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MULTIPLICATIVE NORMS FOR METRIC RINGS

SILVIO AURORA

1. Introduction. In his paper [19], S. Mazur stated two results concerning real normed algebras. The first of these, which asserted that the only normed division algebras over the real field were the real field, the complex field, and the division ring of real quaternions, was essentially proved by Gelfand in [10] and by Lorch in [17]. Elementary proofs of that result have also been given by Kametani [13] and Tornheim [26], while generalizations in various directions have been given by Kaplansky [16], Arens [4] and Ramaswami [23].

The second of the results given by Mazur was that a real normed algebra such that $||xy||=||x||\,||y||$ for all x and y mnst again be isomorphic to the real field, the complex field, or the division ring of real quaternions. This result was generalized in [8] by R.E. Edwards, who showed that the same conclusion holds for a Banach algebra under the weaker hypothesis that $||x||\,||x^{-1}||=1$ for all elements x which have inverses x^{-1}. A. A. Albert has also obtained results in [1], [2] and [3] similar to the second of Mazur's results.

In this paper, the second result of Mazur is generalized for certain types of metric rings. It is shown in section 6 that such rings must be division rings if the condition $||xy||=||x||\,||y||$ for all x and y holds. Similar results hold under the weaker assumption that $||x||\,||x^{-1}||=1$ for every element x which has an inverse x^{-1}. Under suitable additional conditions on the metric rings under discussion, it is shown in § 7 that the results just mentioned may be strengthened to assert that the ring is not only a division ring but is isomorphic to the real field, the complex field, or the division ring of real quaternions. Finally, the results on metric rings are applied to real normed algebras to obtain the results of Mazur and Edwards under weaker assumptions.

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2. Topological rings, metric rings, regular and singular elements. We shall first introduce some pertinent definitions and recall some elementary results concerning topological rings and metric rings. By a topological ring is meant a structure R which is at once a Hausdorff

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space and a ring[1] such that the applications $(a, b) \to a+b$ and $(a, b) \to ab$ of $R \times R$ into R are continuous.

If R is any ring, then a real-valued function $||x||$ defined on R is called a *norm* for R if it satisfies the following conditions:

(i) $||0||=0$ and $||a||>0$ for $a\neq 0$,

(ii) $||a+b|| \leq ||a||+||b||$ for all $a, b \in R$,

(iii) $||-a||=||a||$ for all $a \in R$,

(iv) $||ab|| \leq ||a|| \cdot ||b||$ for all $a, b \in R$. A norm for R is called an *absolute value* for R if it satisfies the following condition, which is clearly stronger than (iv):

(iv′) $||ab||=||a|| \cdot ||b||$ for all $a, b \in R$.

By a *metric ring* (*ring with absolute value*) is meant a ring R together with a norm (absolute value) for R. In any metric ring R the function $d(x, y)=||x-y||$ is a metric for R and induces in the usual way a topology for R relative to which R becomes a topological ring. Every ring admits as a norm the trivial function which takes the value 0 for the zero element of the ring and the value 1 for all other elements; in this case the induced topology is of course the discrete topology. The trivial norm is easily seen to be an absolute value for a ring if and only if the ring contains no proper zero-divisors.

For a finite ring which contains at least two elements it may be observed that the existence of an absolute value is possible only if the ring is a field and the absolute value is the trivial norm. In general, one might expect that the existence of an absolute value for a ring will require rather special properties of that ring. In the case of real normed algebras, for instance, S. Mazur stated in [19, second theorem] that when the norm is an absolute value the algebra must be isomorphic to the field of real numbers, the field of complex numbers, or the division ring of real quaternions. We shall consider below metric rings which satisfy various multiplicative restrictions on the norm such as (iv′), and we shall show that the class of such rings is strongly limited.

By an *isometry* of a metric ring R into a metric ring R_1 is meant a ring homorphism σ of R into R_1 such that $||\sigma x||=||x||$ for all $x \in R$; clearly, σ is necessarily an isomorphism of R into R_1. A metric ring R_1 is said to be an *extension* of the metric ring R provided that there exist an isometry of R into R_1. The notions of *limit*, *convergent sequences*, *fundamental sequences*, *complete metric ring*, and the *completion* of a metric ring are introduced in the standard way and the usual properties of these notions are easily verified.

We now exhibit some metric rings, in each case taking the obvious definitions for the operations of addition and multiplication when these are not specified, and with the ordinary absolute value as the norm in

[1] The rings in this paper are assumed to be associative and to possess a unit element, e.

examples (1)-(5):

(1) The ring of rational integers.
(2) The field of rational numbers.
(3) The field $\mathfrak{R}$ of real numbers.
(4) The field $\mathfrak{C}$ of complex numbers.
(5) The division ring $\mathfrak{Q}$ of real quaternions.
(6) The ring $C(X)$ of all continuous complex-valued functions defined on the compact Hausdorff space X, with the norm given by $\|f\|=\sup|f(x)|$, where the supremum is extended over all $x \in X$.
(7) The ring $\mathscr{D}$ of all complex-valued functions which are defined and continuous on the unit disc $\{\zeta|\,|\zeta|\leq 1\}$ of the complex plane and analytic over the interior, $\{\zeta|\,|\zeta|<1\}$, of that disc. The norm is given by $\|f\|=\sup|f(\zeta)|$, where the supremum is taken for all ζ such that $|\zeta|=1$.
(8) The field Q_p of rational numbers (where p is a fixed prime number) with the norm defined by $\|q\|=p^{-r}$, where r is the uniquely determined integer such that q has a representation $q=p^r(m/n)$ with m and n integers prime to p.
(9) The field P_p of p-adic numbers, which is obtained as the completion of Q_p of example (8).
(10) The ring $C^{(n)}$ of all real-valued functions which are defined on the closed unit interval and for which the first n derivatives exist and are continuous. In this case the norm is defined to be

$$\|f\|=\sum_{r=0}^{n}(r!)^{-1}\sup|f^{(r)}(x)|\ ,$$

where each supremum is extended over all x in the closed unit interval.

All of these rings except those of examples (2) and (8) are complete metric rings; the norm is also an absolute value in all of these rings except those of examples (6) when X contains at least two points, (7) and (10).

The notions of (*left, right*) *inverse* of an element, (*left, right*) *regular* elements, (*left, right*) *singular* elements, and the sets S^{l}, S^r, S, G^{l}, G^r and G are introduced as in [24]. Clearly, $G(G^{l}, G^r)$ is the complement of $S(S^{l}, S^r)$. It is easily verified that $S=S^{l}\cup S^r$ and $G=G^{l}\cap G^r$. Also, G^{l} and G^r are multiplicative semigroups[2] and G is a multiplicative group with e as its identity element.

In many examples the distinction between left regular elements and right regular elements disappears. For example, for a ring R which has no proper idempotents it is true that $G^{l}=G^r$. For, if $a\in G^{l}$

[2] A *semi-group* is understood to be a non-empty system which is closed relative to an associative binary operation.

and $a'a=e$, then aa' is an idempotent distinct from 0, so $aa'=e$ and this means that $a \in G^r$. Similarly, in a ring without proper nilpotents, $G^{\bar{l}}=G^r$. For, in such a ring all idempotents are central by Lemma 1 of [9], so if $a \in G^{\bar{l}}$ where $a'a=e$, then aa' is an idempotent and therefore central. Thus,

$$aa'=a'a(aa')=a'(aa')a=(a'a)(a'a)=e ,$$

so $a \in G^r$, whence $G^{\bar{l}}=G^r$.

If R is a topological ring, its group G of regular elements is the union of a family of disjoint, maximal connected subsets,—the *components* of G. The *principal component*, G_1, is the component which contains the unit element e. It may be shown that G_1 is an invariant subgroup of G such that the cosets modulo G_1 are the components of G.

Following Kaplansky [14] we call a topological ring a *Q-ring* if the set G of its regular elements is an open set[3]. For a complete metric ring it is well known that $G^{\bar{l}}$, G^r and G are open sets so that $S^{\bar{l}}$, S^r and S are closed sets. This is shown in [18], [20] or [24] for the case of Banach algebras, and the present result, which utilizes essentially the same proof, may be found in [14]. Thus, every complete metric ring is a Q-ring.

3. Generalized divisors of zero.

In [25], G. Šilov introduced the concept of a generalized divisor of zero in a Banach algebra. A more detailed study of this concept was presented by Rickart in [24]; the present development of a theory of generalized divisors of zero in a metric ring follows closely the development presented in the latter paper, although the possibility of multiplication by complex scalars permits stronger results in the case of a Banach algebra. Šilov's results demonstrated the existence of generalized divisors of zero in any non-trivial Banach algebra; as a corollary he obtained the result of Mazur mentioned above on Banach algebras with a norm which is an absolute value. Our study of generalized divisors of zero leads in a similar way to a generalization of Mazur's result to the case of certain types of metric rings.

DEFINITION. If a is any element of a metric ring we define $\bar{l}(a)=\inf(||ax||/||x||)$ and $r(a)=\inf(||xa||/||x||)$, where in each case the infimum is taken as x ranges over the non-zero elements of the ring.

The results which follow are easily proven and in many cases follow as in Rickart's paper.

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[3] Kaplansky,s definition is in terms of quasiregular elements, but is easily seen to be equivalent to the present one in rings with unit element.

LEMMA 1. (i) $0 \leq \bar{l}(a) \leq \|a\|$ *for any* a; (ii) $\bar{l}(a)\bar{l}(b) \leq \bar{l}(ab) \leq \|a\| \cdot \bar{l}(b)$ *for any* a *and* b; (iii) $|\bar{l}(a) - \bar{l}(b)| \leq \|a-b\|$ *for any* a *and* b.

COROLLARY. $\bar{l}(x)$ *is a continuous function of* x.

DEFINITION. $Z^{\bar{l}} = \{a | \bar{l}(a) = 0\}$, $Z^r = \{a | r(a) = 0\}$; $Z = Z^{\bar{l}} \cup Z^r$;
$H^{\bar{l}} = \{a | l(a) > 0\}$, $H^r = \{a | r(a) > 0\}$; $H = H^{\bar{l}} \cap H^r$.

It is easily observed that $Z^{\bar{l}}(Z^r, Z)$ is the complement of $H^{\bar{l}}(H^r, H)$. Since the corollary implies that $Z^{\bar{l}}$ (and also Z^r) is closed it follows that $Z = Z^{\bar{l}} \cup Z^r$ is closed. Consequently, $H^{\bar{l}}$, H^r and H are open.

An element of $Z^{\bar{l}}(Z^r, Z)$ is called a *generalized left-divisor* (*right-divisor, divisor*) *of zero*. Clearly, a (left, right) zero-divisor is always a generalized (left, right) divisor of zero. The converse, however, is not always true. For example, let R_1 be the metric ring consisting of the same elements as the ring of example (7), but where the norm of an element distinct from zero is taken as the maximum of 1 and the norm as given in example (7). The topology of R_1 is then the discrete topology. There are no proper zero-divisors in R_1, but the function $f(\zeta) \equiv \zeta - 1$ is a generalized left-divisor of zero in R_1, for if $f_n(\zeta) \equiv \zeta^n + \zeta^{n-1} + \cdots + 1$, then $\|f_n\| = n+1$, while $\|ff_n\| = 2$ since

$$(\zeta - 1)(\zeta^n + \zeta^{n-1} + \cdots + 1) \equiv \zeta^{n+1} - 1;$$

thus, $\|ff_n\| / \|f_n\| = 2/(n+1)$ for $n = 1, 2, \cdots$, so that $\bar{l}(f) = 0$ and f is a generalized left-divisor of zero.

In [24] Rickart defines a left generalized null divisor to be an element s such that there exists a sequence $\{z_n\}$ such that $\|z_n\| = 1$ for all n, and such that $sz_n \to 0$. However, he notes that s is a left generalized null divisor if and only if $\bar{l}(s) = 0$. In a metric ring, it is clear that a left generalized null divisor in the sense of Rickart satisfies the condition $\bar{l}(s) = 0$ and is thus a generalized left-divisor of zero in the sense of this paper. However, a generalized left-divisor of zero in the sense of this paper need not be a left generalized null divisor in the sense of Rickart; for example, the element f in the preceding paragraph is a generalized left-divisor of zero in R_1, but if there were a sequence $\{g_n\}$ of R_1 with $\|g_n\| = 1$ for all n and with $fg_n \to 0$, then for n large fg_n would be zero since R_1 is discrete, and, since R_1 has no proper zero-divisors, either f or g_n would be zero, and this is clearly impossible, so f can not be a left generalized null divisor in the sense of Rickart's definition.

It is nevertheless true that for many metric rings the concepts of

[4] For brevity, right-sided results are often omitted.

left generalized null divisor as defined by Rickart and that of generalized left-divisor of zero as employed in this paper coincide. One can easily show, for instance, that this is the case in a metric ring R such that for any element a distinct from zero there is an element b of $\mathscr{R}(R)$ (this set is introduced later in § 5) such that $||a|| \cdot ||b|| = 1$. It follows also that the concepts coincide in a metric ring R such that for every positive real number r there is an element b of $\mathscr{R}(R)$ such that $||b|| = r$. In particular, this condition holds in any Banach algebra, so that the two concepts coincide in any Banach algebra, as Rickart showed.

If R_2 is the ring of elements of R_1 but with the norm taken such that $||g|| = 1$ for any g distinct from zero, then R_2 is also discrete, so that the topological rings which underlie R_1 and R_2 are identical. However, the element f defined above is a generalized left-divisor of zero in R_1, but not in R_2, for the norm of R_2 is an absolute value, whence $||ax||/||x|| = ||a||$ for all non-zero x in R_2, so $\bar{l}(a) = ||a||$ for any a in R_2, and consequently R_2 can not contain any generalized left-divisors of zero different from zero. This shows that the notion of generalized left-divisors of zero is not a purely topological notion. In particular, this concept differs from that of a *topological zero-divisor* as defined, for example, in [15]. For, while it is easily shown that a topological left zero-divisor in a metric ring is necessarily a generalized left-divisor of zero, the converse is not true since otherwise the element f of R_1 would be a topological left zero-divisor in R_1 and hence in R_2 and hence a generalized left-divisor of zero in R_1.

LEMMA 2. (i) *If* $b \in Z^{\bar{l}}$, *then* $ab \in Z^{\bar{l}}$ *for any* a. (ii) *If* $ab \in Z^{\bar{l}}$, *then* $a \in Z^{\bar{l}}$ *or* $b \in Z^{\bar{l}}$.

LEMMA 3. $Z^{\bar{l}} \subset S^{\bar{l}}$, $Z^r \subset S^r$, $Z \subset S$, $G^{\bar{l}} \subset H^{\bar{l}}$, $G^r \subset H^r$ *and* $G \subset H$.

Lemma 3 shows that the sets $H^{\bar{l}}$, H^r and H are not empty and contain in fact all regular elements. It is also clear that the zero element belongs to the sets $Z^{\bar{l}}$, Z^r and Z; but in many instances these sets contain no element other than zero. For example, the metric rings of examples (1)–(5) possess no generalized divisors of zero other than the zero-element. However, for a complex Banach algebra distinct from $\mathfrak{C}$, G. Šilov showed in [25, lemma] that there always exist generalized divisors of zero distinct from the zero-element.[5] The results which follow give conditions under which certain types of metric rings contain nonzero generalized divisors of zero.

LEMMA 4. *For any metric Q-ring,* H *is the union of the disjoint open sets* G *and* $S \cap H$.

[5] See also the remark by Lorch in [17].

LEMMA 5. *Let R be a metric Q-ring. Let $\{a_n\}$ be a sequence of regular elements of R which converges to an element a in R. If the sequence $\{a_n^{-1}\}$ is bounded,*[6] *then a is a regular element.*

THEOREM 1. *If R is a metric Q-ring, then*[7] $[G] \cap S \subset \bar{Z^l} \cap Z^r$. *If, in addition, $R^{(0)}$ is connected,*[8] *then either R is a division ring or Z contains an element distinct from zero.*

Proof. The first statement follows as in Rickart's paper.

If R is not a division ring, then the closed set S meets $R^{(0)}$. Also, the closed set $[G]$ meets $R^{(0)}$, and $R^{(0)} \subset [G] \cup S$. If $R^{(0)}$ is connected, then $R^{(0)} \cap [G] \cap S$ is not empty, so $[G] \cap S$ contains an element distinct from zero. It follows that $\bar{Z^l} \cap Z^r$ contains a nonzero element, so Z also contains a nonzero element.

4. **Proper rings.** Lemma 3 asserts that the inclusion $Z \subset S$ always holds. Thus, every generalized divisor of zero is a singular element, although, as we see below, a singular element need not be a generalized divisor of zero. Indeed, the generalized divisors of zero possess the special property of permanent singularity; that is, a generalized divisor of zero does not acquire an inverse in any extension of the given ring since it is still a generalized divisor of zero and hence singular in that extension. In the ring $\mathscr{D}$ of example (7), the function $f(\zeta) \equiv \zeta$ is a singular element, but the ring $C(X)$, where X is the unit circle of the complex plane, is readily seen to be an extension of $\mathscr{D}$ in which f is a regular element.[8] Thus, f is not a permanently singular element of $\mathscr{D}$ and so f is not a generalized divisor of zero in $\mathscr{D}$, even though f is a singular element of $\mathscr{D}$. Thus, the inclusion $Z \subset S$ may be a proper inclusion.

DEFINITION. A metric ring R is said to be *proper* provided that $Z = S$, or, equivalently, that $H = G$.

The preceding discussion shows that even a complete metric ring which is connected and locally connected need not be proper; for example, $\mathscr{D}$ is not proper. However, many metric rings are proper, including any ring $C(X)$ of example (6). We see that a proper ring is a division ring if and only if there are no generalized divisors of zero other than zero. In particular, a proper ring with absolute value can have no generalized divisors of zero except zero and is therefore a divi-

[6] A set A is said to be *bounded* if there is a number M such that $\|a\| \leq M$ for all a in A.

