Psi(c)\leqq 1/2\lambda$. As before, it follows the largest value of c it is possible to choose is: $c=V_0$ so that $\frac{1}{2}\|f'+f''\|_\Phi=\frac{1}{2}\|f'-f''\|_\Phi=1$.

Theorem 2. *If $\psi(v)$ is discontinuous, then L_Φ is not strictly convex.*

Proof. Let $v_0=\sup_{\psi(v)=0} v$, $V_0=\sup_{\psi(v)<\infty} v$. By Theorem 1 it may be assumed that $\Psi(v)$ is continuous in the extended sense, so that $\lim_{v\to V_0-}\psi(v)=\infty$. Two cases are distinguished according as $\psi(v)$ is or is not continuous at $v=v_0$.

(A) *v_0 is a point of discontinuity for $\psi(v)$.* Let $v_0<\beta<V_0$ be a point of continuity for $\psi(v)$ and choose β large enough so that the relation $\Psi(\beta)=1/\lambda$ defines $\lambda\leqq\mu(\Delta)/2$ and $\lambda<\infty$. It is then possible to determine sets $A\subseteqq\Delta$, $B\subseteqq\Delta$ such that: $\mu(A)=\mu(B)=\lambda$ and $A\cap B=0$. For each value of a parameter $0<p<1$ define $a_p=[1-p\overline{\psi}(v_0)v_0]/\beta$, $b_p=p\overline{\psi}(v_0)$ and

$$f_p(x)=\begin{cases} \dfrac{a_p}{\lambda} & \text{on} \quad A\,, \\ \dfrac{b_p}{\lambda} & \text{on} \quad B\,, \\ 0 & \text{on} \quad (A\cup B)'\,. \end{cases}$$

By Lemma 1, $\|f_p\|_\Phi=\sup\limits_{\substack{\xi\geqq 0\\ \eta\geqq 0}}(a_p\xi+b_p\eta)$ where $\Psi(\xi)+\Psi(\eta)\leqq 1/\lambda$. Since $\Psi(v)$ is the integral of a positive non-decreasing function not identically zero, $\Psi(v)$ increases continuously to infinity; hence, it is possible to replace the condition $\Psi(\xi)+\Psi(\eta)\leqq 1/\lambda$ by $\Psi(\xi)+\Psi(\eta)=1/\lambda$ with $\xi\geqq v_0$, $\eta\geqq v_0$ for it ξ_1, η_1 satisfy $\Psi(\xi_1)+\Psi(\eta_1)<1/\lambda$ it is always possible to find $\xi_2\geqq\xi_1$, $\eta_2\geqq\eta_1$ so that $\Psi(\xi_2)+\Psi(\eta_2)=1/\lambda$ and $\xi_2\geqq v_0$, $\eta_2\geqq v_0$ while $a_p\xi_2+b_p\eta_2\geqq a_p\xi_1+b_p\eta_1$. Thus $\eta=\eta(\xi)$ is determined as a single-valued function with $v_0\leqq\xi\leqq\beta$. If $I_p(\xi)=a_p\xi+b_p\eta(\xi)$ then $\|f_p\|_\Phi=\sup\limits_{v_0\leqq\xi\leqq\beta} I_p(\xi)$. But $I_p(\xi)$ assumes its maximum subject to $v_0\leqq\xi\leqq\beta$ either for $\xi=v_0$, $\xi=\beta$ or at points for which $d^+I_p(\xi)/d\xi$ and $d^-I_p(\xi)/d\xi$ simultaneously change their signs, where $d^+/d\xi$, $d^-/d\xi$ denote respectively upper and lower derivatives. That is, the maximum must be assumed either at a boundary of the interval $v_0\leqq\xi\leqq\beta$ or at a turning value or a cusp. But

$$\frac{d^-\eta}{d\xi}=\frac{-\bar{\psi}(\xi)}{\psi(\eta(\xi))}$$

so that

$$\frac{d^-I_p(\xi)}{d\xi}=a_p-b_p\frac{\bar{\psi}(\xi)}{\psi(\eta(\xi))}=\frac{1}{\beta}\left[1-p\bar{\psi}(v_0)\left(v_0+\frac{\bar{\psi}(\xi)\beta}{\psi(\eta(\xi))}\right)\right].$$

If p is any value $0<p<P$ where

$$0<P\leqq\min\,[1,\ (\bar{\psi}(v_0)v_0+\psi(\beta)\beta)^{-1}]$$

then $d^-I_p(\xi)/d\xi>0$, $v_0<\xi<\beta$; and since $I_p(v_0)<I_p(\beta)$ it follows: $\|f_p\|_\Phi=I_p(\beta)=1$ for all such p. Choose $0<p'<p''<P/2$ and define

$$f'(x)=f_{p'}(x)\,,\qquad f''(x)=f_{p''}(x)\,.$$

Then $\frac{1}{2}(f'(x)+f''(x))=\frac{1}{2}f_{p'+p''}(x)$ so that $\|f'\|_\Phi=\|f''\|_\Phi=\frac{1}{2}\|f'+f''\|_\Phi=1$. On the other hand $|f'(x)-f''(x)|$ is different from zero on B and therefore its norm is not zero. Thus L_Φ fails to be strictly convex in this case.

(B) v_0 *is a point of continuity for* $\psi(v)$. Let α be a point of discontinuity for $\psi(v)$ so that $\psi(\alpha)<\bar{\psi}(\alpha)\neq 0$. Also, let $\alpha<\beta<V_0$ be a point of continuity for $\psi(v)$ and choose β large enough so that the relation

$\Psi(\alpha)+\Psi(\beta)=1/\lambda$ defines $\lambda \leqq \mu(\Delta)/2$ and $0<\lambda<\infty$. It is then possible to determine sets $A \subseteqq \Delta$, $B \subseteqq \Delta$ such that $\mu(A)=\mu(B)=\lambda$ and $A \cap B = 0$.

Consider the equations

$$\begin{cases} a'\alpha + b'\beta = 1 \\ a'\psi(\beta) - b'\bar{\psi}(\alpha) = 0 \end{cases} \qquad \begin{cases} a''\alpha + b''\beta = 1 \\ a''\psi(\beta) - b''\left(\dfrac{\psi(\alpha)+\bar{\psi}(\alpha)}{2}\right) = 0 \, . \end{cases}$$

Since $\psi(\beta) \geqq \psi(\alpha) > 0$ it follows that the determinants of these equations do not vanish; therefore, the equations may be solved and it may be observed that the values of a', b', a'', b'' are all greater than zero. Define

$$f'(x) = \begin{cases} \dfrac{a'}{\lambda} & \text{on } A \\ \dfrac{b'}{\lambda} & \text{on } B \\ 0 & \text{on } (A \cup B)' \end{cases} \qquad f''(x) = \begin{cases} \dfrac{a''}{\lambda} & \text{on } A \\ \dfrac{b''}{\lambda} & \text{on } B \\ 0 & \text{on } (A \cup B)' \end{cases}$$

By Lemma 1, $\|f'\|_\Phi = \sup\limits_{\substack{\xi \geqq 0 \\ \eta \geqq 0}} (a'\xi + b'\eta)$ where $\Psi(\xi)+\Psi(\eta) \leqq 1$. As in (A) above the condition $\Psi(\xi)+\Psi(\eta) \leqq 1/\lambda$ may be replaced with $\Psi(\xi)+\Psi(\eta)=1/\lambda$ and $\xi \geqq v_0$, $\eta \geqq v_0$ and these relationships determine $\xi=\xi(\eta)$ and $\eta=\eta(\xi)$ as single-valued functions with $v_0 \leqq \xi \leqq \delta$ and $v_0 \leqq \eta \leqq \delta$ respectively where $\Psi(\delta)=1/\lambda$. If $d^+/d\xi$, $d^-/d\xi$ denote respectively upper and lower derivatives, it may be seen:

$$\frac{d^+\eta}{d\xi} = -\frac{\psi(\xi)}{\bar{\psi}(\eta(\xi))}\,, \qquad \frac{d^-\eta}{d\xi} = -\frac{\bar{\psi}(\xi)}{\psi(\eta(\xi))}\,,$$

$$\frac{d^+\xi}{d\eta} = -\frac{\psi(\eta)}{\bar{\psi}(\xi(\eta))}\,, \qquad \frac{d^-\xi}{d\eta} = -\frac{\bar{\psi}(\eta)}{\psi(\xi(\eta))}\,,$$

where $\Psi(\xi)+\Psi(\eta)=1$.

If $I'(\xi)=a'\xi+b'\eta(\xi)$ then $\|f'\|_\Phi = \sup\limits_{v_0 \leqq \xi \leqq \delta} I'(\xi)$. As in (A), $I'(\xi)$ assumes its maximum subject to $v_0 \leqq \xi \leqq \delta$ either for $\xi=v_0$, $\xi=\delta$ or at points for which $d^+I(\xi)/d\xi$ and $d^-I(\xi)/d\xi$ simultaneously change their signs. Now

$$\lim_{\xi \to v_0^+} \left[\frac{d^-I'(\xi)}{d\xi}\right] = a' + b'\left[\frac{d^-\eta(\xi)}{d\xi}\right]_{\xi \to v_0^+} = a' - \frac{\psi(v_0)}{\psi(\delta)}\, b' = a' > 0$$

since $\eta=\delta$ when $\xi=v_0$ and $\psi(\delta)>0$. Thus, $I'(\xi)$ increases in the immediate neighborhood of $\xi=v_0$ and $\xi=v_0$ cannot give a maximum. Also

$$\lim_{\xi\to\delta^-}\left[\frac{d^+I'(\xi)}{d\xi}\right]=a'+b'\left[\frac{d^+\eta(\xi)}{d\xi}\right]_{\xi\to\delta^-}=a'-b'\lim_{\substack{\xi\to\delta^-\\ \eta\to v_0^+}}\frac{\psi(\xi)}{\psi(\eta)}=-\infty$$

since $b'>0$. Thus $I'(\xi)$ decreases in the immediate neighborhood of $\xi=\delta$ to the value $I'(\delta)$ and $\xi=\delta$ cannot give a maximum.

Now

$$\frac{d^+I'(\xi)}{d\xi}=a'-b'\frac{\psi(\xi)}{\bar{\psi}(\eta(\xi))}\quad\text{and}\quad\frac{d^-I'(\xi)}{d\xi}=a'-b'\frac{\bar{\psi}(\xi)}{\psi(\eta(\xi))}\ .$$

Since $a'\psi(\beta)-b'\bar{\psi}(\alpha)=0$ and the condition $\Psi(\xi)+\Psi(\eta(\xi))=1/\lambda$ implies that as ξ increases, η cannot increase and conversely, a critical examination of the expressions above establishes the following relations:

$$\text{if}\quad \xi>\alpha\quad\text{then}\quad\frac{d^+I'(\xi)}{d\xi}\leqq 0\,,\quad\frac{d^-I'(\xi)}{d\xi}\leqq 0\,,$$

$$\text{if}\quad \xi=\alpha\quad\text{then}\quad\frac{d^+I'(\xi)}{d\xi}>0\,,\quad\frac{d^-I'(\xi)}{d\xi}=0\,,$$

$$\text{if}\quad \xi<\alpha\quad\text{then}\quad\frac{d^+I'(\xi)}{d\xi}>0\,,\quad\frac{d^-I'(\xi)}{d\xi}>0\,.$$

Since that $d^+I'(\xi)/d\xi$, $d^-I'(\xi)/d\xi$ can change sign only once, it follows that the value $\xi=\alpha$, $\eta=\beta$ gives unique maximum to $I'(\xi)$. Thus $\|f'\|_\Phi=a'\alpha+b'\beta=1$.

