RETRACTIONS IN SEMIGROUPS

ALEXANDER DONIPHAN WALLACE
RETRACTIONS IN SEMIGROUPS

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Let $S$ be a semigroup (that is, a Hausdorff space together with a continuous associative multiplication) and let $E$ denote the set of idempotents of $S$. If $x \in S$ let

$$L_x = \{ y | yS = x \cup Sx \}$$

and

$$R_x = \{ y | y \cup yS = x \cup xS \} .$$

Put $H_x = L_x \cap R_x$ and for $e \in E$ let

$$H = \cup \{ H_e | e \in E \} ,$$

$$M_e = \{ x | ex \in H \text{ and } xe \in H \} ,$$

$$Z_e = H_e \times (R_e \cap E) \times (L_e \cap E)$$

and

$$K_e = (L_e \cap E) \cdot H_e \cdot (R_e \cap E) .$$

Under the assumption that $S$ is compact we shall prove that $K_e$ is a retract of $M_e$ and that $K_e$ and $Z_e$ are equivalent, both algebraically and topologically. This latter fact sharpens a result announced in [6] and the former settles several questions raised in [7].

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**Lemma 1.** Let $Z = S \times S \times S$ and define a multiplication in $Z$ by

$$(t, x, y) \cdot (t', x', y') = (txy't', x', y) ;$$

then $Z$ is a semigroup and, with this multiplication, the function $f: Z \to S$ defined by $f(t, x, y) = ytx$ is a continuous homomorphism.

The proof of this is immediate. We use only the above defined multiplication in $Z$ and not coordinatewise multiplication. It is clear that $f(Z_e) = K_e$.

Since the sets $H_e$, $e \in E$, are pairwise disjoint groups [1] it is legitimate to define functions

$$\eta_e : H \to E , \quad \theta : H \to H$$

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by "\( \eta(x) \) is the unit of the group \( He \) which contains \( x \)" and "\( \vartheta(x) \) is the inverse of \( x \) in the group \( He \) which contains \( x \)". If \( x \in M_e \) then \( ex, xe \in H \) so that \( \gamma(ex), \gamma(xe) \) are defined. Define \( g : M_e \rightarrow Z \) by 
\[
g(x) = (exe, \eta(ex), \eta(xe))
\]
and note that the continuity of \( \eta \) implies the continuity of \( g \). For \( x \in M_e \) let 
\[
\rho(x) = \eta(xe)x\eta(ex)
\]
so that \( \rho \) is continuous if \( \eta \) is continuous.

**Lemma 2.** For any \( x \in K_e \) we have \( fg(x) = x = \rho(x) \) and \( g(K_e) = Z_e \). The function \( f|Z_e \) takes \( Z_e \) onto \( K_e \) in a one-to-one way and is a homeomorphism if \( \eta \) is continuous. If \( \eta \) is continuous then \( \rho \) retracts \( M_e \) onto \( K_e \).

*Proof.* Let \( t \in H_e, e_1 \in R_e \cap E \) and \( e_2 \in L_e \cap E \). Since \( L_e = L_e \) it is immediate that \( ee = e \) and since \( t \) is an element of the group \( H_e \) whose unit is \( e \) (Green [3]) we also have \( et = t = te \). Similarly we see that \( e_1e = e_1 \). It is important to observe that the sets \( \{L_e|x \in S\} \), \( \{R_e|x \in S\} \) and \( \{H_e|x \in S\} \) are disjointed covers of \( S \) so that, for example \( L_e \cap L_e \neq \emptyset \) implies \( L_e = L_e \). We see that \( ee_1e_1 = e \) and \( e_2e_1e = e_2t \) so that \( ee_1e_2 = t \). We note next that \( te_1 \in H_e \) and thus \( \gamma(te_1) = e_1 \). For \( e \in R_e \cap L_e = R_e \cap L_e \) and \( e^2 = e \), give \( te_1 \in R_e \cap L_e \), in view of Theorem 3 of [2]. But 
\[
R_e \cap L_e = R_e \cap L_e = R_e \cap L_e = H_e_1
\]
and \( H_e_1 \) being a group with unit \( e_1 \) we have, from the definition of \( \gamma \), \( \gamma(te_1) = e_1 \). In a similar fashion we show that \( \gamma(e_1t) = e_2 \). If \( x \in K_e \) then we have \( x = e_2e_1x \) with the above notation and 
\[
f(g(x)) = f(exe, \eta(ex), \eta(xe)) = \gamma(xe)exe \eta(ex) 
\]
\[
= \gamma(e_1x)xe_1x = x.
\]
It will suffice to show in addition that \( g(x) = x \) for \( x \in Z \) since \( f|Z_e = x \) gives \( x = \gamma(x) \). Now let \( x \in R_e = R_e \cap L_e \) so that \( f(x) = e, e_1 \in K_e \) and 
\[
g(f(z)) = (ef(z)e, \eta(e_f(z)), \eta(f(z)e)) = (t, e_1, e_2)
\]
in virtue of the computation given earlier.