[7] If A is any set, the symbols $[A]$ and $A^{(0)}$ denote the topological closure of A and the set of non zero elements of A, respectively.

[8] Compare [**24**].

sion ring. We shall give below some sufficient conditions for a metric ring to be proper; these conditions, in combination with the existence of an absolute value or with some other multiplicative restriction on the norm, will imply that the ring must be a division ring.

THEOREM 2. *If R is a metric Q-ring such that H is connected, then R is proper.*

THEOREM 3. *If R is a metric Q-ring (complete metric ring) such that S is nowhere dense (of first category), then R is proper.*

Proof. Either hypothesis of Theorem 3 insures that S is a closed set. Also, either hypothesis implies that S is nowhere dense, for if S is assumed to be of first category in a complete metric ring, then S is nowhere dense since a closed set of a complete metric space is of first category if and only if it is nowhere dense. The proofs of these two theorems then follow as in [24].

It must be noted that the hypothesis of completeness is needed where it occurs in Theorem 3. For, let R be the set of all functions f which belong to the ring $\mathscr{D}$ of example (7) and for which $f(0)$ is a rational number. It is easily seen that R is a metric Q-ring but is not complete. Also, R is of first category, so the set S for R is also of first category. However, R is not proper, for it contains the singular element $f(\zeta)\equiv\zeta$, which is not a generalized divisor of zero, as was noted at the beginning of this section.

DEFINITION. If R is any ring, then by an *involution* of R is meant a mapping $a \rightarrow a^*$ of R into itself such that:

(i) $(a+b)^*=a^*+b^*$ for all $a, b \in R$,

(ii) $(ab)^*=b^*a^*$ for all $a, b \in R$,

(iii) $(a^*)^*=a$ for all $a \in R$.

That is, an involution of R is an anti-automorphism of period two. For a given involution of R, an element a is said to be *self-adjoint* provided that $a^*=a$.

For $\mathfrak{C}$, for instance, the mapping which associates with each complex number its complex conjugate is an involution. Similarly, the mapping which associates with each quaternion its conjugate is an involution of $\mathfrak{Q}$. In both cases the self-adjoint elements are simply the real numbers. In the case of the ring of all bounded linear operators on a Hilbert space, the mapping which associates with each operator its adjoint is an involution, and the self-adjoint elements are of course the self-adjoint operators. Thus, many rings admit at least one involution. For metric rings, one is naturally interested in the involutions which are closely related to the metric or topological structure of the ring.

DEFINITION. An involution $a \rightarrow a^*$ of a metric ring R is said to be *bounded* provided that there is a positive constant β such that $\|a^*\| \leq \beta\|a\|$ for all a. An involution $a \rightarrow a^*$ of a metric ring R is said to be *real* if no self-adjoint element is an interior point of the set of singular elements.

The involutions described above are all bounded and real, if in the case of the ring of all bounded linear operators on a Hilbert space we take as the norm of an element its bound as an operator.

These definitions differ from the corresponding definitions of Rickart in [24] by the omission of the mention of scalars in the present definitions. Thus, the identity mapping of the field $\mathfrak{C}$ onto itself is a real and bounded involution in the present sense but is not even an involution in the sense of Rickart, since the image of $i \cdot 1$, where i is a scalar, should be $(-i)1$ but is $i \cdot 1$.

For a complex Banach algebra, an involution which is real in the sense of Rickart is also real in the present sense. For, let an involution be real in the sense of Rickart. Then, for any self-adjoint element a the spectrum of a is real. If $\{\lambda_n\}$ is a sequence of non-real complex numbers which converges to zero, then $\{a-\lambda_n \cdot e\}$ is a sequence which converges to a. Since the λ_n are not in the spectrum of a, it follows that $a-\lambda_n \cdot e$ is regular for all n. This shows that a is the limit of a sequence of regular elements and hence is not in the interior of S. Thus, the involution is real in the present sense.

The identity mapping of the ring $\mathscr{D}$ of example (7) is clearly a bounded involution relative to which all elements are self-adjoint. But the function $f(\zeta) \equiv \zeta$ is a singular element of $\mathscr{D}$, and Rouché's Theorem implies that any element of $\mathscr{D}$ whose distance from f is less than 1 is also a singular element; thus, the set of singular elements of $\mathscr{D}$ has a nonempty interior and contains the self-adjoint element f. The involution in question is consequently not real even though it is bounded.

There are also real involutions which are not bounded. For example, let R be the field obtained by adjoining x and y to a given field F, so that R consists of rational expressions in x and y with coefficients in F. If $P(x, y)$ is any irreducible polynomial belonging to $F[x, y]$, then each element of R may be represented in the form $\varphi = P^\mu \cdot M/N$, where M and N are elements of $F[x, y]$ which are not divisible by P, and where μ is a uniquely determined integer which depends only upon φ and P. If $\|\varphi\| = 2^{-\mu}$ where μ is the integer which corresponds to φ, then R becomes a metric ring relative to this norm. The involution of R which maps an expression $f(x, y)$ onto $f(y, x)$ is clearly real since R is not discrete and the only singular element of R is 0. In case $P(x, y)$ does not divide $P(y, x)$, let $Q(x, y) = P(y, x)$, so for any natural number n we have $\|P^n\| = 2^{-n}$, while $\|Q^n\| = 1$. But Q^n is the image of P^n relative to the involution, and $||Q^n||/||P^n||=2^n$, so that the involution is not bounded, although it is real.

For an involution to be both real and bounded, the metric ring in question must be proper, as the theorem which follows shows.

LEMMA 6. *If $a \rightarrow a^*$ is a bounded involution of the metric ring R, then $(\bar{Z^l})^* = Z^r$ and $(Z^r)^* = \bar{Z^l}$.*[9]

Proof. If $||a^*|| \leq \beta ||a||$ for all $a \in R$, then $||a|| = ||a^{**}|| \leq \beta ||a^*||$ for all a, so that

$$\bar{l}(a) \leq ||ax^*||/||x^*|| = ||(xa^*)^*||/||x^*|| \leq \beta^2 ||xa^*||/||x||$$

for all nonzero x. Thus, $\bar{l}(a) \leq \beta^2 r(a^*)$, so $a^* \in Z^r$ implies $a \in \bar{Z^l}$. That is, $(Z^r)^* \subset \bar{Z^l}$, while, similarly, $(\bar{Z^l})^* \subset Z^r$. Taking images relative to the involution, we obtain $Z^r \subset (\bar{Z^l})^*$ and $\bar{Z^l} \subset (Z^r)^*$. By combining the four inclusions, we obtain the desired results.

THEOREM 4. *If R is a metric Q-ring which admits a real, bounded involution, then R is proper.*

Proof. Let $a \rightarrow a^*$ be a real, bounded involution of R. If $a \in \bar{S^l}$, then $a^* \cdot a \in \bar{S^l}$ and $a^* \cdot a$ is self-adjoint. Since the involution is real, $a^* \cdot a \in [G]$. Thus, $a^* \cdot a \in [G] \cap S$. Theorem 1 implies that $a^* \cdot a \in \bar{Z^l} \cap Z^r$. Since $a^* \cdot a \in \bar{Z^l}$, we may conclude from Lemma 2 that $a^* \in \bar{Z^l}$ or $a \in \bar{Z^l}$. That is, $a \in (\bar{Z^l})^* = Z^r$ or $a \in \bar{Z^l}$, so $a \in Z = \bar{Z^l} \cup Z^r$. This shows that $\bar{S^l} \subset Z$. Similarly, $S^r \subset Z$, whence $S = \bar{S^l} \cup S^r \subset Z$. But $Z \subset S$ by Lemma 3, so $Z = S$, and R is thus proper.

5. The sets $\mathscr{L}$(R), $\mathscr{R}$(R), $\mathscr{G}$ and $\mathscr{G}'$. We shall now introduce some sets which measure to some extent how closely the norm of a given metric ring resembles an absolute value.

DEFINITION. The norm of a metric ring is said to be *multiplicative* on a set A if $||ab|| = ||a||\,||b||$ for all $a, b \in A$. (Thus, an absolute value is simply a norm which is multiplicative on the entire ring.) By a *μ-group* is meant a multiplicative group contained in a metric ring and on which the norm of the ring is multiplicative.

DEFINITION. If R is a metric ring, $\mathscr{L}(R) = \{a | a \in R,\ ||ax|| = ||a||\,||x||$

[9] If A is a set in a ring with involution $a \rightarrow a^*$, then the set of all a^*, where a is in A, is denoted by A^*. Note that the statement of the corresponding lemma in [**24**] assumes, but does not use, a *real* involution.

for all $x \in R\}$ and $\mathscr{R}(R)=\{a|a \in R,\ ||xa||=||x||\,||a|| \text{ for all } x \in R\}$.

LEMMA 7. *If R is a metric ring, then:*

(i) $a \in \mathscr{L}(R)$ *if and only if* $\bar{l}(a)=||a||$;

(ii) $0 \in \mathscr{L}(R)$;

(iii) *if* $\mathscr{L}(R) \neq \{0\}$ *then* $||e||=1$;

(iv) $e \in \mathscr{L}(R)$ *if and only if* $||e||=1$;

(v) $\mathscr{L}(R)$ *is a closed set and a multiplicative semigroup*;

(vi) *if* $a,\ ab \in \mathscr{L}(R)$ *where* $a \neq 0$, *then* $b \in \mathscr{L}(R)$.

Proof. If $a \in \mathscr{L}(R)$, then $||ax||/||x||=||a||$ for all $x \neq 0$, so $\bar{l}(a)=||a||$. Conversely, if $\bar{l}(a)=||a||$ then $||a||=\bar{l}(a) \leq ||ax||/||x||$ for any $x \neq 0$, so $||a||\,||x|| \leq ||ax||$ for any x, whence $a \in \mathscr{L}(R)$.

Clearly, $0 \in \mathscr{L}(R)$. Also, $\bar{l}(e)=1$, and since $e \in \mathscr{L}(R)$ if and only if $\bar{l}(e)=||e||$, it follows that $e \in \mathscr{L}(R)$ if and only if $||e||=1$. If $\mathscr{L}(R)$ contains an element $a \neq 0$, then $||a||=||ae||=||a|| \cdot ||e||$, whence $||e||=1$.

$\mathscr{L}(R)$ is the set where the continuous function $||x||-\bar{l}(x)$ vanishes, so $\mathscr{L}(R)$ is a closed set. If $a,\ b \in \mathscr{L}(R)$ then

$$||ab|| \leq ||a|| \cdot ||b|| = \bar{l}(a)\bar{l}(b) \leq \bar{l}(ab) \leq ||ab||$$

by Lemma 1 (i) and (ii), so $||ab||=\bar{l}(ab)$, whence $ab \in \mathscr{L}(R)$. This shows that $\mathscr{L}(R)$ is a multiplicative semigroup.

Finally, if a and ab belong to $\mathscr{L}(R)$ and $a \neq 0$, then

$$||a|| \cdot ||b|| \cdot ||x|| = ||ab|| \cdot ||x|| = ||abx|| = ||a|| \cdot ||bx|| ,$$

so $||b|| \cdot ||x|| = ||bx||$ for any x, whence $b \in \mathscr{L}(R)$.

The sets $\mathscr{L}(R)$ and $\mathscr{R}(R)$ measure the extent to which the norm resembles an absolute value. Indeed, it is easily seen that the norm of R is an absolute value if and only if $\mathscr{L}(R)=R$. For the ring $C(X)$ of example (6) the sets $\mathscr{L}(R)$ and $\mathscr{R}(R)$ coincide and consist of all functions whose absolute value is a constant function. The elements of $\mathscr{L}(R)$ in this case are then regular or equal to zero. In general, it will be useful to consider the set of regular elements of $\mathscr{L}(R)$.

DEFINITION. In a metric ring for which $||e||=1$, let

$$\mathscr{G}=\{a|a \in G,\ ||a|| \cdot ||a^{-1}||=1\}$$

and

$$\mathscr{G}'=\{a|a \in G,\ ||a||=||a^{-1}||=1\}.$$

DEFINITION. If A is any subset of a metric ring R, let $\mathscr{N}(A)=\{||a|| \mid a \in A\}$, $\nu(a)=||a||$ for any $a \in R$.

THEOREM 5. *Let R be a metric ring such that $\|e\|=1$. Then $\mathscr{L}(R)\cap G=\mathscr{G}=\mathscr{R}(R)\cap G$. Also, $\mathscr{G}$ is closed in G and is a subgroup of G. Furthermore, $\mathscr{G}$ is a maximal μ-group.*

Proof. If $a\in\mathscr{L}(R)\cap G$, then $\|a\|\cdot\|a^{-1}\|=\|aa^{-1}\|=1$, so $a\in\mathscr{G}$. Conversely, if $a\in\mathscr{G}$, then

$$\|a\|\cdot\|x\|=\|a\|\cdot\|a^{-1}ax\|\leq\|a\|\cdot\|a^{-1}\|\cdot\|ax\|=\|ax\|$$

for any x, so $a\in\mathscr{L}(R)$, whence $a\in\mathscr{L}(R)\cap G$. Since $\mathscr{L}(R)$ is closed, $\mathscr{G}=\mathscr{L}(R)\cap G$ is closed in G. The proof that $\mathscr{R}(R)\cap G=\mathscr{G}$ is similar to the above.

Since $\mathscr{L}(R)$ and G are semigroups, $\mathscr{G}$ is also a semigroup. Also, e is in $\mathscr{G}$, and $\mathscr{G}$ contains the inverses of all of its elements, so $\mathscr{G}$ is a group. The norm is multiplicative on $\mathscr{L}(R)$ and hence on $\mathscr{G}\subset\mathscr{L}(R)$, so $\mathscr{G}$ is a μ-group. The definition of $\mathscr{G}$ clearly implies that $\mathscr{G}$ is the largest μ-group which is contained in G. But any μ-group which contains $\mathscr{G}$ must be contained in G since G is a maximal multiplicative group, so $\mathscr{G}$ coincides with such a μ-group and is hence a maximal μ-group.

THEOREM 6. *Let R be a metric ring with $\|e\|=1$. Then the restriction of ν to $\mathscr{G}$ is a homomorphism of $\mathscr{G}$ onto the multiplicative group $\mathscr{N}(\mathscr{G})$ and has $\mathscr{G}'$ as its kernel. $\mathscr{G}'$ is the largest multiplicative group on the unit sphere $U=\{x\mid\|x\|=1\}$. If R is also a Q-ring, then $\mathscr{G}'$ and $\mathscr{G}\cup\{0\}$ are closed sets and $\mathscr{G}$ is closed if and only if $\mathscr{G}=\mathscr{G}'$.*

Proof. The restriction of ν to $\mathscr{G}$ is clearly a homomorphism of $\mathscr{G}$ onto $\mathscr{N}(\mathscr{G})$, and the kernel of this homomorphism is $\mathscr{G}\cap U=\mathscr{G}'$. It is also clear that $\mathscr{G}'$ is the largest multiplicative group on U.

Since $\mathscr{G}=\mathscr{L}(R)\cap G$ by the preceding theorem, we have $[\mathscr{G}]\subset\mathscr{L}(R)\cap[G]$ because $\mathscr{L}(R)$ is closed according to Lemma 7(v). If R is a Q-ring, then

$$[\mathscr{G}]\cap S\subset\mathscr{L}(R)\cap[G]\cap S\subset\mathscr{L}(R)\cap Z^{\bar{l}}\cap Z^{r}$$

by Theorem 1. But, if $a\in\mathscr{L}(R)$ then $\bar{l}(a)=\|a\|$ by Lemma 7(i), while if $a\in Z^{\bar{l}}$, $\bar{l}(a)=0$. Thus, if $a\in\mathscr{L}(R)\cap Z^{\bar{l}}\cap Z^{r}$ we have $\bar{l}(a)=\|a\|$ and $\bar{l}(a)=0$, so $a=0$. It follows that $\mathscr{L}(R)\cap Z^{\bar{l}}\cap Z^{r}=\{0\}$, so $[\mathscr{G}]\cap S\subset\{0\}$. But $[\mathscr{G}]\cap G\subset\mathscr{L}(R)\cap G=\mathscr{G}$ by Theorem 5, so $[\mathscr{G}]\subset\mathscr{G}\cup\{0\}$. Then

$$[\mathscr{G}\cup\{0\}]\subset[\mathscr{G}]\cup\{0\}\subset\mathscr{G}\cup\{0\}\ ,$$

so $\mathscr{G}\cup\{0\}$ is closed. $\mathscr{G}'$ is the intersection of the closed sets U and $\mathscr{G}\cup\{0\}$ and is consequently closed.

Finally, if $\mathscr{G}=\mathscr{G}'$ then $\mathscr{G}$ is closed since $\mathscr{G}'$ is closed in a Q-ring by the preceding paragraph. Conversely, suppose $\mathscr{G}$ is closed and contains an element a not in $\mathscr{G}'$. Then theelement s a^n for $n=\pm 1, \pm 2, \cdots$ belong to $\mathscr{G}$, and since $||a||\neq 1$ and $||a^n||=||a||^n$, it follows that there are elements a^n in every neighborhood of 0. Since $\mathscr{G}$ is closed, $0 \in \mathscr{G}$. This is a contradiction. Thus, $\mathscr{G}=\mathscr{G}'$ if $\mathscr{G}$ is closed.

6. Multiplicative conditions on the norm.

We shall now consider several related conditions on the norm of a metric ring. In the sequel it will be assumed that $||e||=1$ in the metric rings under discussion.

M1. The norm of R is an absolute value. (Equivalently, $\mathscr{L}(R)=R$.)

M2. $\mathscr{G}=G$; that is, the norm is multiplicative on G.

M3. $\mathscr{G}$ is open.

M4. $\mathscr{G}$ fails to be nowhere dense in R.

M5. $\mathscr{L}(R)$ fails to be nowhere dense in R.

In the case of M5, Lemma 7(iii) indicates that, for a non-discrete ring, this condition can hold only if $||e||=1$. However, $\mathscr{G}$ has been defined only for metric rings for which $||e||=1$, so that M2, M3 and M4 are meaningless unless $||e||=1$; for that reason we have assumed that $||e||=1$.