If $I''(\xi)=a''\xi+b''\eta(\xi)$ then an analogous argument leads to the relations

$$\text{if}\quad \xi>\alpha\quad\text{then}\quad\frac{d^+I''(\xi)}{d\xi}<0\,,\quad\frac{d^-I''(\xi)}{d\xi}<0\,,$$

$$\text{if}\quad \xi=\alpha\quad\text{then}\quad\frac{d^+I''(\xi)}{d\xi}>0\,,\quad\frac{d^-I''(\xi)}{d\xi}<0\,,$$

$$\text{if}\quad \xi<\alpha\quad\text{then}\quad\frac{d^+I''(\xi)}{d\xi}>0\,,\quad\frac{d^-I''(\xi)}{d\xi}>0\,,$$

so that $\|f''\|_\Phi=a''\alpha+b''\beta=1$.

Consider

$$\frac{f'(x)+f''(x)}{2}=\begin{cases}\dfrac{a'+a''}{2\lambda} & \text{on}\quad A\\[2ex] \dfrac{b'+b''}{2\lambda} & \text{on}\quad B\\[1ex] 0\quad\text{on}\quad (A\cup B)' \end{cases}$$

Let

$$g(x)=\begin{cases}\alpha & \text{on } A\\ \beta & \text{on } B\\ 0 & \text{on } (A\cup B)'\end{cases}$$

then

$$\int_\Delta \Psi(g)\,d\mu=\lambda[\Psi(\alpha)+\Psi(\beta)]=1$$

and

$$\left\|\frac{f'+f''}{2}\right\|_\Phi \geqq \frac{a'+a''}{2}\alpha+\frac{b'+b''}{2}\beta=1\ .$$

Thus by the triangle inequality $\frac{1}{2}\|f'+f''\|_\Phi=1$

A consideration of the defining equation shows that $b'\neq b''$. Therefore $|f'(x)-f''(x)|$ is not zero on B and it may be concluded immediately that $\|f'-f''\|_\Phi\neq 0$. Thus L_Φ fails to be strictly convex in this case also.

LEMMA 2. *If $\psi(v)$ is continuous in the extended sense and $0\leqq v\leqq V_0$, $0\leqq v'\leqq V_0$ then $\Psi(v')\geqq\Psi(v)+\psi(v)(v'-v)$.*

Proof. If $v=v'$ then the relation certainly holds. If $v<v'$ then

$$\Psi(v')=\int_0^{v'}\psi(\bar v)d\bar v=\int_0^v\psi(\bar v)d\bar v+\int_v^{v'}\psi(\bar v)d\bar v\geqq\Psi(v)+\psi(v)(v'-v)$$

If $v>v'$ then

$$\Psi(v')=\int_0^{v'}\psi(\bar v)\,d\bar v=\int_0^v\psi(\bar v)\,d\bar v-\int_{v'}^v\psi(\bar v)d\bar v\geqq\Psi(v)+\psi(v)(v'-v)\ .$$

THEOREM 3. *If*

(i) *$\Psi(v)$ is continuous in the extended sense,*

(ii) *$\psi(v)$ is continuous in the extended sense,*

(iii) *$f(x)\in L_\Phi$ and $|f(x)|=\operatorname{ess\,sup}|f(x)|=M<\infty$ on some set of positive measure and also, when Δ is of infinite measure, $f(x)$ vanishes outside a set of finite measure; then there is a constant $0<C_f<\infty$ and a function $g_f(x)\geqq 0$ such that $\|f\|_\Phi=\int_\Delta f(x)g_f(x)d\mu$, where $\psi(g_f(x))=C_f|f(x)|$ and $\int_\Delta\Psi(g_f)d\mu=1$.*

Proof. We first establish the existence of a constant C_f and a function $g_f(x)$ which satisfy the last two relations of the theorem. Let

$E \equiv \underset{x}{E}[|f(x)|>0]$ and let $S \equiv \underset{x}{S}[|f(x)|=M]$ with $\mu(S)=\delta \neq 0$. Since $\Psi(v)$ is continuous in the extended sense and increases from zero to infinity, there is a value $v' < V_0$ such that $\Psi(v')=1/\delta$. With $C'=\psi(v')M^{-1}$ define $g'(x)=v'$ for $x \in S$ and $g'(x)=\varphi(C'|f(x)|)$ for $x \in S'$. Then $\psi(g'(x)) = C'|f(x)|$ and

$$1 \leqq \int_\Delta \Psi(g'(x))d\mu \leqq \mu(E)\cdot\delta^{-1} < \infty .$$

Two cases will be distinguished according as

$$\text{(A)} \qquad \int_\Delta \Psi(\varphi(C'|f|))d\mu \leqq 1 ,$$

or

$$\text{(B)} \qquad \int_\Delta \Psi(\varphi(C'|f|))\, d\mu > 1 .$$

(A) For each value of the parameter $1 \leqq k < \infty$ define

$$g_k(x)=\begin{cases} \varphi(C'|f(x)|) & \text{for} \quad x \in S' , \\ \min[k\varphi(C'|f(x)|),\, v'] & \text{for} \quad x \in S . \end{cases}$$

The family of functions $g_k(x)$ is then a continuous one satisfying $\psi(g_k(x))=C'|f(x)|$ and increasing with k from $g_1(x)=\varphi(C'|f(x)|)$ to $g'(x)$. The integral $I(k)=\int_\Delta \Psi(g_k)d\mu$ increases continuously from value $\leqq 1$ to values $\geqq 1$. There is then a value k_0 such that $\int_\Delta \Psi(g_{k_0})d\mu=1$. The function $g_{k_0}(x)=g_f(x)$ and the constant $C'=C_f$ are those of the theorem.

(B) Let $C_0=\inf C$ where $\int_\Delta \Psi(\varphi(C|f(x)|))\,d\mu \geqq 1$. Since

$$\lim_{C\to C_0^+} \varphi(C|f(x)|)=\overline{\varphi}(C_0|f(x)|)$$

it follows by Lebesgue's theorem that

$$\int_\Delta \Psi(\overline{\varphi}(C_0|f(x)|)d\mu \geqq 1.$$

Again, let $C^0=\sup C$ where

$$\int_\Delta \Psi(\overline{\varphi}(C|f(x)|))\,d\mu \leqq 1 .$$

By the continuity of $\psi(v)$ it follows that $C^0=C_0$ and since

$$\lim_{C\to C_0^-} \overline{\varphi}(C|f(x)|)=\varphi(C_0|f(x)|)$$

then by Lebesgue's theorem $\int_{\Delta} \Psi(\varphi(C_0|f(x)|))\,d\mu \leqq 1$. For each value of the parameter $0 \leqq k \leqq 1$ define

$$g_k(x) = (1-k)\varphi(C_0|f(x)|) + k\overline{\varphi}(C_0|f(x)|)$$

then the family of functions $g_k(x)$ satisfies $\psi(g_k(x)) = C_0|f(x)|$ and increases continuously with k from $\varphi(C_0|f(x)|)$ to $\overline{\varphi}(C_0|f(x)|)$. The integral $I(k) = \int_{\Delta} \Psi(g_k)\,d\mu$ increases continuously from values $\leqq 1$ so that there is a value k_0 such that $\int_{\Delta} \Psi(g_{k_0})\,d\mu = 1$. The constant $C_0 = C_f$ and function $g_{k_0}(x) = g_f(x)$ are those of the theorem.

It is easily seen in either case (A) or (B) that $0 < C_f$ for if $C_f = 0$ then the corresponding function $g_f(x) \leqq v_0$ and hence $\int_{\Delta} \Psi(g_f)\,d\mu = 0$ which is a contradiction of the proof already made that $\int_{\Delta} \Psi(g_f)\,d\mu = 1$.

Finally, it follows from Lemma 2 that

$$\|f\|_{\Phi} = \int_{\Delta} |f(x)|\, g_f(x)\,d\mu \ .$$

Let $h(x) \geqq 0$ be any function such that $\int_{\Delta} \Psi(h)\,d\mu \leqq 1$. In Lemma 2 let $v = g_f(x)$, $v' = h(x)$; then, integrating over Δ gives

$$\int_{\Delta} \Psi(h)\,d\mu \geqq \int_{\Delta} \Psi(g_f)\,d\mu + C_f \int_{\Delta} |f(x)|(h(x) - g_f(x))\,d\mu$$

or

$$\int_{\Delta} |f(x)|\, g_f(x)\,d\mu \geqq \int_{\Delta} |f(x)|\, h(x)\,d\mu + \frac{1 - \int_{\Delta} \Psi(h)\,d\mu}{C_f} \ .$$

Since $C_f > 0$ we obtain $\|f\|_{\Phi} = \int_{\Delta} |f(x)|\, g_f(x)\,d\mu$.

THEOREM 4. *If*

(i) *the hypotheses of Theorem* 3 *are satisfied,*

(ii) $\|f\|_{\Phi} > 0$,

then

$$\|f\|_{\Phi} = \frac{\int_{\Delta} \Phi(C_f|f|)\,d\mu + 1}{C_f}$$

where C_f is the associated constant of Theorem 3.

Proof. By Young's inequality, for arbitrary $0 \leqq u$, $0 \leqq v$,

$$uv \leqq \Phi(u) + \Psi(v)$$

with equality if and only if at least one of the relations $v = \varphi(u)$ or $u = \psi(v)$ is satisfied. Let $u = C_f |f(x)|$, $v = g_f(x)$ then since $\psi(g_f(x)) = C_f |f(x)|$ the inequality becomes an equality and

$$C_f |f(x)| g_f(x) = \Phi(C_f |f(x)|) + \Psi(g_f(x)) .$$

Since $0 < C_f < \infty$, integration over Δ gives the stated result.