It remains to prove the continuity of \( \eta \) when \( S \) is compact. This was announced in [7] but no proof of this fact has been published. Let \( \mathcal{L} = \{(x, y)|L_x = L_y\} \), \( \mathcal{R} = \{(x, y)|R_x = R_y\} \)
and let \( \mathcal{H} = \mathcal{L} \cap \mathcal{R} \).
LEMMA 3. If $S$ is compact then $\mathcal{H}$, $\mathcal{L}$ and $\mathcal{R}$ are closed.

Proof. Let

$$\mathcal{L}' = \{(x, y) | Sx \subseteq Sy\}$$

and assume that $(a, b) \in S \times S \setminus \mathcal{L}'$. Then $Sb \subseteq S \setminus a$ and hence $Sb \subseteq S \setminus U^*$ for some open set $U$ about $a$ since $Sb$ is closed and $S$ is regular. Again from the compactness of $S$ we can find an open set $V$ about $b$ such that $SV \subseteq S \setminus U^*$. Hence $(U \times V) \cap \mathcal{L}' = \emptyset$ and we may infer that $\mathcal{L}'$ is closed. There is no loss of generality in assuming that $S$ has a unit $\vee$. Hence if $h: S \times S \to S \times S$ is defined by $h(x, y) = (y, x)$ then $h(\mathcal{L}')$ is closed and thus $\mathcal{L} = \mathcal{L}' \cap h(\mathcal{L}')$ is closed. In a similar way it may be shown that $\mathcal{R}$ is closed. Moreover, $\mathcal{H}$ is closed because $\mathcal{H} = \mathcal{L} \setminus \mathcal{R}$.

THEOREM 1 [7]. If $S$ is compact then $H$ is closed, $\eta: H \to E$ is a retraction and $\theta: H \to H$ is a homeomorphism.

Proof. Define $p: S \times S \to S$ by $p(x, y) = x$. Then

$$H = \bigcup \{H_e | e \in E\} = p(H \cap (S \times E))$$

is closed since $\mathcal{H}$ and $E$ are closed. We show next that $\theta$ is continuous and to this end it is enough to prove that $G = \{(x, \theta(x)) | x \in H\}$ in virtue of the fact that $H$ is compact Hausdorff. If $m: S \times S \to S$ is defined by $m(x, y) = xy$ then $\mathcal{H} \cap (H \times H) \cap m^{-1}(E)$ is closed and we will show that this set is the same as $G$. For $(x, \theta(x)) \in G$ implies $m(x, \theta(x)) = x\theta(x) \in E$ in virtue of the definition of $\theta$. Since $x$ and $\theta(x)$ are in the same set $H_e$, $e \in E$, it is clear that $(x, \theta(x)) \in H \times H$ and it is easily seen from the definition of $H_e = L_e \cap R_e$ and $\mathcal{H} = \mathcal{L} \setminus \mathcal{R}$ that also $(x, \theta(x)) \in \mathcal{H}$. Now take $x, y$ such that $xy = e \in E$, $x, y \in H$ and $(x, y) \in \mathcal{H}$. The last fact shows that $H_x = H_y$ and the penultimate condition, together with this shows that $x, y \in H_{e_t}$ for some $e_t \in E$. But $e = xy \in H_{e_t}$ and the fact that $H_{e_t}$ is a group implies that $e = e_t$. Now the uniqueness of inversion in the group $H_e$ shows that $y = \theta(x)$. Hence $\theta$ is continuous and $\eta$ is continuous because $\eta(x) = x\theta(x)$ from the definition of $\eta$ and $\theta$.