It is easily seen that for any metric ring M1 implies M2, M3 implies M4, and M4 implies M5. For a metric Q-ring it is also true that M2 implies M3. Thus, for any metric Q-ring if one of the conditions M1-M5 holds then all of the later ones also hold. Under certain circumstances, two or more of the conditions M1–M5 may be equivalent.

LEMMA 8. *If R is a metric Q-ring, then conditions* M3, M4 *and* M5 *are equivalent in R.*

Proof. By the previous remarks it will suffice to show that when M5 holds then M3 holds. We may assume that R is not discrete, for if R is discrete then M3, M4 and M5 all hold. Now, if M5 holds in R, the closed set $\mathscr{L}(R)$ contains an open sphere Σ which has center $a\neq 0$ and radius $r>0$, so

$$\Sigma=\{x|\ ||x-a||<r\}\ .$$

If

$$\Sigma'=\{x|\ ||x-e||<r/||a||\}$$

is the open sphere with center e and radius $r/||a||$, then $y\in\Sigma'$ implies $||y-e||<r/||a||$, so $||ay-a||=||a||\cdot||y-e||<r$, whence $ay\in\Sigma\subset\mathscr{L}(R)$. Lemma 7(vi) implies that $y\in\mathscr{L}(R)$; this shows that $\Sigma'\subset\mathscr{L}(R)$, so e is an interior point of $\mathscr{L}(R)$. Since R is a Q-ring, e is an interior point of G, so e is an interior point of $\mathscr{L}(R)\cap G=\mathscr{G}$. Since $\mathscr{G}$ is a topolog-

ical group and is therefore homogeneous, $\mathscr{G}$ must be open,[10] so M3 holds for R. This proves the lemma.

LEMMA 9. *If R is a metric Q-ring such that $\mathscr{G}$ meets every component of G, then* M2 *and* M3 *are equivalent in R.*

Proof. If M3 holds, $\mathscr{G}$ is open. By Theorem 5, $\mathscr{G}$ is closed in G, so $\mathscr{G}$ is open and closed in G. Thus, $\mathscr{G}$ contains every component of G which it meets, so if $\mathscr{G}$ meets every component it follows that $\mathscr{G}=G$; that is, M2 holds. Conversely, it has already been pointed out that if M2 holds for a metric Q-ring then M3 also holds.

COROLLARY. *If R is a metric Q-ring such that G is connected, then* M2 *and* M3 *are equivalent in R.*

LEMMA 10. *If R is a metric ring such that G is dense in R, then* M1 *is equivalent to* M2 *in R. In particular, if R is a metric Q-ring (complete metric ring) such that S is nowhere dense (of first category), then* M1 *is equivalent to* M2 *for R.*

Proof. If R is a metric ring in which G is dense, then if M2 holds we have $R=[G]=[\mathscr{G}]\subset\mathscr{L}(R)$, so M1 holds. Thus, M1 is equivalent to M2.

If R is a metric Q-ring and the closed set S is nowhere dense, then G is dense, so that M1 is equivalent to M2 for R. For R a complete metric ring and S of first category, it follows that R is a metric Q-ring and S is nowhere dense since it is a closed first category set of a complete metric space. By the preceding result, M1 is equivalent to M2.

Note. In the presence of condition M1, a metric ring R can have no zero-divisors other than 0, for if $ab=0$, then $||a||\cdot||b||=||ab||=0$, whence $a=0$ or $b=0$. Thus, the ring contains no proper nilpotents or idempotents, and the remarks of § 2 imply that $G^l=G^r=G$, so inverses are always two-sided and unique for such a ring.

The conditions M1–M5 are strong restrictions on the algebraic structure of a metric ring, as this remark on G^l and G^r indicates. Indeed, under suitable conditions they will insure that the given ring is a division ring. Some results in this direction follow.

LEMMA 11. *Let R be a metric ring for which* M1 *holds. Then R is proper if and only if it is a division ring.*

[10] See [**6**].

Proof. If M1 holds for R, then $\bar{l}(a)=\|a\|=r(a)$ for all a in R, by Lemma 7 (i). Thus, $Z=\{0\}$ for this ring. Then $Z=S$ is equivalent to $S=\{0\}$; that is, R is proper if and only if it is a division ring.

THEOREM 7. *Let R be a metric Q-ring such that S is nowhere dense. If* M1 *or* M2 *holds for R, then R is a division ring and its norm is an absolute value.*

Proof. S is nowhere dense, so G is dense, whence M1 is equivalent to M2 by Lemma 10. Also, Theorem 3 implies that R is proper, and it follows from the preceding lemma that R is a division ring. Since M1 must hold in R if M1 or M2 is assumed to hold, it follows that the norm of R is an absolute value.

COROLLARY. *Let R be a complete metric ring such that S is of first category. If* M1 *or* M2 *holds for R, then R is a division ring and its norm is an absolute value.*

THEOREM 8. *Let R be a metric Q-ring such that H is connected. If* M1 *holds for R, then R is a division ring.*

Proof. By Theorem 2, R is proper, so Lemma 11 implies that R is a division ring.

If H is connected and also dense, then R is proper and G, which therefore coincides with H, is connected and dense. Lemmas 8 and 10 and the corollary to Lemma 9 imply that M1–M5 are equivalent, so that if one of the conditions M1–M5 is assumed, then M1 holds, and the theorem just established shows that R is a division ring. This establishes the following corollary.

COROLLARY. *Let R be a metric Q-ring such that H is connected and dense. If one of the conditions* M1–M5 *holds for R, then R is a division ring and its norm is an absolute value.*

THEOREM 9. *Let R be a metric Q-ring which admits a real, bounded involution and for which* M1 *holds. Then R is a division ring.*

Proof. By Theorem 4, R is proper, so Lemma 11 implies that R is a division ring.

LEMMA 12. *Let R be a metric Q-ring which satisfies one of the conditions* M1–M5. *If A is a connected subset of R which does not contain* 0, *then either $A\subset\mathcal{G}$ or A is disjoint from $\mathcal{G}$.*

Proof. Because of the relations among M1–M5, M5 holds, so Lemma 8 implies that M3 holds, whence $\mathscr{G}$ is open. But $\mathscr{G}$ is closed in $R^{(0)}$ since $\mathscr{G}\cup\{0\}$ is closed by Theorem 6. Then $\mathscr{G}$ is open and closed in $R^{(0)}$, so any connected subset A of $R^{(0)}$ must be contained in $\mathscr{G}$ or disjoint from $\mathscr{G}$.

COROLLARY. *If R is a metric Q-ring which satisfies one of the conditions* M1–M5, *then each connected component of $\mathscr{G}$ coincides with a component of G, and, in particular, $\mathscr{G}\supset G_1$.*

THEOREM 10. *Let R be a metric Q-ring such that $R^{(0)}$ is a connected set. If one of the conditions* M1–M5 *holds for R, then R is a division ring with absolute value.*

Proof. Lemma 12 implies that $\mathscr{G}$ contains the connected set $R^{(0)}$. Thus, $R=\mathscr{G}\cup\{0\}$, so R is a division ring with absolute value.

If it is assumed that S is nowhere dense and G is connected in a metric Q-ring in which one of the conditions M1–M5 holds, then Lemma 12 implies that $\mathscr{G}=G$, while G is dense since S is nowhere dense. Thus, $R=[G]$, and $R^{(0)}$ is connected since G is connected. The theorem implies that R is a division ring with absolute value in this case. The assumption of completeness again permits the requirement that S be nowhere dense to be replaced by the requirement that S be of first category.

COROLLARY 1. *Let R be a metric Q-ring (complete metric ring) for which S is nowhere dense (of first category) and G is connected. If one of the conditions* M1–M5 *holds for R, then R is a division ring with absolute value.*

COROLLARY 2. *If R is a metric Q-ring such that $R^{(0)}$ is connected, then precisely one of the following statements is valid:*

(α) *$\mathscr{L}(R)$ is nowhere dense in R.*

(β) *R is a division ring with absolute value.*

COROLLARY 3. *If R is a metric Q-ring (complete metric ring) for which G is a connected set and S is nowhere dense (of first category), then precisely one of the following statements is valid:*

(α) *$\mathscr{L}(R)$ is nowhere dense in R.*

(β) *R is a division ring with absolute value.*

Corollaries 2 and 3 follow immediately from the theorem and Corollary 1, respectively, since if (α) does not hold then M5 holds and therefore (β), which is the conclusion of the theorem and of Corollary 1, must hold.

Corollaries 2 and 3 clearly continue to hold if (α) is replaced by: (α') $\mathscr{S}$ is nowhere dense in R. In Corollaries 1 and 3 the hypothesis that G be connected may be replaced by the hypothesis that $\mathscr{S}$ meet every component of G. Another alternative for these two corollaries is to replace all conditions on G and S by the hypothesis that $\mathscr{S}$ meet every component of $R^{(0)}$.

7. **Division rings with absolute value.** In [21] A. Ostrowski classified the fields which admit an absolute value. However, the property of commutativity played only a minor role in Ostrowski's discussion. We outline below the classification of division rings with absolute value. By combining these results with the results of the preceding section we obtain stronger statements of those results.

DEFINITION. If R is a metric ring such that $\|a+b\| \leq \max(\|a\|, \|b\|)$ for all a and b in R, then R is called a *non-archimedean ring*, and the norm for R is said to be *non-archimedean*. In the contrary case, R is called an *archimedean ring* and the norm of R is said to be *archimedean*.

For any division ring K there is a unique field P, the prime field of K, which is the smallest field contained in K. Then P is either isomorphic to the field of rational numbers, and K is said to have characteristic zero, or P is isomorphic to the field of integers modulo p, where p is a prime number, in which case K is said to have characteristic p. If K is a division ring with absolute value, then the restriction to P of the absolute value of K is an absolute value for P. The classification of the absolute value of K as non-archimedean or archimedean depends only upon its behavior on the prime field of K and, indeed, only upon its behavior on the set of elements of the form ne, where n is a natural number. (If n is a natural number, na denotes the n-fold sum $a+\cdots+a$ (n summands). If n is a negative integer, na is defined as $-[(-n)a]$, while $0a$ denotes 0.) This result, given by Ostrowski in [21], appears in Lemma 13, while a stronger result occurs in Lemma 14.

LEMMA 13. *A division ring K with absolute value is non-archimedean if and only if $\|ne\| \leq 1$ for every natural number n.*

LEMMA 14. *A division ring K with absolute value is non-archimedean if and only if $\|2e\| \leq 1$.*

Note. Lemmas 13 and 14 remain valid if we replace the hypothesis that K is a division ring with absolute value by the hypothesis that K is a commutative metric ring such that $\|a^2\| = \|a\|^2$ for all a in K. Although many metric rings have the property that $\|a^2\| = \|a\|^2$ for all a, the rings of example (10), with n positive, do not have this property. Lemma 14 also holds if 2 is replaced by any other integer greater than 1.

THEOREM 11. *If K is an archimedean division ring with absolute value, and K is complete, then K is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. Furthermore, the norm in K corresponds to the ρth power of the ordinary absolute value, for some ρ such that $0<\rho\leq 1$.*

This theorem and Lemmas 13 and 14 are easily proved. The theorem appears in essence in [21].

COROLLARY. *If K is a complete division ring with absolute value such that $\|2e\|>1$, then K is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. The norm of K corresponds to the ρth power of the ordinary absolute value, for some ρ such that $0<\rho\leq 1$.*

If we note that the completion of an archimedean division ring with absolute value is again an archimedean division ring with absolute value, the theorem implies that any archimedean division ring with absolute value is algebraically and topologically isomorphic to a dense subring of $\mathfrak{R}$, of $\mathfrak{C}$, or of $\mathfrak{Q}$.

The non-archimedean division rings with absolute value constitute a far more varied and extensive class, however. For example, even the locally compact examples are fairly numerous, as may be seen by the list given by Otobe in [22]; all of those examples are of course complete since they are locally compact. We therefore combine the results of § 6 with the preceding results on archimedean division rings for the sake of simplicity.

LEMMA 15. *If K is a non-archimedean division ring with absolute value, then K is totally disconnected.*

COROLLARY 1. *If K is a complete division ring with absolute value, then K is non-archimedean if and only if it is totally disconnected.*

COROLLARY 2. *If K is a complete division ring with absolute value, then K contains a connected set having more than one point if and only if K is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$.*

The field of rational numbers with the ordinary absolute value is archimedean and totally disconnected; this shows the necessity of assuming completeness in Corollary 1. In [7], Dieudonné constructed a connected and locally connected subfield of $\mathfrak{C}$ which is a pure transcendental extension of the field of rational numbers. The field of Dieudonné, with the ordinary absolute value, is then an example of a field which is not complete and which is connected although it is not isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. This shows that Corollary 2 requires the assumption of completeness.

By combining the results just outlined with those of the preceding section, we obtain the results which follow.

THEOREM 12. *Let R be a complete archimedean metric ring such that S is a first category set. If* M1 *or* M2 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$.*

COROLLARY. *Let R be an archimedean metric Q-ring such that S is nowhere dense. If* M1 *or* M2 *holds for R, then R is algebraically and topologically isomorphic to a dense division subring of $\mathfrak{R}$, of $\mathfrak{C}$, or of $\mathfrak{Q}$.*

THEOREM 13. *Let R be a complete metric ring such that H is connected. If* M1 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{C}$, to $\mathfrak{Q}$, or to the field $\mathfrak{F}$ of order* 2 *with the trivial absolute value.*

TEOREM 14. *Let R be a complete metric ring in which H is connected and dense. If one of the conditions* M1–M5 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{C}$ or to $\mathfrak{Q}$.*

THEOREM 15. *Let R be a complete archimedean metric ring which admits a real, bounded involution. If* M1 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$.*

THEOREM 16. *Let R be a complete metric ring such that $R^{(0)}$ is connected. If one of the conditions* M1–M5 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{C}$, to $\mathfrak{Q}$, or to $\mathfrak{F}$.*

THEOREM 17. *Let R be a complete metric ring for which S is of first category and G is connected. If one of the conditions* M1–M5 *holds for R, then R is algebraically and topologically isomorphic to $\mathfrak{C}$ or to $\mathfrak{Q}$.*

COROLLARY. *Let R be a metric Q-ring for which G is connected and S is nowhere dense. If one of the conditions* M1–M5 *holds for R, then R is algebraically and topologically isomorphic to a dense division subring of $\mathfrak{C}$ or of $\mathfrak{Q}$.*

If the requirement of completeness for R in Theorems 13–16 is replaced by the weaker requirement that R be a metric Q-ring, then the conclusion becomes that R is algebraically and topologically isomorphic to a dense division subring of one of the division rings mentioned in the conclusion of that particular theorem. In Theorem 12 and its corollary, and in Theorem 15, the assumption that R is archimedean may be replaced by the assumption that $\|2e\|>1$ or the assumption that R contains a connected set with more than one point.

It is easily seen that completeness is required in these theorems. For, let K be the subfield of $\mathfrak{C}$ constructed in [7] by Dieudonné. Then K is connected and locally connected, K is a dense, proper subfield of $\mathfrak{C}$, and K is a pure transcendental extension of the field of rational numbers. Clearly, K is not isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, to $\mathfrak{Q}$, or to $\mathfrak{F}$. But the set $S=\{0\}$ is nowhere dense in K, while G, H and $K^{(0)}$ coincide and are easily seen to be connected. The identity mapping of K into itself is a real, bounded involution, and M1 holds for K, so that K satisfies all of the hypotheses of these theorems except for completeness Since K does not satisfy the conclusions, completeness is needed.

8. Homogeneous metric rings and rings of quotients. In this section we consider certain types of metric rings which may be embedded in various algebras.

DEFINITION. A metric ring R is said to be *homogeneous* if $\|na\|=|n|\cdot\|a\|$ whenever n is an integer and a is in R. A metric ring R is said to be *weakly homogeneous* if $\|na\|=\|ne\|\cdot\|a\|$ whenever n is an integer and a is in R.

For a homogeneous ring we have $\|ne\|=|n|$, so every homogeneous ring is also weakly homogeneous. However, a weakly homogeneous ring need not be homogeneous; for example, the rings of examples (8) and (9) are weakly homogeneous but are not homogeneous. The rings given in the other examples are all homogeneous. It is clear that a metric ring in which M1 holds must be weakly homogeneous. We can also obtain a sufficient condition for a metric ring to be homogeneous.

LEMMA 16. *If R is a metric ring such that $\|2a\|=2\|a\|$ for every a in R, then R is homogeneous.*

Proof. For any natural number r and for $a\in R$ we have $\|2^r a\|=2^r\|a\|$. Thus, for n a natural number, we have

$$n\|a\|+(2^n-n)\|a\|=2^n\|a\|=\|2^n a\|\leq\|na\|+\|(2^n-n)a\|\leq n\|a\|+(2^n-n)\|a\|,$$

so that $n\|a\|=\|na\|$ for any natural number n and any a in R. It follows easily that $\|na\|=|n|\cdot\|a\|$ for any integer n and any a in R.

If R is any metric ring, and D is a nonempty multiplicative semigroup in R which does not contain 0, which lies in the center of R, and such that $D\subset\mathscr{S}(R)$, then the relation $(a, d)\sim(a', d')$ (if and only if $ad'=a'd$) is an equivalence relation in the set $R\times D$ of ordered pairs

(a, d), where a is in R and d is in D. Let R_D be the set of equivalence classes $[a/d]$ modulo this equivalence relation, with

$$[a/d]+[b/f]=[(af+bd)/df] ,$$

$[a/d]\cdot[b/f]=[ab/df]$, and $||[a/d]||=||a||/||d||$ as the definitions for addition, multiplication and the norm. It is clear that these definitions depend only on the equivalence classes involved and not on the representatives chosen from the classes. It is also easily verified that R_D is a metric ring, and the mapping $x \to [xd/d]$ is an isometry of R into R_D if d is in D. An element d in D may be identified with the element $[d^2/d]$ of R_D which has the inverse $[d/d^2]$ in R_D. We thus obtain the following lemma.