THEOREM 5. *If*
(i) $\Psi(v)$ *is continuous in the extended sense,*
(ii) $\psi(v)$ *is continuous in the extended sense,*
then L_Φ *is strictly convex.*

Proof. Let $f'(x)$, $f''(x)$ be any pair of elements of L_Φ such that $\frac{1}{2}\|f' + f''\|_\Phi = 1$, $\|f'\|_\Phi = 1$, $\|f''\|_\Phi = 1$. Let $f(x) = f'(x) + f''(x)$ and

$$S \equiv \underset{x}{S}[|f(x)| = \text{ess sup } |f(x)|] .$$

If $\mu(S) = 0$ let

$$\underset{n}{E} \equiv \underset{x}{E}\left[|f(x)| \leqq \min\left(n, \left(1 - \frac{1}{n}\right) \text{ess sup } |f(x)|\right)\right] \qquad (n = 1, 2, \cdots) ;$$

if $\mu(S) > 0$ let $\underset{n}{E} = S'$ $(n = 1, 2, \cdots)$. Let Δ_n be a sequence of sets snch that $\Delta_n \subseteqq \Delta_{n+1} \subseteqq \Delta$, $\mu(\Delta_n) < \infty$, $\mu(\Delta_n - E_n) > 0$ and $\lim_n \Delta_n = \Delta$. If $\mu(S) = 0$ define

$$F_n(x) = \begin{cases} \min\left[n, \left(1 - \frac{1}{n}\right) \text{ess sup } |f(x)|\right] & \text{on } (\Delta_n - E_n) , \\ |f(x)| & \text{on } (\Delta_n \cap E_n) , \\ 0 & \text{on } \Delta_n' \end{cases}$$

while if $\mu(S) > 0$ define

$$F_n(x) = \begin{cases} \text{ess sup } |f(x)| & \text{on } (\Delta_n - E_n) \\ |f(x)| & \text{on } (\Delta_n \cap E_n) \\ 0 & \text{on } \Delta_n' \end{cases}$$

observing that since $\|f\|_\Phi < \infty$ then $\text{ess sup } |f(x)| < \infty$ in this case. It follows easily from the definition of the norm in L_Φ that $\|F_n\|_\Phi \to \|f' + f''\|_\Phi$. The functions $F_n(x)$ have been constructed in such a way that they satisfy postulate (iii) of Theorem 3 so that by this theorem and also Theorem

4, there are constants $\frac{1}{2} \leqq C_n < \infty$ and functions $g_n(x)$ satisfying the relations: $\psi(g_n(x)) = C_n F_n(x)$, $\int_\Delta \Psi(g_n) d\mu = 1$ and $\|F_n\|_\Phi = \int_\Delta F_n(x) g_n(x) d\mu$. Since $F_n(x) \leqq F_{n+1}(x)$ and $\int_\Delta \Psi(g_n) d\mu = 1$, $(n=1, 2, \text{etc.})$ it follows that the sequence C_n decreases to a limit $\frac{1}{2} \leqq C < \infty$. Since $\psi(g_n(x)) = C_n F_n(x)$, $F_n(x) \leqq F_{n+1}(x)$ and $\int_\Delta \Psi(g_n) d\mu = 1$ it follows by the monotone properties of $\psi(v)$ that for each arbitrarily chosen but fixed m_0 the sequence $g_n(x)$ ultimately decreases on $(\Delta_m \cap E_m)$. When $\mu(S) > 0$ we see $(\Delta_n \cap E_n) \to (\Delta - S)$ and $(\Delta_n - E_n) \to S$ so that in this event the sequence $g_n(x)$ decreases on S also. When $\mu(S) = 0$ we see as before $(\Delta_n \cap E_n) \to (\Delta - S)$ and $(\Delta_n - E_n) \to S$. Thus the sequence $g_n(x)$ in both cases converges in measure to its limit inferior which we denote by $g(x)$.

By Theorem 3

$$\|F_n\|_\Phi = \int_\Delta |F_n| g_n d\mu = \int_\Delta |f' + f''| g_n d\mu$$

$$\leqq \int_\Delta |f'| g_n d\mu + \int_\Delta |f''| g_n d\mu \leqq \|f'\|_\Phi + \|f''\|_\Phi .$$

Since

$$\|F_n\|_\Phi \to \|f' + f''\|_\Phi = \|f'\|_\Phi + \|f''\|_\Phi ,$$

it follows that

$$\lim_n \int_\Delta |f'| g_n d\mu = \|f'\|_\Phi \quad \text{and} \quad \lim_n \int_\Delta |f''| g_n d\mu = \|f''\|_\Phi .$$

We show that there is a constant $0 < D' < \infty$ such that $\psi(g(x)) = D'|f'(x)|$ almost everywhere. If this were not the case there is a constant $0 < B < \infty$ and sets T_1', T_2' of finite positive measure such that

$$\psi(g(x)) > B|f(x)| \quad \text{on} \quad T_1'$$

$$0 < \psi(g(x)) < B|f(x)| \quad \text{on} \quad T_2' .$$

By Egoroff's theorem we may extract subsets $T_1'' \subseteqq T_1'$, $T_2'' \subseteqq T_2'$ such that the sequence $g_n(x)$ ultimately tends uniformly to $g(x)$ on T_1'' and T_2''. From T_1'', T_2'' we may extract subsets T_1''', T_2''' of positive measure such that the sequence is not only bounded on T_1''' and T_2''' but, since $\int_\Delta \Psi(g_n) d\mu \leqq 1$ and $g(x) = \liminf g_n(x)$, it is also bounded away from V_0. We may again find subsets $T_1 \subseteqq T_2'''$ and $T_2 \subseteqq T_2'''$ such that for suitably small constants $0 < t < \infty$, $0 < \alpha < \infty$:

$$\psi(g_n(x) - t') > B(|f(x)| + \alpha) , \qquad x \in T_1 ,\ 0 \leqq t' \leqq t , \tag{1}$$

$$0<\psi(g_n(x)+t'')<B(|f(x)|)\ ,\qquad x\in T_2\ ,\ 0\leqq t''\leqq t\ ,$$

for all n sufficiently large.

Since $\int_\Delta |f'|g_n\,d\mu\to\|f'\|_\Phi$, for each $0<\varepsilon$ there is an n_ε such such that if $n_\varepsilon\leqq n$ then: $\int_\Delta|f'|h\,d\mu-\int_\Delta|f'|g_n\,d\mu<\varepsilon$ where $h(x)\geqq 0$ is any function with $\int_\Delta \Psi(h)\,d\mu\leqq 1$. Also, since $g(x)$ is bounded away from V_0 and the sequence $g_n(x)$ converges uniformly on T_1 to $g(x)$, there is a constant $0<\beta<\infty$ such that for sufficiently large n

$$\int_{T_1}\psi(g_n(x))\,d\mu\leqq\int_{T_1}\psi(g(x)+\beta)\,d\mu<\infty\ . \tag{2}$$

Let

$$0<\varepsilon<\alpha\mu(T_1)\min\left[t,\ \frac{t\int_{T_2}\psi(g(x))\,d\mu}{\int_{T_1}\psi(g(x)+\beta)\,d\mu}\right]$$

and choose $n\geqq n_\varepsilon$ so that (2) holds. Then, if

$$\int_{T_1}\Psi(g_n)\,d\mu+\int_{T_2}\Psi(g_n)\,d\mu=b \tag{3}$$

and if $0<t_1<\infty$, $0<t_2<\infty$ also satisfy

$$\int_{T_1}\Psi(g_n(x)-t_1)\,d\mu+\int_{T_2}\Psi(g_n(x)+t_2)\,d\mu=b \tag{4}$$

we have by the mean value theorem, for some $0\leqq\theta_1\leqq 1$, $0\leqq\theta_2\leqq 1$,

$$-t_1\int_{T_1}\psi(g_n(x)-\theta_1t_1)\,d\mu+t_2\int_{T_2}\psi(g_n(x)+\theta_2t_2)\,d\mu=0\ . \tag{5}$$

Thus, if

$$t_1=\min\left[t,\ \frac{t\int_{T_2}\psi(g(x))\,d\mu}{\int_{T_1}\psi(g(x)+\beta)\,d\mu}\right]$$

then $t_2\leqq t$. Now by (1)

$$-t_1\int_{T_1}\psi(g_n(x)-\theta_1t_1)\,d\mu\leqq -Bt_1\int_{T_1}|f|\,d\mu-Bt_1\alpha\mu(T_1)$$

$$t_1\int_{T_2}\psi(g_n(x)+\theta_2t_2)\,d\mu\leqq Bt_2\int_{T_2}|f|\,d\mu\ ,$$

so that by (5)

$$-t_1\int_{T_1}|f|\,d\mu+t_2\int_{T_2}|f|\,d\mu\geqq t_1\alpha\mu(T_1)>\varepsilon\ .$$

But if

$$h(x)=\begin{cases}g_n(x)-t_1 & \text{on} \quad T_1\\ g_n(x)+t_2 & \text{on} \quad T_2\\ g_n(x) & \text{on} \quad (T_1\cup T_2)'\end{cases}$$

then by (3) and (4) $\int_\Delta \Psi(h)\,d\mu=\int_\Delta \Psi(g_n)\,d\mu=1$ while

$$\int_\Delta |f'|h\,d\mu=\int_\Delta |f'|g_n\,d\mu-t_1\int_{T1}|f'|\,d\mu+t_1\int_{T_2}|f'|\,d\mu$$
$$\geqq\int_\Delta |f'|g_n\,d\mu+\varepsilon\ ,$$

which contradicts the demonstration already made that $\int_\Delta |f'|h\,d\mu-\int_\Delta |f'|g_n\,d\mu<\varepsilon$. Thus, there is a constant $0<D'<\infty$ such that $\psi(g(x))=D'|f'(x)|$. Similarly, there is a constant $0<D''<\infty$ such that $\psi(g(x))=D''|f''(x)|$.

Since $|f'(x)|=\psi(g(x))/D'$ and $|f''(x)|=\psi(g(x))/D''$ we see $|f'(x)|$ and $|f''(x)|$ differ at most by a constant factor. But $\|f'\|_\Phi=\|f''\|_\Phi=1$ so that this factor is unity. Thus, $f'(x)$ and $f''(x)$ differ at most in sign. But $\psi(g(x))=C|f'(x)+f''(x)|$ so that if $f'(x)=-f''(x)$ at any point, then $\psi(g(x))=D'|f'(x)|=D''|f''(x)|=0$ at this same point. Hence $f'(x)=f''(x)$ almost everywhere and $\|f'-f''\|_\Phi=0$.

Theorems 1, 2 and 5 together have established the necessary and sufficient conditions for the strict convexity of the spaces L_Φ. In order to proceed with the more difficult demonstrations for uniform convexity we shall require the following important proposition relating to the norm of an element in L_Φ.