G. B. Preston raised the question as to the continuity of a certain generalized "inversion"—Suppose that there is a unique function $\alpha: S \to S$ such that $x\alpha(x)x = x$ and $\alpha(x)\alpha(x) = \alpha(x)$ for each $x \in S$. If $S$ is compact then $\alpha$ is continuous. To see this let $\mathcal{N}$ be the set of all $(x, y) \in S \times S$ such that $xyx = x$ and $yxy = y$ and define $\varphi: S \times S \to S \times S$ by $\varphi(x, y) = (xyx, x)$. If $D$ is the diagonal of $S \times S$ then $\varphi^{-1}(D)$ is closed. Similarly $\varphi^{-1}(D)$ is closed where $\varphi(x, y) = (y, yxy)$ and $\mathcal{N} = \varphi^{-1}(D) \cap \varphi^{-1}(D)$ is therefore closed. The uniqueness of $\alpha$ implies that $\{(x, \alpha(x)) | x \in S\} = \mathcal{N}$.
so that \( \alpha \) is continuous if \( S \) is compact. For a discussion of the existence and uniqueness of such functions as \( \alpha \), see [2, pp. 273-274] as well as references therein to Liber, Munn and Penrose, Thierrin, Vagner and the papers of Preston in London Math. Soc., 1954.

From Theorem 1 and Lemma 2 we obtain at once

**Theorem 3.** Let \( S \) be compact and let \( e \in E \); then \( K_e \) is topologically isomorphic with

\[
Z_e = H_e \times (L_e \cap E) \times (R_e \cap E)
\]

and \( K_e \) is a retract of \( M_e \).

It is not asserted that \( K_e \) is a subsemigroup of \( S \). The first corollary is a topologized form of the Rees-Suschkewitsch theorem, see [6], [7] and [2] for a bibliography of relevant algebraic results.

**Corollary 1.** If \( S \) is compact, if \( K \) is the minimal ideal of \( S \) and if \( e \in E \cap K \) then \( K \) is topologically isomorphic with \( eSe \times (Se \cap E) \times (eS \cap E) \) and \( K \) and each "factor" of \( K \) is a retract of \( S \).

**Proof.** We rely, without explicit citation, on the results of [1]. It is immediate that \( M_e = S \). Now \( L_e = Se \), \( Re = eS \) and \( H_e = eSe \) so that (by definition and [1]) \( K_e = Se \cdot eSe \cdot eS \subseteq K \) and, being an ideal, \( K_e = K \). Clearly \( x \rightarrow exe \) retracts \( S \) onto \( eSe \). Now \( Se \subseteq K \subseteq H \) and \( e|Se \) retracts \( Se \) onto \( Se \cap E \).

It is clear, when \( S \) is compact, that \( K \) enjoys all the retraction invariants of \( S \), for example, if \( S \) is locally connected so is \( K \). We do not list these nor do we give here the applications of Corollary 1 that were mentioned in [6].

**Corollary 2.** If \( S \) is a clan [7], if \( K \subseteq E \) and if \( H^n(S) \neq 0 \) for some \( n > 0 \) and some coefficient group, then \( \dim K \geq 2 \).

**Proof.** If \( K \subseteq E \) then \( H_e = \{e\} \) and \( K \) is thus topologically the product \( Se \times eS \) since \( Se, eS \subseteq K \). Now \( H^n(Se) \approx H^n(S) \approx H^n(eS) \) [9] and hence \( Se, eS \) are non-degenerate continua. It follows that \( \dim K \geq 2 \).

It is possible to put some of the above in a more general framework. Let \( T \) be a closed subsemigroup of \( S \) and let

\[
L_e = \{y \mid x \cup Tx = y \cup Ty\}
\]

with similar definitions for \( R_e \) and \( H_e \). If \( e \in E \) then \( H_e \) is a semigroup and \( H_e \) is a group if \( eT \cup Te \subseteq T \). If \( \mathcal{I}, \mathcal{R}, \mathcal{B} \) are defined analogously then \( \mathcal{I} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{I} \). Moreover we have \( \mathcal{I} \circ \mathcal{B} = \mathcal{J} \), where
The problem of retraction of semigroups is addressed when the semigroup is compact [5]. In this case, $\mathcal{K}, \mathcal{L}, \mathcal{R}, \mathcal{L} \circ \mathcal{R}$ and $\mathcal{J}$ are closed. It is easy to see that many of the results of [3] and [2] are valid in this setting. If we define a left $T$-ideal as a non-void set $A$ such that $TA \subseteq A$, then the basic propositions about ideals are also available. Many of these results follow from general theorems on structs [8].

**References**


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