LEMMA 17. *Let R be a metric ring, and D a nonempty multiplicative semigroup in R which does not contain* 0. *Suppose $D \subset \mathscr{L}(R)$ and D is contained in the center of R. Then R_D is a metric ring which is an extension of R such that every element of D has an inverse in R_D.*[11]

COROLLARY. *Let R_1 be a commutative metric ring. Then there is an extension, R, of R_1 such that $\mathscr{L}(R)=\mathscr{G}\cup\{0\}$. In particular, all of the nonzero elements of $\mathscr{L}(R_1)$ have inverses in R.*

Proof. If D is the set of nonzero elements of $\mathscr{L}(R_1)$, then $R=(R_1)_D$ is the required extension of R.

COROLLARY 2. *Let R be a commutative metric ring in which* M1 *holds. Then there is a field, K, with absolute value, such that K is complete and K is an extension of R.*[12]

Proof. If D is the set of nonzero elements of R, then R_D is a field with absolute value. The completion, K, of R_D is the required field.

If K is a field with absolute value, and R is a metric ring which is also an associative linear algebra over K such that $||ka||=||k||\cdot||a||$ for all k in K and a in R, then R is called a *normed algebra* over K. For example, the metric rings of examples (3)–(7) and (10) are normed algebras over $\mathfrak{R}$, while the rings in examples (4), (6) and (7) are normed algebras over $\mathfrak{C}$. It will now be shown that any weakly homogeneous metric ring has an extension which is a normed algebra. Also, for homogeneous metric rings, there is an extension which is a normed algebra over $\mathfrak{R}$.

THEOREM 18. *Let R be a weakly homogeneous metric ring. Then*

[11] Compare the results on algebras of quotients in [24].

[12] Compare the proof of Theorem 2, Corollary 2 in [4], where the technique of embedding in a quotient field is also employed.

there exists an extension of R which is a complete normed algebra over some field K, where either K has the trivial norm, or K is the real field with some power of the ordinary absolute value as its norm, or K is a p-adic field, with some power of the norm given in example (9) *as the norm of K.*

Proof. Let D be the set of nonzero elements of R which have the form ne, with n an integer. Then R_D is an extension of R and contains a subset which is isomorphic to the quotient field, F, for $D \cup \{0\}$. Then R_D is a normed algebra over F, so that the completion of R_D is a normed algebra over the field K, where K is the completion of F; see, for instance, [6]. Thus, R has an extension which is a complete normed algebra over K. If the norm of F is the trivial one, then K coincides with F. In the contrary case, there is a natural number n such that $\|ne\|$ is distinct from 0 and 1. Also, F is a prime field and is therefore isomorphic to the field of rational numbers since the other prime fields are finite and would only admit the trivial absolute value. If $\|ne\| < 1$, we have $\|pe\| < 1$ for some rational prime p. As in Ostrowski's proof, p is unique in that case and the norm is a power of the norm described in example (8), so K is isomorphic to the field of p-adic numbers with the norm taken as some power of the p-adic norm. In case $\|ne\| > 1$ for every natural number n greater than 1, we have the archimedean case, so F is the field of rational numbers with the norm taken as the ρth power of the absolute value, with $0 < \rho \leq 1$. Thus, K consists of the real numbers with the norm given as the ρth power of the absolute value.

COROLLARY. *Let R be a homogeneous metric ring. Then there is an extension of R which is a complete normed algebra over $\mathfrak{R}$.*

Proof. In this case, $\|ne\| = |n| \cdot \|e\| = |n|$ for any integer n, so in the proof of Theorem 18 the norm of an element of F is the usual absolute value. Thus, K is the real field with its usual absolute value.

If K is a complete division ring with absolute value such that $\|2e\| = 2$, then the corollary of Theorem 11 implies that K is algebraically and topologically isomorphic to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$, with the norm corresponding to the ρth power of the ordinary absolute value. K is homogeneous since the condition $\|2e\| = 2$ implies that $\|2a\| = 2\|a\|$ for all a in K. The prime field of K is then the field of rational numbers with the ordinary absolute value as the norm, as the preceding proofs imply. But $\|a\| = |a|^\rho$ for all a in K, while for a rational, $\|a\| = |a|$. Thus, $\rho = 1$, and the following theorem results.

THEOREM 19. *If K is a complete division ring with absolute value* *such that* $||2e||=2$, *then* K *is algebraically isomorphic and isometric to* $\mathfrak{R}$, *to* $\mathfrak{C}$, *or to* $\mathfrak{Q}$.

This result implies that if the hypothesis that $||2e||=2$ is added to Theorems 11–17 and their corollaries the algebraic isomorphism of the conclusions must be an isometry. In a similar vein, Theorem 18 asserts that a weakly homogeneous metric ring R may always be embedded in a complete normed algebra, so a metric ring with absolute value may be embedded in a complete normed algebra; the addition of the strong hypothesis $||2e||=2$ yields a stronger result.

THEOREM 20. *Let* R *be a metric ring with absolute value such that* $||2e||=2$. *Then* R *is algebraically isomorphic and isometric to a subring of* $\mathfrak{Q}$.

Proof. Lemma 16 shows that R is homogeneous, so the corollary of Theorem 18 implies that there is an extension of R which is a complete normed algebra over $\mathfrak{R}$. The construction of this extension R_1 is such that R_1 also has an absolute value. If the real dimension of R_1 as a vector space is greater than one, then $R_1^{(0)}$ is connected, so, by Theorem 16, in the strengthened form just mentioned, R_1 is algebraically isomorphic and isometric to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. If the real dimension of R_1 is one, then R_1 is algebraically isomorphic and isometric to $\mathfrak{R}$. In any event, R is algebraically isomorphic and isometric to a subring of R_1, R_1 is algebraically isomorphic and isometric to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$, each of which is algebraically isomorphic and isometric to a subset of $\mathfrak{Q}$, and the theorem follows.

Note. If r is a fixed integer greater than 1, then the condition $||re||=r$ is equivalent to the condition $||2e||=2$ and may be used as a hypothesis instead of the latter in any of the preceding results.

9. Real and complex normed algebras. The results of the last two sections may now be specialized to the case of normed algebras over $\mathfrak{R}$ or $\mathfrak{C}$. Any normed algebra over $\mathfrak{C}$ may of course be regarded as a normed algebra over $\mathfrak{R}$. A complete normed algebra over $\mathfrak{R}(\mathfrak{C})$ is called a *Banach algebra* (*complex Banach algebra*).

THEOREM 21. *Let* $\mathfrak{A}$ *be a Banach algebra for which one of the conditions* M1–M5 *holds. Then* $\mathfrak{A}$ *is algebraically isomorphic and isometric to* $\mathfrak{R}$, *to* $\mathfrak{C}$, *or to* $\mathfrak{Q}$.

Proof. If $\mathfrak{A}$ has dimension one as a vector space over $\mathfrak{R}$, then $\mathfrak{A}$ is certainly algebraically isomorphic and isometric to $\mathfrak{R}$. If the dimension of $\mathfrak{A}$ is greater than one, then $\mathfrak{A}^{(0)}$ is clearly connected, and the result follows from the strengthened from of Theorem 16 mentioned in the previous section.

COROLLARY 1. *Let* $\mathfrak{A}$ *be a normed algebra over* $\mathfrak{R}$ (*Q-ring which is also a normed algebra over* $\mathfrak{R}$) *such that one of the conditions* M1, M3, M4 *or* M5 (M1–M5) *holds. Then* $\mathfrak{A}$ *is algebraically isomorphic and isometric to* $\mathfrak{R}$, *to* $\mathfrak{C}$, *or to* $\mathfrak{Q}$.

Proof. The completion, $\mathfrak{A}_1$, of $\mathfrak{A}$ is a Banach algebra. Because of the relations among M1–M5, we may assume that M5 holds, and it follows that M5 holds for $\mathfrak{A}_1$. The theorem shows that $\mathfrak{A}_1$ is algebraically isomorphic and isometric to $\mathfrak{R}$, to $\mathfrak{C}$, or to $\mathfrak{Q}$. But $\mathfrak{A}$ is a dense, connected linear subspace of the finitedimensional real vector space $\mathfrak{A}_1$ and therefore coincides with $\mathfrak{A}_1$.

The theorem, with M2 assumed, is essentially the result of Edwards [8; Theorem 1] combined with the first of Mazur's theorems. The corollary, with M1 assumed is the same as Mazur's second theorem in [19].

It may be noted that the corollary does not hold when M2 is assumed and $\mathfrak{A}$ is not a Q-ring. For example, the algebra of all real polynomials $f(x)$ with the norm $\|f\| = \sup|f(x)|$, where the supremum is taken for all x such that $0 \leq x \leq 1$, is a normed algebra over $\mathfrak{R}$ for which G consists only of the constant polynomials distinct from zero; clearly, $\mathscr{G} = G$ for this algebra, so M2 holds, even though this algebra is not even a division ring.

COROLLARY 2. *If* $\mathfrak{A}$ *is a normed algebra over* $\mathfrak{R}$ *which is not isomorphic to* $\mathfrak{R}$, *to* $\mathfrak{C}$, *or to* $\mathfrak{Q}$, *then* $\mathscr{L}(\mathfrak{A})$, $\mathscr{R}(\mathfrak{A})$, $\mathscr{G}$ *and all μ-groups of* $\mathfrak{A}$ *are nowhere dense.*

Proof. The hypothesis implies that M4 and M5 can not hold, so $\mathscr{L}(\mathfrak{A})$, $\mathscr{R}(\mathfrak{A})$ and $\mathscr{G}$ are nowhere dense in $\mathfrak{A}$.

It remains to show that all μ-groups of $\mathfrak{A}$ are nowhere dense. Suppose that A is a μ-group which fails to be nowhere dense. The unit element, j, of the group A is an idempotent, and we have the inclusion $A \subset jA \subset j\mathfrak{A}$, so that $j\mathfrak{A}$ also fails to be nowhere dense. If $\{jx_n\}$ is a sequence of elements of $j\mathfrak{A}$ which converges to an element a in $\mathfrak{A}$, then $\{j^2 \cdot x_n\}$ converges to ja. But since j is an idempotent the sequences $\{jx_n\}$ and $\{j^2 \cdot x_n\}$ coincide, so their limits coincide, whence $a = ja$ is in $j\mathfrak{A}$. This shows that $j\mathfrak{A}$ contains the limit of any convergent sequence of elements of $j\mathfrak{A}$, so $j\mathfrak{A}$ is closed. Because $j\mathfrak{A}$ fails to be nowhere dense it must contain a nonempty open set. But $j\mathfrak{A}$ is a right ideal and therefore, in particular, a topological group relative to addition; the homogeneity of a topological group then implies that $j\mathfrak{A}$ is open. Since

$j\mathfrak{A}$ is open and closed and nonempty in the connected space $\mathfrak{A}$, we see that $j\mathfrak{A}=\mathfrak{A}$. This shows that j has a right inverse, so $j=e$. Now, A is a μ-group which has e as its unit element, so if $a \in A$ then a has an inverse a^{-1} relative to e in A, and $||a||\cdot||a^{-1}||=||aa^{-1}||=||e||=1$, so $a \in \mathscr{G}$. This shows that $A \subset \mathscr{G}$. But it has already been observed that $\mathscr{G}$ is nowhere dense, so the assumption that A fails to be nowhere dense leads to a contradiction. This proves the corollary.

The same proof can be used to show that all μ-groups are nowhere dense in a connected metric ring for which $\mathscr{G}$ is nowhere dense.

In the case of normed algebras over $\mathfrak{C}$, one can also show that the set $\mathscr{G}'$ is generally not too extensive.

THEOREM 22. *If $\mathfrak{A}$ is a normed algebra over $\mathfrak{C}$, then $\mathscr{G}'$ consists exclusively of extreme points of the unit sphere of $\mathfrak{A}$.*

Proof. Suppose that a is an element of $\mathscr{G}'$ which is not an extreme point of the unit sphere. The mapping $x \to xa^{-1}$ is a linear automorphism of the linear space over $\mathfrak{C}$ which underlies $\mathfrak{A}$, and this mapping also preserves distances since a^{-1} belongs to $\mathscr{G}$ and has norm one. Thus, the property of failing to be an extreme point of the unit sphere is preserved, so e, the image of a relative to this mapping, is not an extreme point of the unit sphere.

If $\mathfrak{A}$ were completed, e would also fail to be an extreme point of the unit sphere of the completion, and we therefore assume, without loss of generality, that $\mathfrak{A}$ is complete. Now, e is the midpoint of a segment which lies wholly in the unit sphere of $\mathfrak{A}$, so $e=(b+c)/2$, where $||b||=||c||=1$ and $b\neq c$. Clearly, b and c commute since $c=2e-b$ is in the algebra generated by b and e, so the closed complex normed algebra which is generated by e, b and c is a commutative complex Banach algebra. If $y=(b-c)/2$, then $e-y=c$ and $e+y=b$, whence $||e-y||=||e+y||=1$ in this algebra. But the remark which follows Theorem 1 of [11] asserts that if y is an element of a commutative complex Banach algebra such that $||e-y||=||e+y||=1$, then $y=0$. It follows that $y=0$, so $b=c$. This contradiction shows that a was an extreme point of the unit sphere of $\mathfrak{A}$.

In conclusion, while the results of this paper show that the sets $\mathscr{G}$, $\mathscr{G}'$, $\mathscr{L}(R)$ and $\mathscr{R}(R)$ are usually topologically trivial, they are not algebraically trivial. For, in the case of the algebra $C(X)$ of example (6) where X has at least two points, it is evident that any two points of X may be separated by an element of $\mathscr{G}'$. The Stone-Weierstrass approximation theorem may be used to show that the closed complex subalgebra generated by $\mathscr{G}'$ coincides with $C(X)$.

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ON THE CONSTRUCTION OF R-MODULES AND RINGS WITH POLYNOMIAL MULTIPLICATION

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1. Introduction. Let R be a ring and let R^+ be the additive group of R. If $R^+ = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ is a direct sum of subgroups S_i, then each element of R can be written as an n-tuple $(s_1, s_2, \cdots, s_n)$, $s_i \in S_i$, $i = 1, 2, \cdots, n$, and multiplication in R is given by n mappings

$$f_k : S_1 \times S_2 \times \cdots \times S_n \times S_1 \times S_2 \times \cdots \times S_n \to R^+ , \qquad k = 1, 2, \cdots, n ,$$

where $f_k(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n)$ is the k-th component of the product $(s_1, s_2, \cdots, s_n) \cdot (t_1, t_2, \cdots, t_n)$. The distributive laws in R imply that the mappings f_k are additive in the first n and in the last n arguments. If $S_1, S_2, \cdots, S_n$ are ideals in R, then

$$f_k(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n) = s_k t_k , \qquad k = 1, 2, \cdots, n ,$$

which is a homogeneous quadratic polynomial with integral coefficients in the arguments.

If R is a commutative ring with identity, and if M is a free (left) R-module with basis $e_1, e_2, \cdots, e_n$, then M is an algebra over R if and only if there exist elements $\gamma_{ijk} \in R$ such that multiplication in M is defined by

$$\left(\sum_{i=1}^{n} s_i e_i \right) \cdot \left(\sum_{j=1}^{n} t_j e_j \right) = \sum_{i,j,k=1}^{n} \gamma_{ijk} s_i t_j e_k .$$

The k-th coordinate of the product,

$$f_k(s_1, s_2, \cdots, s_n; t_1, t_2, \cdots, t_n) = \sum_{i,j=1}^{n} \gamma_{ijk} s_i t_j ,$$

is a mapping

$$f_k : \overbrace{R^+ \times R^+ \times \cdots \times R^+}^{2n} \to R^+$$

which is additive in the first n and last n arguments, and which is a homogeneous quadratic polynomial with coefficients in R in the arguments.

These examples suggest the investigation of polynomial mappings with the indicated additive properties, and a discussion of the problem of constructing R-modules and rings which have an additive group which is the direct sum of ideals of a ring R, and for which the multiplication

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is defined by a polynomial mapping.

In § 2 the basic properties of distributive mappings are given. The form of a distributive polynomial mapping is investigated in § 3, and such mappings are characterized in Theorem 2, under the assumption that R is a commutative integral domain. In § 4 and 5 the results of the previous sections are applied to the construction problems mentioned above.

2. Distributive mappings.

Let $S_1, S_2, \cdots, S_k$ be additive semi-groups with identity 0, and let M be an additive abelian group. Let f be a mapping of $S_1 \times S_2 \times \cdots \times S_k$ into M.

DEFINITION. If there exists an integer m, where $1 \leq m \leq k$, such that

(i)
$$f(s_1+s_1', \cdots, s_m+s_m'; s_{m+1}, \cdots, s_k)$$
$$= f(s_1, \cdots, s_m; s_{m+1}, \cdots, s_k) + f(s_1', \cdots, s_m'; s_{m+1}, \cdots, s_k),$$

(ii)
$$f(s_1, \cdots, s_m; s_{m+1}+s_{m+1}', \cdots, s_k+s_k')$$
$$= f(s_1, \cdots, s_m; s_{m+1}, \cdots, s_k) + f(s_1, \cdots, s_m; s_{m+1}', \cdots, s_k'),$$

for all $s_i, s_i' \in S_i$, $i=1, 2, \cdots, k$, the mapping f of $S_1 \times S_2 \times \cdots S_k$ into M is called *m-distributive*.

If $k=m$, only (i) of the definition applies, and the mapping f is a homomorphism of $S_1 \oplus S_2 \oplus \cdots \oplus S_k$ into M. In the examples given in the introduction, $k=2n$, and the mappings are n-distributive.

The following are rather obvious consequences of the definition.

(1) The m-distributive mappings of $S_1 \times S_2 \times \cdots \times S_k$ into M form a subgroup H of the additive abelian group G of all mappings of $S_1 \times S_2 \times \cdots \times S_k$ into M.