THEOREM 6. *If*

(i) $\psi(v)$ *is continuous in the extended sense,*

(ii) $\Psi(v)$ *is continuous in the extended sense,*

(iii) (a) *there is a constant* $0<N<\infty$ *such that* $\Phi(2u)/\Phi(u)\leqq N$, $(0<u)$, *when* Δ *is of infinite measure,*

(b) $\limsup_{u\to\infty}\Phi(2u)/\Phi(u)<+\infty$ *when* Δ *is of finite measure, then for each* $f\in L_\Phi$ *different from zero there is a constant* C_f *and a function* $g_f(x)\geqq 0$ *such that*

$$\|f\|_\Phi=\int_\Delta |f|\,d\mu\ ,$$

$$\psi(g_f(x))=C_f|f(x)| \quad \textit{and} \quad \int_\Delta \Psi(g_f)\,d\mu=1.$$

Proof. Let

$$S=\underset{x}{S}[|f(x)|=\text{ess sup }|f(x)|]\ .$$

If $\mu(S)=0$, let

$$\underset{n}{E}=\underset{x}{E}\left[|f(x)|\leqq\min\left(n,\left(1-\frac{1}{n}\right)\text{ess sup }|f(x)|\right)\right], \qquad (n=1,2,\cdots)\ ;$$

if $\mu(S)>0$ let $\underset{n}{E}=S'$, $(n=1,2,\cdots)$. Let $\varDelta_n$ be a sequence of sets such that $\varDelta_n\leqq\varDelta_{n+1}\leqq\varDelta$, $\mu(\varDelta_n)<\infty$, $\mu(\varDelta_n-E_n)>0$ and $\lim\limits_n \varDelta_n=\varDelta$. If $\mu(S)=0$, define

$$F_n(x)=\begin{cases}\min\left[n,\left(1-\frac{1}{n}\right)\text{ess sup }|f(x)|\right] & \text{on} \quad (\varDelta_n-E_n)\ ,\\ |f(x)| \quad \text{on} \quad (\varDelta_n\cap E_n)\ ,\\ 0 \quad \text{on} \quad \varDelta_n'\end{cases}$$

while if $\mu(S)>0$, define

$$F_n(x)=\begin{cases}\text{ess sup }|f(x)| & \text{on} \quad (\varDelta_n-E_n)\ ,\\ |f(x)| & \text{on} \quad (\varDelta_n\cap E_n)\ ,\\ 0 \quad \text{on} \quad \varDelta_n'\end{cases}$$

observing that in this case $\text{ess sup }|f(x)|<\infty$ since $\|f\|_\Phi<\infty$. The functions $F_n(x)$ satisfy the postulates of Theorems 3 and 4 so that there are constants $\|F_n\|_\Phi^{-1}\leqq C_n<\infty$ and functions $g_n(x)\geqq 0$ such that $\|F_n\|_\Phi^{-1}\leqq C_n<\infty$ and functions $g_n(x)\geqq 0$ such that $\|F_n\|_\Phi=\int_\Delta F_n g_n\,d\mu$ where $\psi(g_n(x))=C_n\cdot F_n(x)$ and $\int_\Delta \Psi(g_n)\,d\mu=1$. Since $F_n(x)\leqq F_{n+1}(x)\leqq|f(x)|$ it follows from the condition $\int_\Delta \Psi(g_n)\,d\mu=1$ that the sequence C_n cannot increase and since $\|f\|_\Phi\geqq\|F_n\|_\Phi$ it has a limit $\|f\|_\Phi^{-1}\leqq C<\infty$. Since $\psi(g_n(x))=C_nF_n(x)$, $F_n(x)\leqq F_{n+1}(x)$ and $\int_\Delta \Psi(g_n)\,d\mu=1$ it follows by the monotone properties of $\psi(v)$ that for each arbitrarily chosen but fixed m the sequence $g_n(x)$ ultimately decreases on $(\varDelta_m\cap E_m)$. When $\mu(S)>0$ we see $(\varDelta_n\cap E_n)\to(\varDelta-S)$ and $(\varDelta_n-E_n)\to S$, so that in this event the sequence $g_n(x)$ decreases on S also. When $\mu(S)=0$ we see $(\varDelta_n\cap E_n)\to(\varDelta-S)$ and $(\varDelta_n-E_n)\to S$. Thus, the sequence $g_n(x)$ in both cases converges in measure to its limit inferior, which we denote by $g(x)$.

(a) Assume that postulate (iii) (a) holds. In this case there is a constant $0<M<\infty$ such that $\overline{\varphi}(2u)\leqq M\varphi(u)$ for $0<u$. Thus, if $\Phi(2u)\leqq N\Phi(u)$ for $0<u$ then $\Phi(4u)\leqq N^2\Phi(u)$. Suppose there were a sequence $0<u_n$ such that for each natural number $\overline{\varphi}(2u_n)>n\varphi(u_n)$, then

$$\Phi(4u_n)\geqq\int_{2u_n}^{4u_n}\varphi(\overline{u})\,d\overline{u}\geqq 2u_n\overline{\varphi}(2u_n)\geqq 2nu_n\varphi(u_n)\geqq 2n\Phi(u_n)$$

since $u_n\varphi(u_n)\geqq\Phi(u_n)$. Now

$$\int_\Delta |f|\overline{\varphi}(2C|f|)\,d\mu\leqq M\int_\Delta |f|\varphi(C|f|)\,d\mu\leqq M\|f\|_\Phi$$

since

$$\int_\Delta \Psi(\varphi(C|f|))\,d\mu\leqq\int_\Delta \liminf \Psi(g_n)\,d\mu\leqq\liminf\int_\Delta \Psi(g_n)\,d\mu=1$$

by Fatou's lemma. By Young's inequality

$$\begin{aligned}\int_\Delta \Psi(\overline{\varphi}(2C|f|)\,d\mu&=2C\int_\Delta |f|\overline{\varphi}(2C|f|)d\mu-\int_\Delta \Phi(2C|f|)d\mu\\&\leqq 2CM\|f\|_\Phi<\infty\ .\end{aligned}$$

But for all n sufficiently large $\overline{\varphi}(2C|f(x)|)\geqq g_n(x)\geqq g(x)$ therefore by the monotone property of $\Psi(v)$ and Lebesgue's theorem

$$1=\liminf\int_\Delta \Psi(g_n)\,d\mu=\int_\Delta \liminf\Psi(g_n)\,d\mu=\int_\Delta \Psi(g)\,d\mu\ .$$

Let $h(x)\geqq 0$ be any function such that $\int_\Delta \Psi(h)\,d\mu\leqq 1$. In Lemma 2 let $v=g(x)$, $v'=h(x)$; then integrating over Δ gives

$$\int_\Delta \Psi(h)\,d\mu\geqq\int_\Delta \Psi(g)\,d\mu+C\int_\Delta |f|(h-g)\,d\mu$$

or

$$\int_\Delta |f|g\,d\mu\geqq\int_\Delta |f|h\,d\mu+\frac{1-\int_\Delta \Psi(h)\,d\mu}{C}\ .$$

Since $C>0$ we have $\|f\|_\Phi=\int_\Delta |f(x)|g(x)\,d\mu$. The function $g(x)$ and the constant C are those of the theorem.

(b) Assume that postulate (iii) (b) holds. Since $\limsup_{u\to\infty}\Phi(2u)/\Phi(u)\leqq N$, there is a u' such that for $u'\leqq u$, $\Phi(2u)/\Phi(u)\leqq 2N$. Then for

$u' < u$, $\Phi(4u) \leq (2N)^2 \Phi(u)$. With appropriate modifications of the corresponding demonstration in (a) above we easily show that there is a constant $0 < M < \infty$ and a value u_1 such that for $u_1 \leq u$, $\bar{\varphi}(u)/\varphi(u) \leq M$. Recalling that Δ is of finite measure, let $v_1 > 0$ be a value such that $\Psi(v_1) \leq 1/\mu(\Delta)$, then

$$\int_\Delta |f| \bar{\varphi}(2C|f|)\, d\mu \leq \int_\Delta |f| \bar{\varphi}(u_1)\, d\mu + M \int_\Delta |f| \varphi(C|f|)\, d\mu$$
$$\leq \left[\frac{\bar{\varphi}(u_1)}{v_1} + M \right] \|f\|_\Phi$$

since $\varphi(C|f(x)|) \leq g(x)$ and by Fatou's lemma

$$\int_\Delta \Psi(g)\, d\mu = \int_\Delta \liminf \Psi(g_n)\, d\mu \leq \liminf \int_\Delta \Psi(g_n)\, d\mu = 1 \ .$$

By Young's inequality

$$\int_\Delta \Psi(\bar{\varphi}(2C|f|))\, d\mu = 2C \int_\Delta |f| \bar{\varphi}(2C|f|)\, d\mu - \int_\Delta \Phi(2C|f|)\, d\mu$$
$$\leq 2C \int_\Delta |f| \bar{\varphi}(2C|f|)\, d\mu \leq 2C \left[\frac{\bar{\varphi}(u_1)}{v_1} + M \right] \|f\|_\Phi < \infty \ .$$

But $\bar{\varphi}(2C|f(x)|) \geq g_n(x) \geq g(x)$ for all n sufficiently large, so that by Lebesgue's theorem and the monotone property of $\Psi(v)$

$$\int_\Delta \Psi(g)\, d\mu = \int_\Delta \liminf \Psi(g_n)\, d\mu = \liminf \int_\Delta \Psi(g_n)\, d\mu = 1 \ .$$

The remainder of the proof is as in (a) above. The constant C and the function $g(x)$ are those of the theorem.

The above theorem may be generalized in several ways. The author has secceeded in obtaining a number of analogous conclusions [8] when the function $\psi(v)$ is discontinuous and when the hypotheses relative to the function $\Phi(u)$ do not hold. It is interesting to observe that for spaces in which conditions (iii) (a) or (iii) (b) do not apply, there is always an element f of the space for which the norm is not attained; that is to say, there is no function $h(x) \geq 0$ such that $\|f\|_\Phi = \int_\Delta |f| h\, d\mu$ with $\int_\Delta \Psi(h)\, d\mu = 1$. In this case, however, there is a constant $0 < C$ such that

$$\int_\Delta \Psi(\bar{\varphi}(C|f|))\, d\mu = 1 - a \ ,$$

where $0 < a < 1$ is a constant; for any larger constant $D > C$ the integral $\int_\Delta \Psi(\varphi(D|f|)d\mu$ is infinite. It is further remarkable that in this case

$$\|f\|_\Phi = \int_\Delta |f|\bar{\varphi}(C|f|)d\mu + aC^{-1}.$$

The proofs and complete statements of these propositions will not be presented since they are not essential to the discussions relating to convexity. Theorem 4 admits an obvious generalization not only to spaces which satisfy the postulates of Theorem 6, but to the more general case when only the first of these conditions holds. The problem of determining the constant C which appears in all of these theorems in terms of elementary properties of $f(x)$ has hot met with a suitable and satisfying solution despite the author's attempts to find one.

We proceed now to a consideration of the necessary and sufficient conditions for uniform convexity of Orlicz spaces. It was remarked in the introduction that every uniformly convex space is strictly convex but that the converse statement need not be true; therefore, it is clear that any necessary condition for strict convexity must be also a necessary condition for uniform convexity. Thus by Theorems 1 and 2 we must assume at least that $\Psi(v)$ is continuous in the extended sense and $\psi(v)$ is continuous in the extended sense. For a similar reason, the following theorems furnish us with further necessary conditions.