If M is a ring, then the set of mappings G is an M-module in the usual way, and the set of m-distributive mappings H is a submodule of G.

(2) The mappings in H satisfy the relation

$$f(s_1, \cdots, s_m; s_{m+1}, \cdots, s_k)$$
$$= \sum_{j=m+1}^{k} \sum_{i=1}^{m} f(0, \cdots, 0, s_i, 0, \cdots, 0; 0, \cdots, 0, s_j, 0, \cdots, 0)$$

for all $s_i \in S_i$, $i=1, 2, \cdots, k$.

Statement (2) is proved by induction from (i) and (ii) of the definition.

(3) The mappings in H satisfy

$$f(s_1, \cdots, s_m; 0, \cdots, 0)=f(0, \cdots, 0; s_{m+1}, \cdots, s_k)=0$$

for all $s_i \in S_i$, $i=1, 2, \cdots, k$.

Statement (3) is a generalization of the fact that the distributive laws in a ring imply $a\cdot 0=0\cdot a=0$.

3. Polynomial functions. Let $S_1, S_2, \cdots, S_k$ be subsemigroups (not necessarily distinct) of the additive group R^+ of a ring R, all of which contain the element 0 of R. Let R^* be any ring containing R, and let

$$f(x_1, x_2, \cdots, x_k)=\sum a_{j_1 j_2 \cdots j_k} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}$$

be a polynomial in $R^*[x_1, x_2, \cdots, x_k]$. Then f defines a mapping of $S_1 \times S_2 \times \cdots \times S_k$ into R^* where

$$f(s_1, s_2, \cdots, s_k)=\sum a_{j_1 j_2 \cdots j_k} s_1^{j_1} s_2^{j_2} \cdots s_k^{j_k}, \qquad s_i \in S_i,\ i=1, 2, \cdots, k .$$

The set S of all such mappings (polynomial functions) is a submodule of the left R^*-module G of all mappings of $S_1 \times S_2 \times \cdots \times S_k$ into R^*. As above, we let H be the set of m-distributive mappings of $S_1 \times S_2 \times \cdots \times S_k$ into R^*, so that H is a submodule of G. Consequently the set of mappings $H \cap S$ is a submodule of G.

THEOREM 1. *Each mapping $f \in H \cap S$ is defined by a polynomial of the form*

$$\text{(A)} \qquad f(x_1, x_2, \cdots, x_k)=\sum_{l=m+1}^{k} \sum_{i=1}^{m} \sum_{\substack{j_i, j_l=1 \\ j_i+j_l \leq t}}^{t-1} a_{j_i j_l}^{(i,l)} x_i^{j_i} x_l^{j_l} .$$

Proof. Let f be defined by a polynomial in $R^*[x_1, x_2, \cdots, x_k]$ of degree t. Since $f \in H$, we have by (2), Section 2

$$\begin{aligned} f(s_1, s_2, \cdots, s_k) &= \sum_{l=m+1}^{k} \sum_{i=1}^{m} f(0, \cdots, 0, s_i, 0, \cdots, 0; 0, \cdots, 0, s_l, 0, \cdots, 0) \\ &= \sum_{l=m+1}^{k} \sum_{l=1}^{m} \sum_{\substack{j_i, j_l=0 \\ j_i+j_l \leq t}}^{t} a_{0, \cdots, 0, j_i, 0, \cdots, 0, j_l, 0, \cdots, 0} s_i^{j_i} s_l^{j_l} , \end{aligned}$$

for all $s_i \in S_i$, $i=1, 2, \cdots, k$. The latter expression can be written

$$\sum_{l=m+1}^{k} \sum_{i=1}^{m} \sum_{\substack{j_i, j_l=1 \\ j_i+j_l \leq t}}^{t-1} a_{0, \cdots, 0, j_i, 0, \cdots, 0, j_l, 0, \cdots, 0} s_i^{j_i} s_l^{j_l}$$

$$+ \sum_{i=1}^{m} \sum_{j_i=1}^{t} a_{0, \cdots, 0, j_i, 0, \cdots, 0}\, s_i^{j_i}$$

$$+\sum_{l=m+1}^{k}\sum_{j_l=1}^{t} a_{0,\cdots,0,j_l,0,\cdots,0}s_l^{j_l}+a_{0,0,\cdots,0}\ .$$

By (3), Section 2,

$$0=f(0,0,\cdots,0)=a_{0,0,\cdots,0}\ ;$$

$$0=f(0,\cdots,0,s_i,0,\cdots,0;0,\cdots,0)=a_{0,0,\cdots,0}+\sum_{j_i=1}^{t}a_{0,\cdots,0,j_i,0,\cdots,0}s_i^{j_i}$$

$$=\sum_{j_i=1}^{t}a_{0,\cdots,0,j_i,0,\cdots,0}s_i^{j_i}$$

for all $s_i\in S_i$, $i=1, 2, \cdots, m$;

$$0=f(0,\cdots,0;0,\cdots,0,s_l,0,\cdots,0)$$

$$=a_{0,0,\cdots,0}+\sum_{j_l=1}^{t}a_{0,\cdots,0,j_l,0,\cdots,0}s_l^{j_l}$$

$$=\sum_{j_l=1}^{t}a_{0,\cdots,0,j_l,0,\cdots,0}s_l^{j_l}$$

for all $s_l\in S_l$; $l=m+1, \cdots, k$. Denoting $a_{0,\cdots,0,j_i,0,\cdots,0,j_l,0,\cdots,0}$ by $a^{(i,l)}_{j_i,j_l}$, we have

$$f(s_1,s_2,\cdots,s_k)=\sum_{l=m+1}^{k}\sum_{i=1}^{m}\sum_{\substack{j_i,j_l=1\\ j_i+j_l\leq t}}^{t-1}a^{(i,l)}_{j_i,j_l}s_i^{j_i}s_l^{j_l}$$

for all $s_i\in S_i$, $i=1, 2, \cdots, k$, which completes the proof.

The following examples show that for an arbitrary ring R, the converse of Theorem 1 does not hold, and that Theorem 1 is the best possible theorem in the sense that there exist rings for which every polynomial function defined by a polynomial of form (A) is m-distributive.

EXAMPLE 1. Let $R=I$, the ring of ordinary integers, let $R^*=R$, and let $S_1=S_2=R^+$. Let $f\colon S_1\times S_2\rightarrow R$ be defined by $f(x_1,x_2)=x_1^2x_2$. Then f is defined by a polynomial of form (A) with $m=1$. However $f\notin H$ for $f(1+1;1)=f(2,1)=4$, and $f(1;1)+f(1;1)=1+1=2$.

EXAMPLE 2. Let R be the ring with additive group $R^+=\{u\}$, the cyclic group of order 9, and with multiplication defined by $(iu)\cdot(ju)=3iju$. Then R is a commutative ring [2] such that $R^3=0$, $R^2\neq 0$.

Let f be any mapping of $S_1\times S_2\times\cdots\times S_k$ into an extension R^* of R, where $S_1, S_2, \cdots, S_k$ are any subsemigroups of R^+ containing 0, such that f is defined by a polynomial of form (A). Then

$$f(s_1, s_2, \cdots, s_k)=\sum_{l=m+1}^{k}\sum_{i=1}^{m}\sum_{\substack{j_i, j_l=1\\ j_i+j_l\leqq t}}^{t-1} a^{(i,l)}_{j_i,j_l} s_i^{j_i} s_l^{j_l}$$

$$=\sum_{j=m+1}^{k}\sum_{i=1}^{m} a^{(i,l)}_{1,1} s_i s_l ,$$

since $R^3=0$. It is evident that f is m-distributive, that is, $f\in H\cap S$.

In the sequel we will be concerned with m-distributive polynomial mappings of $S_1\times S_2\times\cdots\times S_k$ into R. Since a polynomial with coefficients in an extension R^* of R may have its values in R, we obtain a larger class of mappings by allowing the coefficients of $f(x_1, x_2, \cdots, x_k)$ to be in $R^*\supseteqq R$. For example, polynomials with (ordinary) integral coefficients have values in R, and if R does not have an identity, we may consider the coefficients to be in an extension R^* of R. Moreover it is a consequence of the theorem that if R is an ideal in R^*, then f has values in R.

The following lemma is well known (see for example [6, pp. 65–66]), but is given here in the form in which it is most useful for our purposes.

LEMMA. *Let*

$$f=\sum a_{j_1, j_2, \cdots, j_k} x_1^{j_1} x_2^{j_2}\cdots x_k^{j_k}\in R^*[x_1, x_2, \cdots, x_k]$$

where R^* is a commutative integral domain, and let f be of degree m_i in x_i, $i=1, 2, \cdots, k$. Let $(s_i^{(1)}, s_i^{(2)}, \cdots, s_i^{(n_i)})$ be a set of distinct elements of R^* where $n_i>m_i$, $i=1, 2, \cdots, k$, such that $f(s_1^{(l_1)}, s_2^{(l_2)}, \cdots, s_k^{(l_k)})=0$ for $l_i=1, 2, \cdots, n_i$, $i=1, 2, \cdots, k$. Then $f=0\in R^*[x_1, x_2, \cdots, x_k]$.

THEOREM 2. *Let R^* be a commutative integral domain, let R be a subring of R^*, and let $S_1, S_2, \cdots, S_k$ be non-zero ideals in R. A mapping f from $S_1\times S_2\times\cdots\times S_k$ into R^* is in $H\cap S$ if and only if f is defined by a polynomial of the form*

$$\text{(B)}\qquad f(x_1, x_2, \cdots, x_k)=\sum_{l=m+1}^{k}\sum_{i=1}^{m}\sum_{s_i, s_l=0}^{r} a^{(i,l)}_{p^{s_i}, p^{s_l}} x_i^{p^{s_i}} x_l^{p^{s_l}}$$

when R has characteristic $p>0$, and by

$$\text{(C)}\qquad f(x_1, x_2, \cdots, x_k)=\sum_{l=m+1}^{k}\sum_{i=1}^{m} a_{il} x_i x_l$$

when R has characteristic zero.

Proof. Let f be defined by a polynomial of form (B) when R has characteristic $p>0$. Then

$$f(s_1+s_1', \cdots, s_m+s_m'; s_{m+1}, \cdots, s_k)$$

$$= \sum_{l=m+1}^{k} \sum_{i=1}^{m} \sum_{s_i, s_l=0}^{r} a^{(i,l)}_{p^{s_i}, p^{s_l}} (s_i+s_i')^{p^{s_i}} s_l^{p^{s_l}}$$

$$= \sum_{l=m+1}^{k} \sum_{i=1}^{m} \sum_{s_i, s_l=0}^{r} a^{(i,l)}_{p^{s_i}, p^{s_l}} (s_i^{p^{s_i}} + s_i'^{p^{s_i}}) s_l^{p^{s_l}}$$

$$= f(s_1, \cdots, s_m; s_{m+1}, \cdots, s_k) + f(s_1', \cdots, s_m'; s_{m+1}, \cdots, s_k) ,$$

so that f satisfies (i) of the definition for m-distributiveness. Similarly (ii) is satisfied, so that $f \in H \cap S$.

It is immediate that a mapping f defined by a polynomial of form (C) is m-distributive.

Conversely, we divide the proof into three parts.

1. *R is infinite and has characteristic $p > 0$.*

If $f \in H \cap S$, then f is defined by a polynomial of form (A) by Theorem 1. Then we have for each i $(1 \leqq i \leqq m)$ and for each l $(m < l \leqq k)$,

$$f(0+0, \cdots, s_i+s_i', \cdots, 0+0; 0, \cdots, s_l, \cdots, 0)$$

$$= \sum_{\substack{j_i, j_l=1 \\ j_i+j_l \leqq t}}^{t-1} a^{(i,l)}_{j_i, j_l} (s_i+s_i')^{j_i} s_l^{j_l}$$

$$= f(0, \cdots, s_i, \cdots, 0; 0, \cdots, s_l, \cdots, 0)$$

$$+ f(0, \cdots, s_i', \cdots, 0; 0, \cdots, s_l, \cdots, 0)$$

$$= \sum a^{(i,l)}_{j_i, j_l} s_i^{j_i} s_l^{j_l} + \sum a^{(i,l)}_{j_i, j_l} s_i'^{j_i} s_l^{j_l} ,$$

for all $s_i, s_i' \in S_i$, $s_l \in S_l$. Therefore we have the identity

$$\sum_{j_i=2, j_l=1}^{t-1} a^{(i,l)}_{j_i, j_l} \left[j_i s_i^{j_i-1} s_i' + \frac{j_i(j_i-1)}{2!} s_i^{j_i-2} s_i'^2 + \cdots + \frac{j_i(j_i-1)}{2!} s_i^2 s_i'^{j_i-2} + j_i s_i s_i'^{j_i-1} \right] s_l^{j_l} = 0 . \tag{3.1}$$

Since R is an infinite integral domain, each ideal $S_i \neq 0$ is infinite. Therefore the polynomial in $R^*[x, y, z]$ which has the same coefficients as the above expression, vanishes for infinitely many values of each argument x, y, z in R^*. By the lemma, each coefficient is zero. Now the coefficient of $x^{j_i-r} y^r z^{j_l}$ $(0 < r < j_i;\ 1 < j_i < t;\ 0 < j_l < t)$ is $\binom{j_i}{r} a^{(i,l)}_{j_i, j_l} = 0$. If j_i is not a power of p, then at least one of the binomial coefficients $\binom{j_i}{r}$, $r = 1, 2, \cdots, j_i - 1$, is prime to p. Since R, and consequently R^*, has characteristic p, this implies that $a^{(i,l)}_{j_i, j_l} = 0$, for j_i and j_l in the stipulated ranges, whenever j_i is not a power of p.

Using (ii) of the definition of an m-distributive mapping, a similar argument shows that $a^{(i,l)}_{j_i,j_l}=0$ for $j_i=1, 2, \cdots, t-1$; $j_l=2, 3, \cdots, t-1$ whenever j_l is not a power of p.

Since the above argument holds for each i and each l, the polynomial of form (A) which defines f has all coefficients zero except for coefficients $a^{(i,l)}_{p^{s_i},p^{s_l}}$, $s_i=0, 1, 2, \cdots$, $s_l=0, 1, 2, \cdots$. Thus f is defined by a polynomial of form (B).

2. *R is finite and has characteristic $p>0$.*

Since R is a commutative integral domain, R is a finite field $GF(p^n)$ and each ideal $S_i\neq 0$ in R is R itself. Since $s^{p^n}=s$ for all $s\in R$, each polynomial function of $S_1\times S_2\times\cdots\times S_k$ into R^* is defined by a polynomial of form (A) of degree at most p^{n-1} in each argument. Since the degree in each argument is less than the number of elements in each $S_i=R$, the lemma can be applied to the identity 3.1, and the proof of 1. is valid in this case also.

3. *R has characteristic zero.*

Since R and each ideal $S_i\neq 0$ in R have infinitely many elements, the proof of 1. can be followed to obtain

$$\binom{j_i}{r}a^{(i,l)}_{j_i,j_l}=0 \qquad \text{and} \qquad \binom{j_l}{r}a^{(i,l)}_{j_i,j_l}=0\ ,$$

for j_i, j_l, and r in the ranges previously stipulated. Since R, and consequently R^*, has characteristic zero, this implies that $a^{(i,l)}_{j_i,j_l}=0$ except for $j_i=j_l=1$. Consequently f is defined by a polynomial of form (C).

The following result was obtained in the proof of the theorem.

COROLLARY. *Let $R=GF(p^n)$ and R^* be a commutative integral domain containing R. A mapping f of*

$$\overbrace{R\times R\times\cdots\times R}^{k\text{ terms}}$$

into R^* is in $H\cap S$ if and only if f is defined by a polynomial of form (B) *with $r=n-1$.*

4. Application to the construction of R-modules. Let $S\neq 0$ be an ideal in a ring R. The set of $(k-1)$-tuples $V=\{(s_2, s_3, \cdots, s_k), s_i\in S\}$ with equality, addition and left scalar multiplication defined componentwise is a left R-module. The group of the module is the direct sum

$$\overbrace{S^+\oplus S^+\oplus\cdots\oplus S^+}^{k-1\text{ terms}}.$$

For $r \in R$, $s_i \in S$, the i-th component rs_i of the scalar product $r(s_2, s_3, \cdots, s_k)$ is a 1-distributive polynomial function f of the arguments $r; s_2\, s_3, \cdots, s_k$. In this section we characterize the most general polynomial function f for which $V = S^+ \oplus S^+ \oplus \cdots \oplus S^+$ is an R-module, where R is a commutative integral domain with characteristic zero.

Now V is a left R-module if and only if there exists a mapping f from $R \times V$ into V which satisfies the module identities

$$(\mathrm{M}_1) \qquad f(r_1, v_1 + v_2) = f(r_1, v_1) + f(r_1, v_2)\ ,$$

$$(\mathrm{M}_2) \qquad f(r_1 + r_2, v_1) = f(r_1, v_1) + f(r_2, v_1)\ ,$$

$$(\mathrm{M}_3) \qquad f(r_1 r_2, v_1) = f(r_1, f(r_2, v_1))\ ,$$

for every $r_1, r_2 \in R$ and every $v_1, v_2 \in V$. Denoting the components of $f(r, v) = f(r; s_2, \cdots, s_k)$ by $f_i(r; s_2, \cdots, s_k)$, $i = 2, 3, \cdots, k$, we observe that f is given by a set of $k-1$ mappings f_i from

$$\overbrace{R \times S \times S \times \cdots \times S}^{k \text{ terms}}$$

into $S \subseteqq R$. Setting $R = S_1$, $S = S_2, \cdots, S = S_k$ to agree with the notation of the preceding sections, the identities (M_1) and (M_2) are just the conditions (i) and (ii) that each mapping f_i be 1-distributive. Interpreting M_3 for the components f_i we have

$$(4.1) \quad f_i(r_1 r_2; s_2, \cdots, s_k) = f_i(r_1; f_2(r_2; s_2, \cdots, s_k), \cdots, f_k(r_2; s_2, \cdots, s_k))$$

for every $r_1, r_2 \in R$ and every $s_i \in S$; $i = 2, 3, \cdots, k$.

We now assume that R^* is an ideal-preserving extension of R, that is, R^* is a ring containing R with the property that if S is an ideal in R, then S is an ideal in R^*. For example, there exists a ring with identity containing R which is an ideal-preserving extension of R. Let f_i, $i = 2, 3, \cdots, k$, be a mapping from $R \times V$ into R^* defined by a polynomial

$$(4.2) \qquad f_i(x_1; x_2, \cdots, x_k) = \sum a_{j_1 j_2 \cdots j_k} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}$$

with coefficients in R^*. Denote the system consisting of the group V and the mappings f_i defined by (4.2) by (V, f_i). We obtain the following application of Theorem 2.

THEOREM 3. *Let R^* be a commutative integral domain with characteristic zero which is an ideal-preserving extension of R. Then (V, f_i) is a left R-module with scalar multiplication defined by $r \cdot (s_2, s_3, \cdots, s_k) = (f_2, f_3, \cdots, f_k)$ if and only if each f_i is defined by a polynomial of the form*

$$f_i(x_1;\, x_2,\, \cdots,\, x_k)=\sum_{l=2}^{k} a_l^{(i)} x_1 x_l\ , \qquad a_l^{(i)}\in R^*\ , \tag{4.3}$$

such that the matrix $A=(a_l^{(i)})$ *is idempotent; that is* $r\cdot(s_2, s_3, \cdots, s_k)=r(s_2, s_3, \cdots, s_k)A'$, *where the right member is an ordinary matrix product in which* A' *is the transpose of the matrix* A.

Proof. If (V, f_i) is a left R-module, then by the foregoing discussion, the mappings f_i are 1-distributive polynomial mappings with values in $S\subseteqq R^*$. By Theorem 2, with $S_1=R$, $S_2=S_3=\cdots=S_k=S$, and $m=1$, each f_i is defined by a polynomial of form (C)

$$f_i(x_1;\, x_2,\, \cdots,\, x_k)=\sum_{l=2}^{k} a_{1l}^{(i)} x_1 x_l=\sum_{l=2}^{k} a_l^{(i)} x_1 x_l\ .$$

Since each f_i must satisfy the identity (4.1) we have

$$\sum_{l=2}^{k} a_l^{(i)}(r_1 r_2)s_l=\sum_{l=2}^{k} a_l^{(i)} r_1\Big[\sum_{j=2}^{k} a_j^{(l)} r_2 s_j\Big]$$
$$=\sum_{l=2}^{k}\sum_{j=2}^{k} a_j^{(i)} a_l^{(j)} r_1 r_2 s_l\ ,$$

for every $r_1, r_2\in R$ and every $s_l\in S$. This implies $a_l^{(i)}=\sum_{j=2}^{k} a_j^{(i)} a_l^{(j)}$ or that the matrix $A=(a_l^{(i)})$ is idempotent. Since

$$f_i(r;\, s_2,\, \cdots,\, s_k)=\sum_{l=2}^{k} a_l^{(i)} rs=r\sum_{l=2}^{k} a_l^{(i)} s_l\ ,$$

we have $r\cdot(s_2, \cdots, s_k)=r(s_2, \cdots, s_k)A'$ where the right member is an ordinary matrix product.

Conversely, it is readily observed that if f_i is defined by (4.3) with $A=(a_l^{(i)})$ idempotent, then f_i has values in S since S is an ideal in R^*, f_i is 1-distributive, and f_i satisfies (4.1). Therefore (V, f_i) is a left R-module.

If we specialize to the case where $R=F$ is a field, we have $S_2=S_3=\cdots=S_k=F$ and $R^*=F$, so that (V, f_i) is the group of $(k-1)$-tuples with elements in F for which scalar multiplication is defined by (4.2). Theorem 3 characterizes the (V, f_i) which are F-modules, and we let (V, A) denote the F-module (V, f_i) with scalar multiplication defined by (4.3) where $A=(a_l^{(i)})$ is idempotent. Let $E_m=\begin{pmatrix} I_m & 0\\ 0 & 0\end{pmatrix}$, where $0\leqq m\leqq k-1$. The following theorem completely classifies the F-modules (V, f_i).

THEOREM 4. *The left* F*-module* (V, A) *is* F*-isomorphic to the* F*-module* (V, E_m) *for some* m, $0\leqq m\leqq k-1$. *Moreover* (V, E_m) *is not* F*-isomorphic to* (V, E_n) *if* $m\neq n$.

Proof. If A is similar to B, then (V, A) is F-isomorphic to (V, B). For in (V, A),

$$r\cdot(s_2, s_3, \cdots, s_k)=r(s_2, s_3, \cdots, s_k)A' ,$$

and in (V, B),

$$r\cdot(s_2, s_3, \cdots, s_k)=r(s_2, s_3, \cdots, s_k)B'=r(s_2, s_3, \cdots, s_k)PA'P^{-1}$$

for some non-singular matrix P. The mapping φ defined by

$$\varphi[(s_2, s_3, \cdots, s_k)]=(s_2, s_3, \cdots, s_k)P^{-1}$$

is an F-isomorphism.

Since A is idempotent, A is similar to E_m for some m, $0\leq m\leq k-1$ [1, p. 88], which completes the proof of the first part of the theorem.

In (V, E_m),

$$r\cdot(s_2, s_3, \cdots, s_k)=(rs_2, rs_3, \cdots, rs_{m+1}, 0, \cdots, 0) ,$$

so that the submodule $1\cdot(V, E_m)=(s_2, s_3, \cdots, s_{m+1}, 0, \cdots, 0)$ is the vector space over F of dimension m. Any F-isomorphism of (V, E_m) onto (V, E_n) induces an F-isomorphism of $1\cdot(V, E_m)$ onto $1\cdot(V, E_n)$, but if $m\neq n$ these submodules cannot be F-isomorphic since they are vector spaces of different dimensions over F.

COROLLARY. *The F-modules (V, A) and (V, B) are F-isomorphic if and only if A and B have the same rank.*

In the above discussion, the (V, f_i) were all $(k-1)$-tuples for a fixed k. We now consider (V_k, f_i) and (V_l, f_i), $k\neq l$. By Theorem 4, it is sufficient to consider (V_k, E_m), $0\leq m\leq k-1$ and (V_l, E_n), $0\leq n\leq l-1$.

THEOREM 5. *The F-modules (V_k, E_m) and (V_l, E_n) are F-isomorphic if and only if $m=n$ and either $k=l$ or F^+ has infinite rank.*[1]

Proof. Suppose first that φ is an F-isomorphism of (V_k, E_m) onto (V_l, E_n). Then as in Theorem 4, $1\cdot(V_k, E_m)$ and $1\cdot(V_l, E_n)$ are F-isomorphic vector spaces of dimension m and n respectively over F. Hence $m=n$. Assume that $k\neq l$, and let M and N be the submodules of (V_k, E_m) and (V_l, E_m) respectively which are annihilated by $1\in F$. Then φ induces an isomorphism of M onto N as additive groups.

$$M=\{(0, \cdots, 0, s_{m+1}, \cdots, s_{k-1}), s_i\in F\}=\overbrace{F^+\oplus\cdots\oplus F^+}^{k-1-m}$$

[1] The additive group F^+ of a field F of characteristic 0 is a divisible torsion-free group and therefore is the direct sum of α copies of the additive group of rational numbers. The cardinal number α, which is an invariant, is called the rank of F^+ [4, pp. 10–11].

and

$$N=\{(0, \cdots, 0, s_{m+1}, \cdots, s_{l-1}), s_i \in F\}=\overbrace{F^+ \oplus \cdots \oplus F^+}^{l-1-m} .$$

If F^+ has finite rank, then M and N have different rank, and are not isomorphic. Hence F^+ has infinite rank.

Conversely, if $m=n$ and $k=l$, there is nothing to prove. Suppose, then, that $m=n$ and that F^+ has infinite rank. Now $(V_k, E_m)=1\cdot(V_k, E_m)\oplus M$ and $(V_l, E_m)=1\cdot(V_l, E_m)\oplus N$, where M and N each have the decomposition into a direct sum of copies of F^+ given above. Since F^+ has infinite rank, M and N have the same rank and are isomorphic as additive groups. But since F annihilates M and N, this isomorphism is an F-isomorphism. Finally, $1\cdot(V_k, E_m)$ is F-isomorphic to $1\cdot(V_l, E_m)$ since they are vector spaces of the same dimension.

5. Application to the construction of rings. As in the previous section, we let $S\neq 0$ be an ideal in a ring R and consider the set of n-tuples $V=\{(s_1, s_2, \cdots, s_n), s_i \in S\}$ with equality and addition defined componentwise. Now V is a ring if and only if there exists a mapping f from $V\times V$ into V which satisfies

$$(R_1) \qquad f(v_1+v_2, v_3)=f(v_1, v_3)+f(v_2, v_3)$$

$$(R_2) \qquad f(v_1, v_2+v_3)=f(v_1, v_2)+f(v_1, v_3)$$

$$(R_3) \qquad f(f(v_1, v_2), v_3)=f(v_1, f(v_2, v_3))$$

for every $v_1, v_2, v_3 \in V$.

Denoting the components of $f(v_1, v_2)=f(s_1, \cdots, s_n; t_1, \cdots, t_n)$ by $f_i(s_1, \cdots, s_n; t_1, \cdots, t_n)$, $i=1, 2, \cdots, n$, f is given by a set of n mappings f_i from

$$\overbrace{S\times S\times \cdots \times S}^{2n \text{ terms}}$$

into $S\subseteq R$. The identities R_1 and R_2 are just the conditions (i) and (ii) that each mapping f_i be n-distributive. In this application, $k=2n$, and $S_i=S$, $i=1, 2, \cdots, k$ in the notation of § 2. Interpreting R_3, the associative law, for the components f_i, we obtain

$$(5.1) \quad f_i(f_1(s_1, \cdots, s_n; t_1, \cdots, t_n), \cdots, f_n(s_1, \cdots, s_n; t_1, \cdots, t_n); u_1, \cdots, u_n)$$
$$=f_i(s_1, \cdots, s_n; f_1(t_1, \cdots, t_n; u_1, \cdots, u_n), \cdots, f_n(t_1, \cdots, t_n; u_1, \cdots, u_n))$$

for every $s_i, t_j, u_k \in S$.

We assume that R^* is an ideal-preserving extension of R and that each f_i, $i=1, 2, \cdots, n$ is defined by a polynomial

(5.2) $$f_i(x_1, \cdots, x_n; y_1, \cdots, y_n) = \sum a_{j_1 \cdots j_n k_1 \cdots k_n} x_1^{j_1} \cdots x_n^{j_n} y_1^{k_1} \cdots y_n^{k_n}$$

with coefficients in R^*. Denote the system consisting of the group V and the mappings f_i defined by (5.2) by (V, f_i, n). We obtain the following application of Theorem 2.

THEOREM 6. *Let R^* be a commutative integral domain which is an ideal preserving extension of R. Then (V, f_i, n) is a ring with multiplication defined by $(s_1, \cdots, s_n)\cdot(t_1, \cdots, t_n) = (f_1, \cdots, f_n)$ if and only if each f_i, $i=1, 2, \cdots, n$ satisfies* (5.1) *and is defined by a polynomial of the form*

(5.3) $$f_i(x_1, \cdots, x_n; y_1, \cdots, y_n) = \sum_{l=1}^{n} \sum_{j=1}^{n} \sum_{s_j, s_l = 0}^{r} {}^{(i)}a_{s_j, s_l}^{(j,l)} x_j^{p^{s_j}} y_l^{p^{s_l}} ,$$

or

(5.4) $$f_i(x_1, \cdots, x_n; y_1, \cdots, y_n) = \sum_{l=1}^{n} \sum_{j=1}^{n} a_{jl}^{(i)} x_j y_l ,$$

according as R has characteristic $p > 0$ or 0.

Proof. If (V, f_i, n) is a ring, then we have observed above that the mappings f_i are n-distributive mappings with values in $S \subseteqq R^*$. Since the f_i are polynomial mappings into R^*, it follows from Theorem 2, that they are defined by polynomials of form (B) or (C) according as the characteristic of R is $p > 0$ or 0. We have seen that the associative law implies (5.1).

Conversely, if multiplication in (V, f_i, n) is defined by $(s_1, \cdots, s_n)\cdot(t_1, \cdots, t_n) = (f_1, \cdots, f_n)$, where each f_i is defined by (5.3) or (5.4) according as the characteristic of R is $p > 0$ or 0, then by Theorem 2, each f_i is n-distributive. Thus, multiplication in (V, f_i, n) is distributive with respect to addition. Since each f_i satisfies (5.1), multiplication is associative, and (V, f_i, n) is a ring.

EXAMPLE 3. Let R be a field F with characteristic zero. Then $R^* = F$, $S = F$, and $(V, f, 1)$ is the group F^+ and the mapping f defined by $f(x; y) = \sum a_{jk} x^j y^k$, $a_{jk} \in F$. By Theorem 6, $(V, f, 1)$ is a ring with multiplication defined by $s \cdot t = \sum a_{jk} s^j t^k$ only if f is defined by $f(x; y) = axy$, $a \in F$. If $a \neq 0$, $(V, f, 1)$ is isomorphic to F under the correspondence $sa^{-1} \leftrightarrow s$, so that we can conclude that the only non-trivial rings with additive group F^+ and with multiplication defined by a polynomial function of $F \times F$ into F are fields isomorphic to F [3, p. 177].

EXAMPLE 4. Let R be the finite field $GF(3^2)$. Then $R^* = GF(3^2)$,

$S=GF(3^2)$, and $(V, f, 1)$ is a ring only if multiplication is defined by (see the Corollary to Theorem 2).

$$s \cdot t = f(s;\ t) = a_{00}st + a_{01}st^3 + a_{10}s^3t + a_{11}s^3t^3\ , \qquad a_{ij} \in GF(3^2)\ .$$

Selecting $a_{00}=a_{10}=1$, $a_{01}=a_{11}=0$, $f(s;\ t)=st+s^3t$, and $f(s;\ t)$ satisfies (5.1). Hence $(V, f, 1)$ is a ring. Let ξ be the primitive eighth root of unity which generates the multiplicative group of $GF(3^2)$. Then $\xi^2 \cdot 1 = f(\xi^2;\ 1) = \xi^2 + \xi^6 = \xi^2(1+\xi^4) = 0$. Hence $(V, f, 1)$ has zero divisors, and in this case we have an example of a non-trivial ring with additive group $GF(3^2)^+$ and with polynomial multiplication which is not isomorphic to $GF(3^2)$.

It should be remarked in conclusion, that when R has characteristic zero and (V, f_i, n) is a ring, the multiplication rule (5.4) is the same as that for an algebra over R^* (see Introduction); and if R^* has an identity, (V, f_i, n) can be regarded as a subalgebra of an ordinary algebra of dimension n over R^*. Hence the coefficients $a_{jl}^{(i)}$ of the polynomials f_i play the same role as the multiplication constants of an algebra, and the associative law (5.1) can be interpreted as a matrix identity [**5**, p. 294].

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AN ULTRASPHERICAL GENERATING FUNCTION

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1. **Introduction.** Let $P_n^{(\alpha,\alpha)}(v)$ denote ultraspherical polynomials and let

$$(1)\qquad \begin{aligned} w&=2(v-t)(1-2vt+t^2)^{-1/2}\,,\\ g&=1-2vt+t^2\,,\\ y&=-tu(1-2vt+t^2)^{-1/2}\,,\\ r&=(1-2yw+y^2)^{1/2}\,, \end{aligned}$$

with the roots to be those assuming the value 1 for $t=0$. Then this note will prove that

$$(2)\qquad g^{-\alpha-1/2}\,{}_2F_1\left[\begin{matrix}c,\ 1+2\alpha-c; & \dfrac{1-y-r}{2}\\ 1+\alpha\quad ; & \end{matrix}\right]{}_2F_1\left[\begin{matrix}c,\ 1+2\alpha-c; & \dfrac{1+y-r}{2}\\ 1+\alpha\quad ; & \end{matrix}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_3F_2\left[\begin{matrix}-n,\ c,\ 1+2\alpha-c; & u\\ 1+\alpha,\ 1+2\alpha\quad ; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,,$$

valid for t sufficiently small. In (2), c is an arbitrary parameter. Equation (2) is a direct generalization of Rice's result given in [8, equ. 2.14], to which it reduces for $\alpha=0$. (A different generalization of Rice's result is given in [3].) For c the non-positive integer $-k$, the left side of (2) reduces to a product of ultraspherical polynomials:

$$(3)\qquad g^{-\alpha-1/2}\frac{k!k!}{(1+\alpha)_k(1+\alpha)_k}P_k^{(\alpha,\alpha)}(r+y)P_k^{(\alpha,\alpha)}(r-y)$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_3F_2\left[\begin{matrix}-n,\ -k,\ 1+2\alpha+k; & u\\ 1+\alpha,\ 1+2\alpha\quad ; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

In addition, this note will show other results on ultraspherical polynomials. Further, it will provide a new way of deriving some results of Weisner. These will be shown later.