THEOREM 7. [5] *Every uniformly convex space is reflexive.*

THEOREM 8. [6] *Necessary and sufficient conditions that an Orlicz space be reflexive are that there exist a constant* $0 < N < \infty$, *such that*

(a) $\Phi(2u)/\Phi(u) \leqq N$ *and* $\Psi(2v)/\Psi(v) \leqq N$, $(0 < u,\ 0 < v)$ *when* Δ *is of infinite measure;*

(b) $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$ *and* $\limsup_{u\to\infty} \Psi(2v)/\Psi(v) \leqq N$ *when* Δ *is of finite measure.*

The conditions implied in Theorems 1, 2 and 8 must be supplemented with an additional necessary condition in order to insure uniform convexity. This is expressed in the next theorem.

THEOREM 9. *A necessary condition that* L_Φ *should be uniformly convex is that for every constant* $0 < a < \infty$ *there is a constant* $1 < K_a < \infty$ *such that* (a) *when* Δ *is of infinite measure then* $\varphi(u+au)/\varphi(u) > K_a$, $(0 < u)$; *and* (b) *when* Δ *is of finite measure then* $\liminf_{u\to\infty} \varphi(u+au)/\varphi(u) > K_a$.

Proof. By Theorems 1, 2 and 8 and our above remarks we may and shall assume that $\Psi(v)$ and $\psi(v)$ are continuous in the extended sense and that $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$ and $\limsup_{v\to\infty} \Psi(2v)/\Psi(v) \leqq N$ for some constant $0<N<\infty$. We see then that $\psi(v)\to\infty$ for if $\sup\psi(v)\leqq A$, $0<A<\infty$ then $\varphi(u)=\infty$, $A<u$ so that $\Phi(2u)=\int_0^{2u}\varphi(\bar{u})d\bar{u}=\infty$ for $A/2<u$ which contradicts $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$. Similarly the condition $\limsup_{v\to\infty} \Psi(2v)/\Psi(v) \leqq N$ implies that $\varphi(v)\to\infty$.

Suppose there were a value $0<a<\infty$ and a sequence u_n such that alternatively according to the respective hypotheses

(a) $$\frac{\varphi(u_n+au_n)}{\varphi(u_n)}\to 1$$

(b) $$\liminf_{u_n\to\infty} \frac{\varphi(u_n+au_n)}{\varphi(u_n)}=1 .$$

There is then a sequence of pairs: $\{v_n=\varphi(u_n),\ \bar{v}_n=\varphi(u_n+au_n)\}$ such that $\bar{v}_n/v_n\to 1$. Let $\lambda_n=1/(\Psi(v_n)+\Psi(\bar{v}_n))$ and define w_n by $2\Psi(w_n)=1/\lambda_n$; then $\bar{v}_n\geqq w_n\geqq v_n$ and $\bar{v}_n/w_n\to 1$, $v_n/w_n\to 1$. We remark that in the second case $u_n\to\infty$ so that $v_n\to\infty$ and ultimately $\lambda_n<\mu(\varDelta)/2$. Determine sets: A_n, B_n of positive measure such that $A_n\cap B_n=0$, $\mu(A_n)=\mu(B_n)=\mu_n$ $=\min[\mu(\varDelta)/2,\ \lambda_n]$; and define functions $f_n'(x)$, $f_n''(x)$ respectively as

$$f'(x)=\begin{cases}\dfrac{(1+a)}{[(1+a)\bar{v}_n+v_n]\mu_n} & \text{on } A_n\ ,\\[2ex] \dfrac{1}{[(1+a)\bar{v}_n+v_n]\mu_n} & \text{on } B_n\ ,\\[2ex] 0 & \text{on } (A_n\cup B_n)'\end{cases}$$

$$f''(x)=\begin{cases}\dfrac{1}{[(1+a)\bar{v}_n+v_n]\mu_n} & \text{on } A_n\ ,\\[2ex] \dfrac{(1+a)}{[(1+a)\bar{v}_n+v_n]\mu_n} & \text{on } B_n\ ,\\[2ex] 0 & \text{on } (A_n\cup B_n)'\ .\end{cases}$$

With $C_n'=C_n''=[(1+a)\bar{v}_n+v_n]\mu_n u_n$ we see: $\psi(g_n')=C_n'f_n'$, $\psi(g_n'')=C_nf_n''$ where

$$g_n'(x)=\begin{cases}\bar{v}_n & \text{on } A_n\\ v_n & \text{on } B_n\\ 0 & \text{on } (A_n\cup B_n)'\end{cases}\qquad g_n''(x)=\begin{cases}v_n & \text{on } A_n\\ \bar{v}_n & \text{on } B_n\\ 0 & \text{on } (A_n\cup B_n)'\end{cases}$$

and for all n sufficiently large so that $\lambda_n=\mu_n$ we have

$$\int_{\Delta} \Psi(g_n')\,d\mu = \lambda_n(\Psi(v_n)+\Psi(\bar{v}_n))=1\ ,$$

$$\int_{\Delta} \Psi(g_n'')\,d\mu = \lambda_n(\Psi(v_n)+\Psi(\bar{v}_n))=1\ .$$

Thus, by Theorem 6

$$\|f_n'\|_{\Phi} = \int_{\Delta} f_n' g_n'\,d\mu = \frac{(1+a)\bar{v}_n+v_n}{(1+a)\bar{v}_n+v_n} = \frac{\mu_n}{\mu_n} = 1$$

$$\|f_n''\|_{\Phi} = \int_{\Delta} f_n'' g_n''\,d\mu = \frac{(1+a)\bar{v}_n+v_n}{(1+a)\bar{v}_n+v_n} = \frac{\mu_n}{\mu_n} = 1\ .$$

Now

$$\frac{f_n'(x)+f_n''(x)}{2} = \begin{cases} \dfrac{(1+a/2)}{[(1+a)\bar{v}_n+v_n]\mu_n} & \text{on } A_n \cup B_n \\ 0 & \text{on } (A_n \cup B_n)' \end{cases}$$

so that by Lemma 1 and Theorem 6

$$\left\|\frac{f_n'+f_n''}{2}\right\|_{\Phi} = \frac{(1+a/2)w_n}{[(1+a)\bar{v}_n+v_n]} \cdot \frac{2\lambda_n}{\lambda_n} \to 1$$

since $\bar{v}_n/w_n \to 1$ and $v_n/w_n \to 1$. Again

$$|f_n'(x)-f_n''(x)| = \begin{cases} \dfrac{a}{[(1+a)\bar{v}_n+v_n]} \cdot \dfrac{1}{\lambda_n} & \text{on } A_n \cup B_n \\ 0 & \text{on } (A_n \cup B_n)' \end{cases}$$

so that

$$\|f_n'-f_n''\|_{\Phi} = \frac{a}{(1+a/2)}\left\|\frac{f_n'+f_n''}{2}\right\|_{\Phi} \to \frac{a}{1+a/2} > 0$$

and L_Φ is not uniformly convex.

LEMMA 3. *Let $0<\varepsilon<1/4$ and $1<K_\varepsilon<T<\infty$, $0<b$ be constants such that alternatively*

(a) $$K_\varepsilon < \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} < T\ , \qquad (0<u)\ ,$$

(b) $$K_\varepsilon < \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} < T\ , \qquad (b<u)\ ,$$

then there is a constant $0<L_\varepsilon$ such that

$$\Psi(v') \geqq \Psi(v) + \psi(v)(v'-v) + L_\varepsilon \Phi(|u'-u|)$$

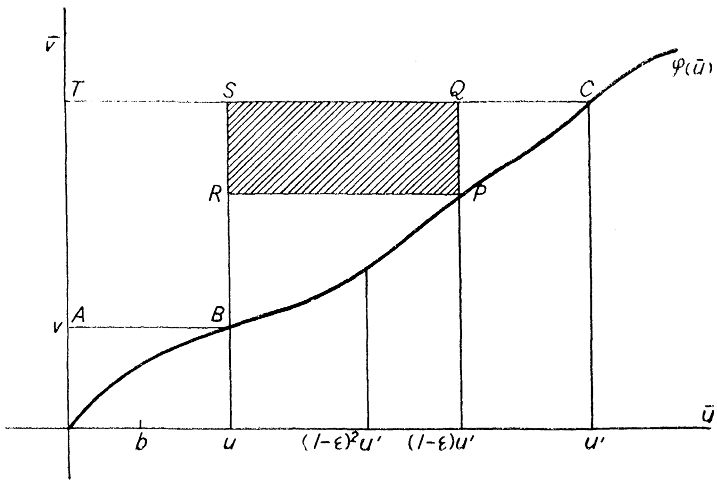
when respectively

(a) $$|u'-u| \geqq \frac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2} u > 0$$

(b) $$|u'-u| \geqq \max\left[\mathrm{b}, \frac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2} u\right]$$

where (u, v), (u', v') *are related by either* $v=\varphi(u)$ *or* $u=\psi(v)$ *and* $v'=\varphi(u')$ *or* $u'=\psi(v')$.

Proof. Assume $u' \geqq u$ and consider the first diagram



We note first

$$\text{(a)}\quad (1-\varepsilon)^2 u' \geqq u > 0\,; \qquad \text{(b)}\quad (1-\varepsilon)^2 u' \geqq u \geqq b$$

according to the respective hypotheses. Since $\varphi(u)$ is a monotone non-decreasing function we find from the definition of $\Psi(v)$

$$\Psi(v') - \Psi(v) = \text{Area } (OCT) - \text{Area } (OBA)$$

$$\geqq \text{Area } (ABST) + \text{Area } (PQRS)$$

so that:

$$\Psi(v') \geqq \Psi(v) + \psi(v)(v'-v) + \overline{QP}\cdot\overline{PR}\,.$$

Observing that respectively

(a) $$\frac{u'}{(1-\varepsilon)} > u' \geqq u > 0\,,$$

(b) $$\frac{u'}{(1-\varepsilon)} > u' \geq u > b ,$$

by corresponding hypotheses with $u'/(1-\varepsilon)$ instead of u, we see that $\varphi((1-\varepsilon)u') \leq (1/K_\varepsilon)\varphi(u')$. Thus

$$\overline{QP} = \varphi(u') - \varphi((1-\varepsilon)u') \geq \left(1 - \frac{1}{K_\varepsilon}\right)\varphi(u') .$$

Also

$$\overline{PR} = (1-\varepsilon)u' - u \geq [(1-\varepsilon) - (1-\varepsilon)^2]u' = \varepsilon(1-\varepsilon)u' .$$