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2. **A preliminary result.** It will be established in this section that

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$$(4)\quad \sum_{n=0}^{\infty} \frac{(b)_n}{n!} {}_2F_1\left[\begin{matrix} -n, a; \\ b \quad ; \end{matrix} x\right] {}_{p+1}F_q\left[\begin{matrix} -n, c_1, c_2, \cdots, c_p; \\ d_1, d_2, \cdots, d_q \quad ; \end{matrix} u\right] t^n$$

$$= (1-t)^{a-b}(1-t+xt)^{-a} \sum_{n=0}^{\infty} \frac{(c_1)_n \cdots (c_p)_n (-tu)^n (b)_n}{(d_1)_n \cdots (d_q)_n (1-t)^n n!} {}_2F_1\left[\begin{matrix} -n, a; \\ b \quad ; \end{matrix} \frac{x}{xt+1-t}\right],$$

for

$$|t| < 1, \ |tu/(1-t)| < 1, \ xt+1-t \neq 0, \ p \leq q .$$

Start with

$$(5)\quad (1-t)^{a-b-k}(xt+1-t)^{-a} {}_2F_1\left[\begin{matrix} -k, a; \\ b \quad ; \end{matrix} \frac{x}{1-t+xt}\right]$$

$$= (1-t)^{-b-k}(1-x)^{-a} {}_2F_1\left[\begin{matrix} b+k, a; \\ b \quad ; \end{matrix} \frac{x}{(x-1)(1-t)}\right]$$

$$= (1-x)^{-a} \sum_{n=0}^{\infty} \frac{(b+k)_n t^n}{n!} {}_2F_1\left[\begin{matrix} b+n+k, a; \\ b \quad ; \end{matrix} \frac{x}{x-1}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(b+k)_n t^n}{n!} {}_2F_1\left[\begin{matrix} -n-k, a; \\ b \quad ; \end{matrix} x\right].$$

Multiply the first and last lines of (5) by

$$(6)\quad \frac{(b)_k (c_1)_k (c_2)_k \cdots (c_p)_k (-tu)^k}{(d_1)_k (d_2)_k \cdots (d_q)_k k!}$$

and sum on k from 0 to ∞. A shift of indices will then give equation (4). The restrictions given insure the absolute convergence of the various series which are multiplied together.

It should be here noted that (4) includes two results by Weisner as special cases. See [7, equ's. 4.3 and 4.6]. The first follows from (4) by taking

$$(7)\quad p=1, \ q=1, \ c_1=d, \ d_1=b$$

and summing the result by Chaundy's equation 25 in [4].

The second Weisner result follows from (4) by taking

$$(8)\quad p=0, \ q=1, \ d_1=b,$$

and summing by the formula of Rainville as quoted in [5, p. 267, equ. 25].

3. Proof of (2). The use of a quadratic transformation [6, p. 9] on a standard form of the ultraspherical polynomials converts them into

$$(9)\qquad P_n^{(\alpha,\alpha)}(x)=\frac{(1+\alpha)_n}{n!}z^{-n}{}_2F_1\left[\begin{matrix}-n,\ \alpha+1/2;\\ 2\alpha+1\quad;\end{matrix}\ 1-z^2\right]$$

with $2x=z+1/z$. This is equivalent to a formula by Weisner [7, p. 1038]. Let

$$(10)\qquad v=\frac{1}{2}(2-x)(1-x)^{-1/2},\quad a=\alpha+1/2,\quad b=2\alpha+1,$$

replace t by $t(1-x)^{-1/2}$ in (4), and let w, g, y, r be defined by (1). Then (4) becomes

$$(11)\qquad \sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_{p+1}F_q\left[\begin{matrix}-n,\ c_1,\ \cdots,\ c_p;\\ d_1,\ \cdots,\ d_q\quad;\end{matrix}\ u\right]P_n^{(\alpha,\alpha)}(v)t^n$$

$$=g^{-\alpha-1/2}\sum_{n=0}^{\infty}\frac{(c_1)_n\cdots(c_p)_n}{(d_1)_n\cdots(d_q)_n}y^n\frac{(1+2\alpha)_n}{(1+\alpha)_n}P_n^{(\alpha,\alpha)}(w).$$

In (11), take

$$(12)\qquad p=2,\ q=2,\ d_1=1+\alpha,\ d_2=1+2\alpha,\ c_1=c,\ c_2=1+2\alpha-c$$

and apply the formula given in [2, equ. 17]. Result (2) above follows immediately.

For an additional result from (11), take

$$(13)\qquad p=0,\quad q=2,\quad d_1=1+\alpha,\quad d_2=1+2\alpha,$$

and use the result from Bateman [1], that

$$(14)\qquad {}_0F_1\left(-;1+\alpha;\ \frac{y(w-1)}{2}\right){}_0F_1\left(-;1+\alpha;\ \frac{y(w+1)}{2}\right)$$

$$=\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\alpha)}(w)y^n}{(1+\alpha)_n(1+\alpha)_n}.$$

This gives

$$(15)\qquad g^{-\alpha-1/2}{}_0F_1\left(-;1+\alpha;\ \frac{y(w-1)}{2}\right){}_0F_1\left(-;1+\alpha;\ \frac{y(w+1)}{2}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_1F_2\left[\begin{matrix}-n\quad;\\ 1+\alpha,\ 1+2\alpha;\end{matrix}\ u\right]P_n^{(\alpha,\alpha)}(v)t^n.$$

Two further results are obtainable from (11) on ultraspherical polynomials. However they are both special cases of the results by Weisner mentioned above, and so are merely presented here for completeness. For the first, take in (11)

$$p=q=1\,,\quad d_1=1+2\alpha\,,\quad c_1=a\,, \tag{16}$$

and sum the result by [2, equ. (18)] to get

$$g^{-\alpha-1/2}(1-yw)^{-a}{}_2F_1\left[\begin{matrix}a/2,\ (a+1)/2; & \dfrac{y^2(w^2-1)}{(1-yw)^2}\\ 1+\alpha \quad ; & \end{matrix}\right] \tag{17}$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_2F_1\left[\begin{matrix}-n,\ a; & u\\ 1+2\alpha\ ; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

If a is a non-positive integer $-k$ then (17) becomes

$$g^{-\alpha-1/2}\frac{r^k k!}{(1+\alpha)_k}P_k^{(\alpha,\alpha)}\left(\frac{1-yw}{r}\right) \tag{18}$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_2F_1\left[\begin{matrix}-n,\ -k; & u\\ 1+\alpha\quad ; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

For the other result of Weisner's, in (11) take

$$p=0\,,\quad q=1\,,\quad d_1=1+2\alpha\,, \tag{19}$$

and sum to get:

$$g^{-\alpha-1/2}e^{yw}{}_0F_1\left(-\,;1+\alpha;\ \frac{y^2(w^2-1)}{4}\right) \tag{20}$$

$$=\sum_{n=0}^{\infty}\frac{(1+2\alpha)_n}{(1+\alpha)_n}\,{}_1F_1\left[\begin{matrix}-n\ ; & u\\ 1+2\alpha\,; & \end{matrix}\right]P_n^{(\alpha,\alpha)}(v)t^n\,.$$

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ON THE CASIMIR OPERATOR

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The Casimir operator is an important tool in the study of associative [4], Lie [4] and alternative algebras [7]. However its use has been for algebras of characteristic 0. We give a new definition of the Casimir operator for associative, Lie and alternative algebras, which keeps desirable properties of the usual Casimir operator and which is useful for arbitrary characteristic.

We show that under certain conditions our Casimir operator is the identity transformation and for non-degenerate alternative (or associative) algebras we show that it is the transformation into which the identity element of the algebra maps. We apply our results to obtain the first Whitehead lemma for non-degenerate alternative algebras of arbitrary characteristic. We also obtain a special case of the Levi theorem for Lie algebras of prime characteristic.

1. The Casimir Operator. Let $\mathfrak{A}$ be an associative, Lie or alternative algebra with basis $e_1, e_2, \cdots, e_n$ over an arbitrary field $\mathfrak{F}$. For uniformity we use the notation $x \rightarrow S_x$ for a representation of $\mathfrak{A}$, where if $\mathfrak{A}$ is alternative we mean the S_x part of a representation $x \rightarrow (S_x, T_x)$. If $\mathfrak{A}$ is a Lie or associative algebra, $f(x, y) = t(S_x S_y)$ where t is the trace function, is an invariant symmetric bilinear form. In [7, p. 444] it is shown that if $\mathfrak{A}$ is alternative this form is invariant if $\mathfrak{F}$ is not of characteristic 2. For arbitrary characteristic we have

$$
\begin{aligned}
t(S_x S_{yz}) &= t(S_x S_y S_z + S_x T_y S_z - S_x S_z T_y) \\
&= t(S_x S_y S_z + S_x T_y S_z - T_y S_x S_z) = t(S_{xy} S_z) .
\end{aligned}
$$

Similarly $t(T_x T_y)$ is invariant.

We call $\mathfrak{A}$ *non-degenerate* if $t(R_x R_y)$ is non-degenerate where R is the representation of right multiplications. It can be shown that this is equivalent to the non-degeneracy of the bilinear form $t(L_x L_y)$ of the left multiplications. It is well known that if $\mathfrak{A}$ is a non-degenerate alternative (or associative) algebra it is a direct sum of simple algebras. Dieudonne [3] has shown that this is also true for Lie algebras.

If $\mathfrak{A}$ is semi-simple and $\mathfrak{F}$ is of characteristic 0, the usual Casimir operator Γ_S^* for the representation S is defined as follows: Let $\mathfrak{N}$ be the set of all x of $\mathfrak{A}$ such that $t(S_x S_y) = 0$ for all y of $\mathfrak{A}$. Then $\mathfrak{A} = \mathfrak{N} \oplus \mathfrak{C}$ where $\mathfrak{N}$ and $\mathfrak{C}$ are semi-simple ideals of $\mathfrak{A}$. Let $e_1', e_2', \cdots, e_k'$ be the

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complementary basis to a basis $e_1, e_2, \cdots, e_k$ of $\mathfrak{C}$ such that[2] $t(S_iS'_j)=\delta_{ij}$ (Kroneker's delta). (Note that the complementary basis depends on the representation.) Then $\Gamma^*_S=\sum_{i=1}^{k} S_iS'_i$.

For arbitrary $\mathfrak{F}$ we define a new Casimir operator Γ_S for each non-degenerate $\mathfrak{A}$. This will include every semi-simple $\mathfrak{A}$ of characteristic 0, since $\mathfrak{A}$ is non-degenerate in this case. We use the same complementary basis $e'_1, e'_2, \cdots, e'_n$ such that $t(R_iR'_i)=\delta_{ij}$ for every representation (or anti-representation) and define

$$\Gamma_S=\sum_{i=1}^{n} S_iS'_i \;. \tag{1}$$

If $\mathfrak{A}$ is alternative we also define $\Gamma_T=\sum_{i=1}^{n} T_iT'_i$.

Unlike Γ^*_S, Γ_S does not automatically reduce to zero when $t(S_xS_y)=0$ for all x, y of $\mathfrak{A}$. In fact it follows from Corollary 3.1 below that for alternative algebras $\Gamma_S\neq 0$ if $S\neq 0$. We note also that for the representation $x\to R_x$ we have $\Gamma^*_R=\Gamma_R$.

Analogous to the corresponding result for Γ^*_S for Lie and associative algebras [4, p. 682] and for alternative algebras [7, p. 445] we have the following theorem.

THEOREM 1. *Let Γ_S be the Casimir operator* (1) *for a representation $x\to S_x(x\to(S_x, T_x))$ of a non-degenerate Lie or associative (alternative) algebra $\mathfrak{A}$ over an arbitrary field. Then Γ_S commutes with S_x (and T_x) for all x of $\mathfrak{A}$.*

Except for the commutativity of Γ_S and T_x which will be proved along with Lemma 3.2, the proof is similar to those in the references.

We also have the following result which follows from the properties of the complementary basis.

THEOREM 2. *Let $\mathfrak{A}$ be a non-degenerate associative, Lie or alternative algebra over an arbitrary field. Then the Casimir operators Γ_R and Γ_L of the right and left multiplications of $\mathfrak{A}$ are both the identity transformation.*

2. Application to alternative (and associative) algebras. Since every associative algebra is an alternative algebra, the results of this section hold for associative algebras.

In place of the identities (4) of [6] used in the definition of a representation $x\to(S_x, T_x)$ of an alternative algebra $\mathfrak{A}$, we will use the

[2] For simplification we write S_{e_i} as S_i and $S_{e'_i}$ as S'_i.

equivalent (except for characteristic 2) identities

$$(2) \qquad S_x^2 = S_{x^2}\,, \quad T_x^2 = T_{x^2} \qquad \text{for all } x \text{ of } \mathfrak{A},$$

in order to insure that the *semi-direct sum* [6, p. 3] or *split null extension* $\mathfrak{S} = \mathfrak{A} + \mathfrak{M}$ of $\mathfrak{A}$ and the representation space $\mathfrak{M}$ is an alternative algebra for arbitrary characteristic.

THEOREM 3. *For every representation S of a non-degenerate alternative algebra $\mathfrak{A}$, $\Gamma_S = S_e$ where $e = \Sigma e_i e_i'$ is the identity element of $\mathfrak{A}$.*

The proof follows from Theorem 2 and the properties of the complementary basis.

COROLLARY 3.1. *If $S \neq 0$ the matrix of Γ_S can be taken to have the form* $\operatorname{diag}(I, 0)$. *Hence if in addition the representation is irreducible, Γ_S is the identity transformation.*

Proof. By (2), $S_e^2 = S_e$ and the result follows.

COROLLARY 3.2. *$\Gamma_S S_x = S_x$ for all x of $\mathfrak{A}$.*

Proof. Assume $S \neq 0$ and take Γ_S to have the form $\operatorname{diag}(I, 0)$. Then the matrix of S_x must have the form $\operatorname{diag}(S_x', S_x'')$ where I and S_x' have the same order. By identity (4) of [6] we have $T_x\Gamma_S - \Gamma_S T_x = S_x - S_x\Gamma_S$. Hence $S_x'' = 0$ and $T_x = \operatorname{diag}(T_x', T_x'')$ and so $S_x\Gamma_S = S_x$. This completes the proof of Theorem 1, for we also have $T_x\Gamma_S = \Gamma_S T_x$.

Evidently all of the above results also hold when S is replaced by T.

Now for a non-degenerate alternative algebra $\mathfrak{A}$ with neither S nor $T=0$ we may apply Corollary 3.1 and Theorem 1 to take

$$(3) \qquad \begin{gathered} \Gamma_S = \operatorname{diag}(I^{(1)}, I^{(2)}, 0^{(3)}, 0^{(4)})\,, \quad \Gamma_T = \operatorname{diag}(I^{(1)}, 0^{(2)}, I^{(3)}, 0^{(4)}) \\ S_x = \operatorname{diag}(S_x^{(1)}, S_x^{(2)}, S_x^{(3)}, 0^{(4)})\,, \quad T_x = \operatorname{diag}(T_x^{(1)}, T_x^{(2)}, T_x^{(3)}, 0^{(4)}) \end{gathered}$$

where the superscript (i) indicates the matrix has order k_i and each I is an identity matrix and $S_x^{(3)} = 0^{(3)}$, $T_x^{(2)} = 0^{(2)}$. Also $x \to (S_x^{(i)}, T_x^{(i)})$, $(i=1, 2, 3)$ are representations of $\mathfrak{A}$ with respective Casimir operators

$$(4) \qquad \begin{gathered} \Gamma_S^{(1)} = \Gamma_T^{(1)} = I^{(1)}\,; \quad \Gamma_S^{(2)} = I^{(2)}\,, \quad \Gamma_T^{(2)} = 0^{(2)}\,; \\ \Gamma_S^{(3)} = 0^{(3)}\,, \quad \Gamma_T^{(3)} = I^{(3)}\,. \end{gathered}$$

Thus the representation space $\mathfrak{M}$ can be expressed as $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$

$+\mathfrak{M}_3+\mathfrak{M}_4$ where $\mathfrak{M}_i$ is an invariant subspace of dimension k_i and hence is an ideal of the split-null extension $\mathfrak{S}=\mathfrak{A}+\mathfrak{M}$. It also follows that $\mathfrak{M}_2$ and $\mathfrak{M}_3$ are in the *nucleus* [2] of $\mathfrak{S}$.

We are now able to obtain the following generalization of the first Whitehead lemma (see [8]) for alternative algebras of characteristic zero [6, Theorem 3].

THEOREM 4. *Let $\mathfrak{A}$ be a non-degenerate alternative algebra over an arbitrary field and let $x \rightarrow (S_x, T_x)$ be a representation of $\mathfrak{A}$ acting in a space M. Let $\mathfrak{S}$ be the split null extension $\mathfrak{S}=\mathfrak{A}+\mathfrak{M}$ and let $h(x)$ be a linear mapping of $\mathfrak{A}$ into $\mathfrak{M}$ such that*

$$h(xy)=xh(y)+h(x)y=h(x)S_y+h(y)T_x \tag{5}$$

for all x, y of $\mathfrak{A}$. Then $h(x)$ is an inner derivation of $\mathfrak{S}$. If $\mathfrak{A}$ is not of characteristic 2 then[3]

$$h(x)=[x, g]+\frac{x}{2}\sum_{i=1}^{n}\{[R_i', R_{h(e_i)}]+[L_i', L_{h(e_i)}]\} \tag{6}$$

where g is in the nucleus of $\mathfrak{S}$; R, L are right and left multiplications in $\mathfrak{S}$ and $e_1', e_2', \cdots, e_n'$ are a complementary basis to a basis $e_1, e_2, \cdots, e_n$ of $\mathfrak{A}$.

Proof. If either S or T is zero the theorem follows similarly to the associative characteristic zero case, so assume neither is. Since $\mathfrak{M}_i$ is invariant,

$$h(x)=h_0(x)=h_1(x)+h_2(x)+h_3(x)$$

where $h_j(x)$ is a linear mapping of $\mathfrak{A}$ into $\mathfrak{M}_j$ ($\mathfrak{M}_0=\mathfrak{M}$) such that

$$h_j(xy)=xh_j(y)+h_j(x)y=h_j(x)S_y+h_j(y)T_x\ .$$

Then we have

$$h_j(x)\Gamma_S=\sum_{i=1}^{n}\{h_j(xe_i)e_i'-xh_j(e_i)\cdot e_i'\}=\sum_{i=1}^{n}\{h_j(e_i)(e_i'x)-xh_j(e_i)\cdot e_i'\}\ .$$

Consequently for $j=0, 1, 2, 3$

$$h_j(x)\Gamma_S=x\sum_{i=1}^{n}\{L_i'L_{h_j(e_i)}-R_{h_j(e_i)}R_i'\}\ . \tag{7}$$

Similarly

$$h_j(x)\Gamma_T=x\sum_{i=1}^{n}\{R_i'R_{h_j(e_i)}-L_{h_j(e_i)}L_i'\}\ . \tag{8}$$

[3] We use $[P, Q]$ to denote the commutator $PQ-QP$.