Hence

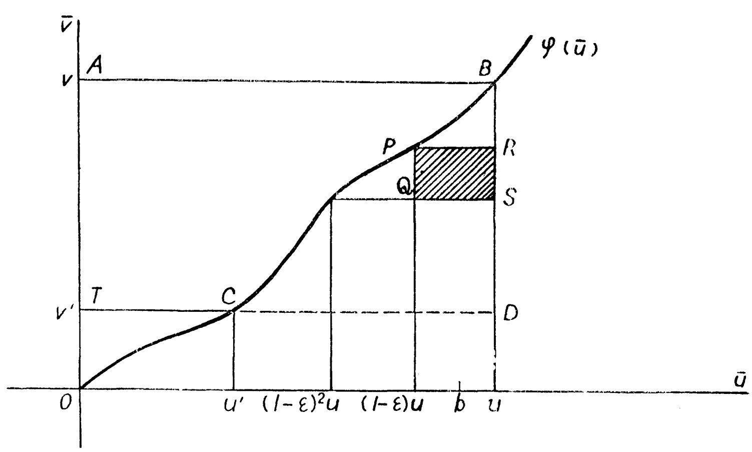
$$\overline{QP} \cdot \overline{PR} \geq \left(1 - \frac{1}{K_\varepsilon}\right)\varepsilon(1-\varepsilon)u'\varphi(u')$$

$$\geq \left(1 - \frac{1}{K_\varepsilon}\right)\varepsilon(1-\varepsilon)\Phi(u') \geq \left(1 - \frac{1}{K_\varepsilon}\right)\varepsilon(1-\varepsilon)\Phi(|u'-u|) .$$

Thus with $P_\varepsilon = (1 - 1/K_\varepsilon)\varepsilon(1-\varepsilon) > 0$ we have

$$\Psi(v') \geq \Psi(v) + \psi(v)(v'-v) + P_\varepsilon \Phi(|u'-u|) .$$

Assume $u' < u$ and consider the second diagram.



We note first that

$$u - u' \geq \frac{(2\varepsilon - \varepsilon^2)u}{(1-\varepsilon)^2} \geq (2\varepsilon - \varepsilon^2)u$$

so that $(1-\varepsilon)^2 u \geq u'$. Since $\varphi(u)$ is monotone nondecreasing, from the definition of $\Psi(v)$ we find:

$$\Psi(v)-\Psi(v')=\text{Area }(OBA)-\text{Area }(OCT)$$
$$\leq \text{Area }(ABDT)-\text{Area }(PQRS)$$

so that

$$\Psi(v')\geq \Psi(v)-\text{Area }(ABDT)+\text{Area }(PQRS)$$
$$=\Psi(v)+\psi(v')(v'-v)+\overline{PQ}\cdot\overline{RP}\,.$$

But

$$\overline{PQ}=\varphi((1-\varepsilon)u)-\varphi((1-\varepsilon)^2u)$$
$$\geq\left(1-\frac{1}{K_\varepsilon}\right)\varphi((1-\varepsilon)u)\geq\left(1-\frac{1}{K_\varepsilon}\right)\frac{1}{T}\varphi(u)$$

where, if we are considering the second set of hypotheses, we make use of the fact that $b\leq u$. Also $\overline{RP}=\varepsilon u$; therefore

$$\Psi(v')\geq\Psi(v)+\psi(v)(v'-v)+\left(1-\frac{1}{K_\varepsilon}\right)\frac{\varepsilon}{T}u\varphi(u)$$
$$\geq\Psi(v)+\psi(v)(v'-v)+\left(1-\frac{1}{K_\varepsilon}\right)\frac{\varepsilon}{T}\Phi(u)$$
$$\geq\Psi(v)+\psi(v)(v'-v)+Q_\varepsilon\Phi(|u'-u|)$$

where $Q_\varepsilon=(1-1/K_\varepsilon)(\varepsilon/T)>0$.

Taking $L_\varepsilon=\min(P_\varepsilon, Q_\varepsilon)$ we have the stated result.

THEOREM 10. *Let $\psi(v)$ be continuous, $u=\psi(v)$ and $u'=\psi(v')$ and let $0<\varepsilon<1/4$, $1<R_\varepsilon\leq N<\infty$ be constants such that alternatively*

(a) (i) $\dfrac{\Phi(2u)}{\Phi(u)}\leq N\,,\qquad(0<u)$

(ii) $R_\varepsilon<\dfrac{\varphi(u)}{\varphi((1-\varepsilon)u)}\,,\qquad(0<u)$

(iii) $|u'-u|\geq\dfrac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2}u>0$

or

(b) (i) $\limsup\limits_{u\to\infty}\dfrac{\Phi(2u)}{\Phi(u)}\leq N\,,$

(ii) $\limsup_{u\to\infty} \dfrac{\varphi(u)}{\varphi((1-\varepsilon)u)} > R_\varepsilon$,

(iii) $|u'-u| \geqq \max\left[\dfrac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2},\ \dfrac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2}u\right]$

then there is a constant $L_\varepsilon > 0$ such that

$$\Psi(v') \geqq \Psi(v) + \psi(v)(v'-v) + L_\varepsilon \Phi(|u'-u|) .$$

Proof. (*a*) By the same reasoning employed in Theorem 6, we may use hypothesis (a) (i) to show that there is a constant $0 < M < \infty$ such that $\varphi(2u)/\varphi(u) \leqq M$, $0 < u$. Writing $(1-\varepsilon)^2u$ for u and noting that $\varphi(2(1-\varepsilon)^2u) \geqq \varphi((1-\varepsilon)u)$, $0 < \varepsilon < 1/4$ we have

$$M \geqq \frac{\varphi(2(1-\varepsilon)^2u)}{\varphi((1-\varepsilon)^2u)} \geqq \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} , \qquad (0 < u) .$$

Again writing $(1-\varepsilon)u$ for u in (ii) we have

$$R_\varepsilon < \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} , \qquad (0 < u) .$$

With $M = T$, $R_\varepsilon = K_\varepsilon$ we may apply Lemma 3 to obtain the stated result.

(b) As in the proofs of Theorem 6 we may use (b) (i) to show that there is a constant $0 < M < \infty$ such that $\limsup_{u\to\infty} \varphi(2u)/\varphi(u) \leqq M$; this implies that for each $0 < \varepsilon < 1/4$ there is a value $u_1 < \infty$ such that if $(1-\varepsilon)^2u_1 < u$ then $\varphi(2u)/\varphi(u) \leqq 2M$.

Writing $(1-\varepsilon)^2u$ for u we see

$$\frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} \leqq \frac{\varphi(2(1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} \leqq 2M , \qquad (u_1 \leqq u) .$$

Since $\psi(v)$ is continuous, it follows that if $0 < b$ is any constant then $\varphi((1-\varepsilon)^2b) > 0$; since $\varphi(2u)/\varphi(u) \leqq 2M$ when $(1-\varepsilon)^2u_1 \leqq u$ it follows that $\varphi(2u_1) < \infty$; therefore

$$\frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} \leqq \frac{\varphi(2u_1)}{\varphi((1-\varepsilon)^2b)} < \infty \qquad (b \leqq u \leqq u_1) .$$

Thus with

$$T = 1 + \max\left[2M,\ \frac{\varphi(2u_1)}{\varphi((1-\varepsilon)^2b)}\right]$$

we have

$$\frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} < T \qquad (b \leqq u) .$$

The second hypothesis implies that there is a $u_2 < \infty$ such that

$$\frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} > \frac{R_\varepsilon+1}{2} > 1 \qquad (u_2 \leqq u) .$$

Let $\eta=(2\varepsilon-\varepsilon^2)/(1-\varepsilon)^2$, then $\eta>0$. Let

$$S=\inf_{\eta<u<u_2} \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} .$$

Suppose $S=1$, then there would exist a sequence $\eta \leqq u_n \leqq u_2$ such that $\varphi((1-\varepsilon)u_n)/\varphi((1-\varepsilon)^2u_n) \to 1$. From this sequence a subsequence u_{n1} could be extracted which either increases or decreases to a limit $\eta \leqq u' \leqq u_2$. If u_{n1} increases to u', then the left continuity of $\varphi(u)$ implies that $\varphi((1-\varepsilon)u')/\varphi((1-\varepsilon)^2u')=1$; while if u_{n1} decreases to u' then the right continuity of $\overline{\varphi}(u)$ implies that $\overline{\varphi}((1-\varepsilon)u')/\overline{\varphi}((1-\varepsilon)^2u')=1$. In either event this would imply that $\psi(v)$ had a discontinuity at alternatively $v=\varphi((1-\varepsilon)u')$ or $v=\overline{\varphi}((1-\varepsilon)u')$, since $\psi(v) \leqq (1-\varepsilon)^2u' < (1-\varepsilon)u' \leqq \overline{\psi}(v)$. Since $\psi(v)$ is continuous by hypothesis, we conclude: $S>1$. If we let $K_\varepsilon=\min[S, (R_\varepsilon+1)/2]$, $b=(2\varepsilon-\varepsilon^2)/(1-\varepsilon)^2$ and T as above, we see that the hypotheses of Lemma 3 are satisfied and we have established the proposition.

We shall suppress the proofs of the two following lemmas since they may be found readily in the reference cited.

LEMMA 4. [10] *If $f(x) \in L_\Phi$ and $f(x) \neq 0$ on a set of positive measure, then*

$$\int_\Delta \Phi\left(\frac{|f(x)|}{\|f\|_\Phi}\right) d\mu \leqq 1 .$$

LEMMA 5. [10] *If $f(x) \in L_\Phi$ and if there is a constant $0<N<\infty$ such that* (a) *$\Phi(2u)/\Phi(u) \leqq N$, $0<u$ when Δ is of infinite measure; or* (b) *$\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$ when Δ is of finite measure, then*

$$\int_\Delta \Phi(|f(x)|)\, d\mu < \infty .$$

LEMMA 6. *If $\{f_n(x)\}$ is a sequence of elements of L_Φ such that $\int_\Delta \Phi(|f_n(x)|)\, d\mu \to 0$ and if there is a constant $0<N<\infty$ such that either* (a) *$\Phi(2u)/\Phi(u) \leqq N$, $0<u$ when Δ is of infinite measure; or*

(b) $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leq N$ *and also* $\psi(v)$ *is continuous, when* Δ *is of finite measure, then* $\|f_n\|_\Phi \to 0$.

Proof. (a) let $p \geq 1$ be any positive integer and choose n_p sufficiently small so that $\int_\Delta \Phi(|f_n(x)|)d\mu \leq 1/N^p$ for all $n_p \leq n$. Then

$$\int_\Delta \Phi(2^p|f_n|)d\mu \leq N^p \int_\Delta \Phi(|f_n|)d\mu \leq 1$$

so that if $g_n(x) \geq 0$ and $\int_\Delta \Psi(g_n)d\mu \leq 1$ by Young's inequality

$$\int_\Delta 2^p|f_n|g_n d\mu \leq \int_\Delta \Phi(2^p|f_n|)d\mu + \int_\Delta \Psi(g_n)d\mu \leq 2$$

so that $\|f_n\|_\Phi \leq 2/2^p$, $(n_p \leq n)$. Since p may be chosen arbitrarily large, the proposition is demonstrated.

(b) Since $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leq N$ it follows that there is a $u' < \infty$ such that: $\Phi(2u)/\Phi(u) \leq 2N$, $u' \leq u$ and $\Phi(2u') < \infty$. Since $\psi(v)$ is continuous and $\psi(0)=0$, it follows that $\varphi(u) > 0$, $0 < u$ and hence $\Phi(u) > 0$, $0 < u$; therefore, if $0 < u'' \leq u'$ be any number we see $\Phi(2u)/\Phi(u) \leq \Phi(2u')/\Phi(u'') < \infty$ when $u'' \leq u \leq u'$ so that if $N_{u''} = \max[2N, \Phi(2u')/\Phi(u'')]$ we have $\Phi(2u)/\Phi(u) \leq N_{u''} < \infty$, $u'' < u$. Let $p \geq 1$ be any number and choose $0 < u'' \leq 1/2^p$; let $S_n = \underset{x}{E}[|f_n(x)| \geq u'']$. If $g_n(x) \geq 0$ and $\int_\Delta \Psi(g_n)d\mu \leq 1$ then by Young's inequality

$$\begin{aligned}\int_\Delta 2^p|f_n|g_n d\mu &\leq \int_\Delta \Phi(2^p|f_n|)d\mu + \int_\Delta \Psi(g_n)d\mu \\ &\leq \int_{S_n} \Phi(2^p|f_n|)d\mu + \int_{S_n'} \Phi(2^p|f_n|)d\mu + 1 \\ &\leq (N_{u''})^p \int_{S_n} \Phi(|f_n|)d\mu + \Phi(1)\mu(\Delta) + 1 .\end{aligned}$$

By choosing n sufficiently large, we have $\int_{S_n} \Phi(|f_n|)d\mu \leq (N_{u''})^{-p}$ so that

$$\|f_n\|_\Phi \leq \frac{2+\Phi(1)\mu(\Delta)}{2^p} .$$

Taking p sufficiently large we see that $\|f_n\|_\Phi \to 0$ since $\mu(\Delta) < \infty$.

LEMMA 7. *If* α, β *are real, then*

$$|\alpha-\beta|=|\alpha|+|\beta|-|\alpha+\beta|+||\alpha|-|\beta|| .$$

Proof. If $\alpha\geqq\beta\geqq 0$ then $|\alpha-\beta|=||\alpha|-|\beta||$ and $|\alpha|+|\beta|=|\alpha+\beta|$. If $\alpha\geqq 0\geqq\beta$ then $|\alpha|+|\beta|=|\alpha-\beta|$ and $|\alpha+\beta|=||\alpha|-|\beta||$. If $0>\alpha\geqq\beta$ then $|\alpha|+|\beta|=|\alpha+\beta|$ and $|\alpha-\beta|=||\alpha|-|\beta||$. The remaining cases in which $\beta\geqq\alpha$ hold by symmetry.

LEMMA 8. *If $\eta\leqq 1$ then $\Phi(\eta u)\leqq\eta\Phi(u)$ and if $\eta\geqq 1$ then $\Phi(\mu u)\geqq\eta\Phi(u)$.*

Proof. Since $\varphi(u)$ is monotone nondecreasing if $\eta\leqq 1$ we have

$$\Phi(\eta u)=\int_0^{\eta u}\varphi(\bar{u})d\bar{u}\leqq\eta\int_0^u\varphi(\bar{u})d\bar{u}=\eta\Phi(u) .$$

If $\eta\geqq 1$ and $\xi=1/\eta\leqq 1$ then $\Phi(\xi u')\leqq\xi\Phi(u')$; so that, if $\xi u'=u$, we have $\eta\Phi(u)\leqq\Phi(\eta u)$.

LEMMA 9. (a) *If there is a constant $0<N<\infty$ such that $\Phi(2u)/\Phi(u)\leqq N$ then*

$$\Phi(u_1+u_2)\leqq N[\Phi(u_1)+\Phi(u_2)] ,$$

for arbitrary $0\leqq u_1$, $0\leqq u_2$;

(b) *if for each $0<u''$ there is a constant $0<N_{u''}<\infty$ such that $\Phi(2u)/\Phi(u)\leqq N_{u''}$, $u''\leqq u$, then*

$$\Phi(u_1+u_2)\leqq\Phi(2u'')+N_{u''}[\Phi(u_1)+\Phi(u_2)]$$

for arbitrary $0\leqq u_1$, $0\leqq u_2$.

Proof. (a) Let $u_3=\max[u_1, u_2]$ so that $u_1+u_2\leqq 2u_3$. Then

$$\Phi(u_1+u_2)\leqq\Phi(2u_3)\leqq N\Phi(u_3)\leqq N[\Phi(u_1)+\Phi(u_2)] .$$

(b) Let $u_3=\max[u_1, u_2]$. If $u_3\leqq u''$, then

$$\Phi(u_1+u_2)\leqq\Phi(2u_3)\leqq\Phi(2u'') .$$

If $u_3>u''$ then

$$\Phi(u_1+u_2)\leqq\Phi(2u_3)\leqq N_{u''}\Phi(u_3)\leqq N_{u''}[\Phi(u_1)+\Phi(u_2)] .$$

THEOREM 10. *Let $\Psi(v)$ and $\psi(v)$ continuous and let $0<\varepsilon<1/4$, $1<R_\varepsilon\leqq N<\infty$ be constants such that alternatively*

(a) *when Δ is of infinite measure*

$$\text{(i)}\quad \Phi(2u)/\Phi(u) \leqq N\,, \qquad (0 < u)\,,$$

$$\text{(ii)}\quad R_\varepsilon < \frac{\varphi(u)}{\varphi((1-\varepsilon)u)}\,, \qquad (0 < u)\,;$$

or

(b) *when Δ is of finite measure*

$$\text{(i)}\quad \limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N\,,$$

$$\text{(ii)}\quad \limsup_{u\to\infty} \frac{\varphi(u)}{\varphi((1-\varepsilon)u)} > R_\varepsilon$$

then L_Φ is uniformly convex.

Proof. We first assume that $f_n'(x) \geqq 0$, $f_n''(x) \geqq 0$, that $\|f_n'\|_\Phi = \|f_n''\|_\Phi = 1$ and that $\frac{1}{2}\|f_n' + f_n''\|_\Phi \to 1$; and we shall prove that $\|f_n' - f_n''\|_\Phi \to 0$.

Let $\eta = (2\varepsilon - \varepsilon^2)/(1-\varepsilon)^2$; we observe that $\lim_{\varepsilon\to 0} \eta = 0$. By Theorem 9 there is a constant $0 < L_\eta < \infty$ such that, corresponding to the alternative hypotheses, when

(a) $$|u' - u| \geqq \eta u > 0\,,$$

(b) $$|u' - u| \geqq \max(\eta, \eta u)\,,$$

then

$$(*)\qquad \Psi(v') \geqq \Psi(v) + \psi(v)(v' - v) + L_\eta \Phi(|u' - u|)$$

where $u = \psi(v)$ and $u' = \psi(v')$. By Theorem 6, let $1 < D_n < \infty$, $1 < C_n' < \infty$ be constants, $g_n(x) \geqq 0$, $h'(x) \geqq 0$ be functions such that

$$\psi(g_n(x)) = D_n\left(\frac{f_n'(x) + f_n''(x)}{2}\right)\,, \qquad \int_\Delta \Psi(g_n)\,d\mu = 1$$

$$\psi(h_n'(x)) = C_n' f_n'(x)\,, \qquad \int_\Delta \Psi(h_n')\,d\mu = 1\,.$$

Let alternatively

(a) $$E_n(\eta) \equiv \underset{x}{E}\left[\left|C_n' f_n'(x) - \frac{D_n}{2}\left(f_n'(x) + f_n''(x)\right)\right| \geqq \eta C_n' f_n'(x)\right]$$

(b) $$E_n(\eta) \equiv \underset{x}{E}\left[\left|C_n' f_n'(x) - \frac{D_n}{2}\left(f_n'(x) + f_n''(x)\right)\right| \geqq \max\left[\eta, \eta C_n' f_n'(x)\right]\right].$$

Write $v' = g_n(x)$, $v = h_n'(x)$ in $(*)$ above and integrate over $E_n(\eta)$ to obtain

$$\int_{E_n(\eta)} \Psi(g_n)\,d\mu \geqq \int_{E_n(\eta)} \Psi(h_n')\,d\mu + \int_{E_n(\eta)} C_n' f_n'(x)(g_n(x) - h_n'(x))\,d\mu$$

$$+ L_\eta \int_{E_n(\eta)} \Phi\left(\left|C_n' f_n'(x) - \frac{D_n}{2}(f_n'(x) + f_n''(x))\right|\right) d\mu\,.$$

But, by Lemma 2, $\Psi(v') \geqq \Psi(v) + \psi(v)(v' - v)$; so that making the same substitutions as before and integrating over $E_n'(\eta)$ we may assert

$$\int_{E_n'(\eta)} \Psi(g_n)\,d\mu \geqq \int_{E_n'(\eta)} \Psi(h_n')\,d\mu + \int_{E_n'(\eta)} C_n' f_n'(x)(g_n(x) - h_n'(x))\,d\mu\,.$$

Hence

$$\int_\Delta \Psi(g_n)\,d\mu \geqq \int_\Delta \Psi(h_n')\,d\mu + C_n' \int_\Delta C_n' f_n'(x)(g_n(x) - h_n'(x))\,d\mu$$

$$+ L_\eta \int_{E_n(\eta)} \Phi\left(\left|C_n' f_n'(x) - \frac{D_n}{2}(f_n'(x) + f_n''(x))\right|\right) d\mu\,,$$

so that

$$C_n' \int_\Delta f_n'(x)(h_n'(x) - g_n(x))\,d\mu \geqq L_\eta \int_{E_n(\eta)} \Phi\left(\left|C_n' f_n' - \frac{D_n}{2}(f_n' + f_n'')\right|\right) d\mu\,.$$

By Lemma 8, since $1 \leqq C_n' < \infty$ we have

$$(+) \qquad \int_\Delta f_n'(x)(h_n'(x) - g_n(x))\,d\mu \geqq L_\eta \int_{E_n(\eta)} \Phi\left(\left|f_n' - \frac{D_n}{2C_n'}(f_n' + f_n'')\right|\right) d\mu\,.$$

Now

$$\|f_n' + f_n''\|_\Phi = \int_\Delta (f_n'(x) + f_n''(x)) g_n(x)\,d\mu$$

$$= \int_\Delta f_n'(x) g_n(x)\,d\mu + \int_\Delta f_n''(x) g_n(x)\,d\mu \leqq \|f_n'\|_\Phi + \|f_n''\|_\Phi\,.$$

But

$$\int_\Delta f_n'(x) g_n(x)\,d\mu \leqq \int_\Delta f_n'(x) h_n'(x)\,d\mu \leqq \|f_n'\|_\Phi$$

and

$$\int_\Delta f_n''(x) g_n(x)\,d\mu \leqq \|f_n''\|_\Phi$$

and also

$$\|f_n' + f_n''\|_\Phi \to \|f_n'\|_\Phi + \|f_n''\|_\Phi$$

so that:

$$\int_{\Delta} f_n'(x)(h_n'(x)-g_n(x))\,d\mu \to 0 .$$

Thus by (+) since $0<L_\eta<\infty$,

$$(**)\qquad \int_{E_n(\eta)} \Phi\left(\left|f_n' - \frac{D_n}{2C_n}(f_n'+f_n'')\right|\right)d\mu \to 0 .$$

on $E_n'(\eta)$ alternatively

$$\text{(a)}\qquad \left|f_n'(x) - \frac{D_n}{2C_n'}(f_n'(x)+f_n''(x))\right| \leqq \eta f_n'(x) ;$$

$$\text{(b)}\qquad \left|f_n'(x) - \frac{D_n}{2C_n'}(f_n'(x)+f_n''(x))\right| \leqq \max\left[\frac{\eta}{C_n'},\ \eta f_n'(x)\right] \leqq \max[\eta,\ \eta f_n'(x)]$$

so that, since $\eta \leqq 1$, by Lemma 4 and 8 we have alternatively as $\varepsilon \to 0$

$$\text{(a)}\qquad \int_{E_n'(\eta)} \Phi\left(\left|f_n' - \frac{D_n}{2C_n'}(f_n'+f_n'')\right|\right)d\mu \leqq \int_{E_n'(\eta)} \Phi(\eta f_n')\,d\mu$$

$$\leqq \eta \int_{E_n'(\eta)} \Phi(f_n')\,d\mu \leqq \eta \int_{\Delta} \Phi(f_n')\,d\mu \leqq \eta \to 0$$

$$\text{(b)}\qquad \int_{E_n'(\eta)} \Phi\left(\left|f_n' - \frac{D_n}{2C_n'}(f_n'+f_n'')\right|\right)d\mu \leqq \int_{E_n'(\eta)} \Phi(\eta)\,d\mu$$

$$+\int_{E_n'(\eta)} \Phi(\eta f_n')\,d\mu \leqq \Phi(\eta)\mu(\Delta) + \int_{\Delta} \Phi(\eta f_n')\,d\mu$$

$$\leqq \Phi(\eta)\mu(\Delta) + \eta \int_{\Delta} \Phi(f_n')\,d\mu \leqq \Phi(\eta)\mu(\Delta) + \eta \to 0$$

since $\mu(\Delta)$ is finite. Combining these results with (**) we see that

$$\int_{\Delta} \Phi\left(\left|f_n' - \frac{D_n}{2C_n'}(f_n'+f_n'')\right|\right)d\mu \to 0 .$$

Thus by Lemma 6

$$\left\|f_n' - \frac{D_n}{2C_n'}(f_n'+f_n'')\right\|_{\Phi} \to 0$$

and this implies in turn that

$$\frac{D_n}{C_n'}\left\|\frac{f_n'+f_n''}{2}\right\|_{\Phi} \to \|f_n'\|_{\Phi} .$$

But $\frac{1}{2}\|f'_n+f''_n\|_\Phi \to 1=\|f'_n\|_\Phi$ so that $D_n/C'_n \to 1$. It then follows that

$$\left\|f'_n-\left(\frac{f'_n+f''_n}{2}\right)\right\|_\Phi \to 0$$

from which we have immediately that

$$\|f'_n-f''_n\|_\Phi \to 0 \ .$$

We now prove the theorem for the general case when the functions $f'_n(x), f''_n(x)$ are not necessarily positive. We use the equivalent definition of uniform convexity which has been noted in the introduction. Let $\|f'_n\|_\Phi=\|f''_n\|_\Phi=1$ and suppose $\|f'_n+f''_n\|_\Phi \to 2$. We define

$$F'_n(x)=\begin{cases} |f'_n(x)| & \text{if} \quad (f'_n(x)+f''_n(x)) \text{ has the sign of } f'(x) \\ 0 & \text{otherwise ;} \end{cases}$$

$$F''_n(x)=\begin{cases} |f''_n(x)| & \text{if} \quad (f'_n(x)+f''_n(x)) \text{ has the sign of } f''_n(x) \ , \\ 0 & \text{otherwise .} \end{cases}$$

Clearly

$$0 \leqq F'_n(x) \leqq |f'_n(x)| \ , \qquad 0 \leqq F''_n(x) \leqq |f''_n(x)| \ ,$$
$$F'_n(x)+F''_n(x) \geqq |f'_n(x)+f''_n(x)|$$

and

$$2|F'_n(x)-F''_n(x)| \geqq |f'_n(x)-f''_n(x)| \ ,$$

so that $\|F'_n\|_\Phi \leqq 1$, $\|F''_n\|_\Phi \leqq 1$, $\liminf \|F'_n+F''_n\|_\Phi \geqq 2$ and $\|f''_n-f''_n\| \leqq 2\|F'_n-F''_n\|_\Phi$. Our result for positive functions applied to $F'_n(x)$ and $F''_n(x)$ now gives $\|f'_n-f''_n\|_\Phi \to 0$ and L_Φ is uniformly convex.

THEOREM 11. *Necessary and sufficient conditions that L_Φ be uniformly convex are*

(a) *in case Δ is of infinite measure*

(i) *$\Psi(v)$ is continuous,*

(ii) *$\psi(v)$ is continuons,*

(iii) *there is a constant $0<N<\infty$ such that $\Phi(2u)/\Phi(u) \leqq N$, $\Psi(2v)/\Psi(v) \leqq N$, $(0<u,\ 0<v)$.*

(iv) *for each constant $0<\varepsilon<1/4$ there is a constant $1<R_\varepsilon<\infty$ such that $\varphi(u)/\varphi((1-\varepsilon)u)>R_\varepsilon$, $(0<u)$;*

r:

(b) *in case Δ is of finite measure*

(i) *$\Psi(v)$ is continuons,*

(ii) *$\psi(v)$ is continuous,*

(iii) *there is a constant $0<N<\infty$ such that*

$$\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N\,, \quad \textit{and} \quad \limsup_{v\to\infty} \Psi(2v)/\Psi(v) \leqq N$$

(iv) *for each constant* $0 < \varepsilon < 1/4$ *there is a constant* $1 < R_\varepsilon < \infty$ *such that* $\liminf_{u\to\infty} \varphi(u)/\varphi((1-\varepsilon)u) > R_\varepsilon$.

Proof. The theorem is simply a summary of the results of Theorems 1, 2, 7, 8 and 9 and of Theorem 10.

It is interesting to remark (a) that the condition $\varphi((1+\varepsilon)u)/\varphi(u) > R_\varepsilon > 1$ implies that $\Psi(2v)/\Psi(v) \leqq N$ for some constant $0 < N < \infty$, and (b) that the condition $\liminf_{u\to\infty} \varphi((1+\varepsilon)u)/\varphi(u) > R_\varepsilon > 1$ implies that $\limsup_{v\to\infty} \Psi(2v)/\Psi(v) \leqq N$ for some constant $0 < N < \infty$; but the implications converse to (a) and (b) are untrue. To prove the direct statement we choose an integer $0 < p$ such that $((R_\varepsilon+1)/2)^p > 2$. Now, respectively (a) for all $0 < u$,

$$\frac{\varphi((1+\varepsilon)u)}{\varphi(u)} > \left(\frac{R_\varepsilon+1}{2}\right);$$

and (b) there is a value $0 < u_\varepsilon$ such that if $u_\varepsilon \leqq u$, then

$$\frac{\varphi((1+\varepsilon)u)}{\varphi(u)} > \left(\frac{R_\varepsilon+1}{2}\right).$$

Then if (a) $0 < u$, (b) $u_\varepsilon < u$, we see that

$$\varphi((1+\varepsilon)^p u) \geqq \left(\frac{R_\varepsilon+1}{2}\right)^p \varphi(u) \geqq 2\varphi(u)\,.$$

Letting $v = \varphi(u)$ we have $(1+\varepsilon)^p \psi(v) \geqq \psi(2v)$ when alternatively:
(a) $0 < v$,
(b) $v_\varepsilon < v$ where $v_\varepsilon = \varphi(u_\varepsilon)$. But then

$$\text{(a)} \qquad (1+\varepsilon)^p \Psi(v) = (1+\varepsilon)^p \int_0^v \psi(\bar{v})\,d\bar{v} \geqq \int_0^v \psi(2\bar{v})\,dv \geqq \tfrac{1}{2}\Psi(2v)$$

where $0 < v$,

$$\text{(b)} \qquad (1+\varepsilon)^p \Psi(v) = (1+\varepsilon)^p \int_{v_\varepsilon}^v \psi(\bar{v})\,d\bar{v} + (1+\varepsilon)^p \Psi(v_\varepsilon)$$

$$\geqq \int_{v_\varepsilon}^v \psi(2\bar{v})\,d\bar{v} + (1+\varepsilon)^p \Psi(v_\varepsilon) > \tfrac{1}{2}\Psi(2v) - \tfrac{1}{2}\Psi(2v_\varepsilon)$$

where $v_\varepsilon \leqq v$, and since $\Psi(2v_\varepsilon) < \infty$ and $\Psi(v) \to \infty$ we see that $\limsup_{v\to\infty} \Psi(2v)/\Psi(v) \leqq 2(1+\varepsilon)^p < \infty$. To prove the converse construct the following function. Let $u_1=0$, $v_1=0$, $u_n=2^n$, $v_n=2^n$, $u_n'=(1+\varepsilon)u_n$, $v_n'=(2^n+\frac{1}{2})$; $(n=2, 3, \cdots)$. Join the points (u_1v_1) to (u_2v_2); (u_nv_n) to $(u_n'v_n')$; (u_n', v_n') to (u_{n+1}, u_{n+1}) each by straight line segments and let this function be $\varphi(u)$. Then

$$\frac{\varphi((1+\varepsilon)u_n)}{\varphi(u_n)}=\frac{\varphi(u_n')}{\varphi(u_n)}=\frac{v_n'}{v_n}=\frac{2^n+\frac{1}{2}}{2^n}\to 1$$

while $\psi(2v)\leqq 4\psi(v)$, $(0<v)$ and therefore $\Psi(2v)/\Psi(v)\leqq 8$, $(0<v)$. It is also clear that condition (i) is implied by the condition $\limsup_{v\to\infty}\Psi(2v)/\Psi(v)\leqq N$ and consequently by (iv). Thus, if we wished to do so, we might delete any statement relative to the function $\Psi(v)$ from Theorem 11. It is true, however, that the remaining conditions are independent for none of them is implied by any combination of the other hypotheses.

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