By (3) and (4) we have

$$h(x)=h_1(x)\Gamma_S+h_2(x)\Gamma_S+h_3(x)\Gamma_T .$$

Hence by (7) and (8) $h(x)=xD$ where

$$D=\sum_i \{L_i'L_{h_1(e_i)}-R_{h_1(e_i)}R_i'\}+\sum_i \{L_i'L_{h_2(e_i)}-R_{h_2(e_i)}R_i'\} + \sum_i \{R_i'R_{h_3(e_i)}-L_{h_3(e_i)}L_i'\} .$$

To show that D is inner it suffices to show that for x, y in $\mathfrak{S}$, $L_xL_y-R_yR_x$ is in the Lie algebra $\mathfrak{L}(\mathfrak{S})$ of linear transformations generated by the right and left multiplications of $\mathfrak{S}$. This is true since $L_xL_y-R_yR_x=2[R_y, L_x]+L_{yx}-R_{yx}$.

Now let $\mathfrak{A}$ have characteristic $\neq 2$ and use (7) and (8) to get

$$h(x)(\Gamma_S+\Gamma_T)=x\left\{\sum_i [R_i', R_{h(e_i)}]+\sum_i [L_i', L_{h(e_i)}]\right\} .$$

Then by (7) and the nucleus property of $\mathfrak{M}_2$ we have[4] $h_2(x)\Gamma_S=[x, v_2]$ where $v_2=\sum_i h_2(e_i)e_i'$ is in $\mathfrak{M}_2$. Similarly $h_3(x)\Gamma_T=[x, v_3]$ where v_3 is in $\mathfrak{M}_3$. But

$$h(x)(\Gamma_S+\Gamma_T)+h_2(x)\Gamma_S+h_3(x)\Gamma_T=2h(x)$$

hence

$$h(x)=[x, g]+x\left\{\frac{1}{2}\sum_i [R_i', R_{h(e_i)}]+\frac{1}{2}\sum_i [L_i', L_{h(e_i)}]\right\}$$

where $g=\frac{1}{2}(v_2+v_3)$ is in the nucleus of $\mathfrak{S}$.

As is the case for similar theorems, the first part of Theorem 4 can be stated in the following form.

THEOREM 5. *Let $\mathfrak{A}$ be a non-degenerate subalgebra of an alternative algebra $\mathfrak{B}$ over an arbitrary field. Then any derivation of $\mathfrak{A}$ into $\mathfrak{B}$ can be extended to an inner derivation of $\mathfrak{B}$.*

3. **Application to Lie algebras.** We obtain the following special case of the generalization of the Levi theorem to algebras of prime characteristic.

THEOREM 6. *Let $\mathfrak{L}$ be a Lie algebra over an arbitrary field with radical $\mathfrak{R}\neq\mathfrak{L}$ such that $\mathfrak{L}\mathfrak{R}=0$ and $\mathfrak{L}/\mathfrak{R}$ is non-degenerate. Then there is an algebra $\mathfrak{S}$ (which is isomorphic to $\mathfrak{L}/\mathfrak{R}$ and is a direct sum of*

[4] This actually $=-v_2x$ since $xv_2=0$.

simple algebras) such that $\mathfrak{L}$ *is the direct sum* $\mathfrak{L}=\mathfrak{S}\oplus\mathfrak{R}$.

Proof. Let $e_1, e_2, \cdots, e_n$ be a basis for $\mathfrak{L}$ such that $e_1, e_2, \cdots, e_k$ are a basis for a subspace $\mathfrak{B}$ and $e_{k+1}, \cdots, e_n$ are a basis for $\mathfrak{R}$. Then the right multiplication of each x of $\mathfrak{L}$ has the form

$$R_x=\begin{bmatrix} P_x & Q_x \\ 0 & 0 \end{bmatrix} \tag{9}$$

where $P_x=Q_x=0$ if x is in $\mathfrak{R}$ and P_x is the right multiplication of the image $\bar{x}$ of x in $\mathfrak{L}/\mathfrak{R}$. Now if $\Gamma_P=\sum_{i=1}^{k} P_iP_i'$ is the Casimir operator (1) for the representation P of $\mathfrak{L}/\mathfrak{R}$, then by Theorem 2, Γ_P is the identity I and hence

$$\Gamma=\sum_{i=1}^{k} R_iR_i'=\begin{bmatrix} I & Q \\ 0 & 0 \end{bmatrix}.$$

By using the properties of the complementary basis of $\mathfrak{L}/\mathfrak{R}$ and the fact that the Lie algebra of right multiplications of the elements of $\mathfrak{B}$ is isomorphic to $\mathfrak{L}/\mathfrak{R}$ it can be shown that Γ commutes with R_x for all x of $\mathfrak{L}$.

We now show that the associative algebra $\mathfrak{L}^*$ generated by the R_x for all x of $\mathfrak{L}$ is isomorphic to the associative algebra $\mathfrak{P}^*$ generated by the P_x. Certainly by (9) there is a homomorphism of $\mathfrak{L}^*$ onto $\mathfrak{P}^*$ which maps any polynomial $p(R_x, R_y, \cdots)$ into $p(P_x, P_y, \cdots)$. Now if $p(R_x, R_y, \cdots)=0$ then $p(P_x, P_y, \cdots)=0$ since Γ commutes with $p(R_x, R_y, \cdots)$. Hence $\mathfrak{L}^* \simeq \mathfrak{P}^*$.

Now $\mathfrak{L}/\mathfrak{R}$ is a direct sum of simple algebras and therefore [**1**, Lemma 2], $\mathfrak{P}^*$ (and hence $\mathfrak{L}^*$) is semi-simple. Consequently [**1**, Lemma 2] $\mathfrak{L}$ is a direct sum of an algebra $\mathfrak{S}$, which is a direct sum of simple algebras, and an abelian algebra $\mathfrak{R}_1$. But we must have $\mathfrak{R}_1=\mathfrak{R}$ completing the proof.

It is to be noted that it is easy to give examples of prime characteristic where all but the non-degeneracy of $\mathfrak{L}/\mathfrak{R}$ of the hypothesis is satisfied but for which the conclusion is false.

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MICHIGAN STATE UNIVERSITY

REPRESENTATION THEOREMS FOR CERTAIN FUNCTIONAL OPERATORS

R. E. EDWARDS

1. Introduction. Almost all the operators arising in applications of the Heaviside operational calculus share two properties. The precise formulation of these properties may vary, but their general nature is, in the first case, a commutativity rule relating to the operation of semi-translation, whilst in the second case it is a condition of continuity of some sort. Possible precise formulations of these conditions are typified by postulates (O_1), (O_2) and (O_2'), which appear subsequently. Verification of the opening remark is to be found by glancing at the diverse illustrations of the technique to be found for example throughout [4].

It is the aim of the present paper to base proofs of general representation theorems upon such characteristic properties. The appropriate theorems will depend of course on the topologies envisaged in the continuity condition. Because of this, neither theorem proved here applies to all conceivable "operational expressions": an outlaw expression would be $\exp(hp)(h>0)$, for instance. Modifications are possible, however, and would lead to theorems covering wider ranges of operational expressions.

As is well known, if the operands are restricted suitably, the operational calculus can be formulated in terms of the one-sided Laplace transform. Special attention is given to this case, and the corresponding representation theorem can be looked upon as a solution of the problem of factor functions for the Laplace transformation. The methods employed were suggested by those used in [3] to study factor functions for the Fourier transformation.

The general nature of all results obtained is very close to one given by L. Schwartz [5, p. 18, Théorème X].

2. Classes of functions and operators. The widest class of functions to be considered will be denoted by $\mathscr{F}$ and will consist of those functions $f=f(t)$ which are defined and locally integrable on the half-line $R_+=\{t:t>0\}$. Functions which are equal a.e. are identified. A fundamental operator mapping $\mathscr{F}$ into itself is "semi-translation by s", where $s\geqq 0$: this is denoted by U_s and is defined by

$$(2.1)\qquad U_sf(t)=\begin{cases} f(t-s) & \text{for } t>s\,, \\ 0 & \text{for } 0<t\leqq s\,. \end{cases}$$

The first of the two characteristic properties to be postulated about

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operators T is

(O_1) *T commutes with U_s for each $s \geqq 0$.*

The second, which reads

(O_2) *T is continuous from $\mathscr{F}$ into $\mathscr{F}$,*

is interpreted relative to the topology of convergence in mean over each bounded interval of R_+. This topology on $\mathscr{F}$ is defined by the family of seminorms

$$p_n(f)=\int_0^n |f(t)|\,dt \qquad (n=1, 2, \cdots) \tag{2.2}$$

and makes $\mathscr{F}$ into a Fréchet space.

The first of the representation theorems may now be stated.

THEOREM 1. *Let T be a linear operator mapping $\mathscr{F}$ into itself which satisfies (O_1) and (O_2). Then T is given by truncated convolution with a certain Radon measure μ concentrated on the closed half-line $t \geqq 0$, that is,*

$$Tf(t)=\mu * f(t)=\int_0^t f(t-s)\,d\mu(s) \tag{2.3}$$

for f in $\mathscr{F}$. Conversely, if μ is such a measure, (2.3) *defines an operator T satisfyiug (O_1) and (O_2).*

The measure μ may fail to be absolutely continuous; for this reason some care is needed in defining the right members of (2.3). This is dealt with in the proof of Theorem 1, to be given in § 3.

The second theorem pays special attention to the subspace $\mathscr{E}$ of $\mathscr{F}$ composed of functions f for which

$$q_n(f)=\int_0^\infty e^{-nt}|f(t)|\,dt < +\infty \tag{2.4}$$

holds for some n which may depend on f. $\mathscr{E}$ is practically the largest domain for the Laplace transformation

$$\hat{f}(p)=\int_0^\infty e^{-pt}f(t)\,dt\,; \tag{2.5}$$

if f satisfies (2.4), then $\hat{f}(p)$ is defined for $\Re p \geqq n$. Many of the operational expressions $F(p)$ appearing in applications of the Heaviside method act on $\mathscr{E}$ according to the ritual: take the Laplace transform, multiply by $F(p)$, and then invert the Laplace transform. The operational expression $F(p)$ thus acts as a "factor function". Detailed consideration of such factor functions is deferred until § 5.

In order to state the second representation theorem it is necessary to introduce a topology on $\mathscr{E}$. If $\mathscr{E}_n (n=1, 2, \cdots)$ denotes the subspace of $\mathscr{E}$ defined by the inequality (2.4), then $\mathscr{E}_n \subset \mathscr{E}_{n+1}$ and $\mathscr{E} = \bigcup_{n=1}^{\infty} \mathscr{E}_n$. Furthermore q_n is a norm on $\mathscr{E}_n$ relative to which the latter is a Banach space. Accordingly, on $\mathscr{E}$ one may introduce the inductive limit topology defined by the $\mathscr{E}_n$ and the q_n; see [1, p. 61]: this is the finest locally convex topology on $\mathscr{E}$ which induces on each $\mathscr{E}_n$ a topology less fine than that defined by the norm q_n. We shall denote by (O_2') the condition which results from (O_2) by replacing therein the Fréchet space $\mathscr{F}$ by the space $\mathscr{E}$ equipped with the said inductive limit topology.

THEOREM 2. *Let T be a linear operator mapping into itself which satisfies (O_1) and (O_2'). Then T admits a representation* (2.3), *where now the measure μ satisfies a condition*

$$\int e^{-nt} d|\mu|(t) < +\infty \tag{2.6}$$

for some n (which may depend on μ, that is, on T); and conversely.

It may be noted here and now that Theorem 2 applies in particular to any T satisfying (O_1) and (O_2) which happens to map $\mathscr{E}$ into $\mathscr{E}$. This is so because any such T has a restriction to $\mathscr{E}$ which is necessarily continuous for $\mathscr{E}$'s topology, which assertion is most easily established by applying the generalised closed graph theorem [2, p. 36, Exercice 13]. Condition (O_2) is easily seen to imply that the restriction of T to $\mathscr{E}$ has a closed graph when considered as a map of $\mathscr{E}$ into itself[1].

3. Proof of Theorem 1.

The first thing is to define $\mu * f$ for $f \in \mathscr{F}$ and any measure μ concentrated on the half-line $t \geqq 0$. An analogous process works in connection with Theorem 2 for functions $f \in \mathscr{E}$ and measures μ satisfying (2.6) for some n.

In the present case we note that for fixed f in $\mathscr{F}$, $U_s f$ is a continuous function with values in $\mathscr{F}$ and that $p_n(U_s f)=0$ for $s \geqq n$. It is therefore certain that the abstract integral

$$\int U_s f \cdot d\mu(s) \tag{3.1}$$

exists as an element of $\mathscr{F}$: this element is $\mu * f$. To see how this

[1] It is necessary merely to observe that, for each n, the topology of $\mathscr{F}$ induces on $\mathscr{E}_n$ a topology less fine than that defined by q_n. So, by definition, the inductive limit topology is finer than that induced on $\mathscr{E}$ by $\mathscr{F}$'s topology. This being so, it is trivial to verify that the restriction of T to $\mathscr{E}$ has a closed graph.

definition is related to the "pointwise" one, we note that the dual of $\mathscr{F}$ may be identified with the space of bounded, measurable functions φ on R_+ which vanish a.e. outside bounded intervals, the linear form associated with such a φ being given by

$$\langle f, \varphi \rangle = \int_0^\infty f(t)\varphi(t)dt . \tag{3.2}$$

Now the definition of (3.1) is such that for all φ one has

$$\langle \mu * f, \varphi \rangle = \int \langle U_s f, \varphi \rangle d\mu(s) ,$$

so that by (3.2)

$$\int_0^\infty \mu * f(t) \cdot \varphi(t)dt = \int d\mu(s) \int_s^\infty f(t-s)\varphi(t)dt$$

for all φ. If $f(t-s)$, qua function of s, is integrable for μ over bounded intervals, and if $\int_0^t f(t-s)d\mu(s)$ is locally integrable (Lebesgue), the integral on the right can be rewritten as

$$\int_0^\infty \varphi(t)dt \int_0^t f(t-s)\, d\mu(s) .$$

Comparison shows that, under these conditions, $\mu * f$ is the function defined a.e. as $\int_0^t f(t-s)d\mu(s)$. This latter definition covers in particular the truncated convolution of two functions in $\mathscr{F}$.

Consider then the operator T defined by $Tf = \mu * f$. By what has been said, T maps $\mathscr{F}$ into itself. Linearity of T is obvious. Since also $U_s U_a = U_a U_s$ for $a \geqq 0$, $s \geqq 0$, and U_a is continuous on $\mathscr{F}$, the abstract definition gives at once

$$TU_a f = \int U_s U_a f \cdot d\mu(s) = \int U_a U_s f \cdot d\mu(s) = U_a \int U_s f \cdot d\mu(s) = U_a T f ;$$

thus T satisfies (O_1).

To prove the continuity of T it is merely necessary to take stock of the fact that $p_n(U_s f)$ vanishes for $s \geqq n$ and is everywhere at most $p_n(f)$. As a consequence,

$$p_n(Tf) \leqq \int p_n(U_s f) d|\mu|(s) \leqq m \cdot p_n(f) ,$$

where m is the $|\mu|$-measure of the interval $0 \leqq s \leqq n$. Thus (O_2) is satisfied. The converse part of Theorem 1 is thus established.

Suppose now that T satisfies (O_1) and (O_2). If f and g belong to $\mathscr{F}$ we have

$$T(f * g) = T\left(\int_0^\infty U_s g \cdot f(s) ds\right) = \int_0^\infty T U_s g \cdot f(s) ds$$
$$= \int_0^\infty U_s T g \cdot f(s) ds = f * Tg .$$

This is applied to a sequence $g = g_\nu (\nu = 1, 2, \cdots)$ forming an "approximate identity" for the truncated convolution. A simple example of such a sequence is furnished by the functions

$$g_\nu(t) = \begin{cases} \nu & \text{for } 0 < t < 1/\nu , \\ 0 & \text{for } t > 1/\nu . \end{cases}$$

It is easily verified that $f * g_\nu \to f$ in $\mathscr{F}$, and that $p_n(g_\nu) \leqq 1$ for all n and all ν. Since $T(f * g_\nu) = f * Tg_\nu$, if we let ν tend to infinity there follows

$$Tf = \lim_{\nu \to \infty} (f * h_\nu) ,$$

where $h_\nu = Tg_\nu$. Now the sequence (g_ν) is bounded in $\mathscr{F}$ and T is continuous; so the sequence (h_ν) is likewise bounded in $\mathscr{F}$, that is,

$$\mathrm{Sup}_\nu \int_0^n |h_\nu(t)| dt < +\infty$$

for each n. By dropping terms if necessary, we may assume that the sequence (h_ν) converges weakly to a measure μ concentrated on the half-line $t \geqq 0$. Accordingly, if f is continuous, $f * h_\nu(t)$ will converge pointwise to $\int_0^t f(t-s) d\mu(s)$ for each t. However, $f * h_\nu \to Tf$ in $\mathscr{F}$, and it follows at once that the two limits must coincide. Thus $Tf = \mu * f$ holds at any rate for f continuous. Such functions are dense in $\mathscr{F}$, and both members of this equality are continuous on $\mathscr{F}$. So equality holds for all f. This completes the proof of Theorem 1.

4. Proof of Theorem 2. The general plan of the proof is very similar to that of Theorem 1. As before, the existence of the abstract integrals is dealt with first. In this connection it is useful to note the inequality

$$q_n(\mu * f) \leqq q_n(\mu) \cdot q_n(f) , \tag{4.1}$$

where $q_n(\mu)$ denotes the left member of (2.6), provided both factors on the right are finite. Thus if μ satisfies (2.6) for a certain n, and if f belongs to $\mathscr{E}_N$ for some integer N, then (4.1) shows that $\mu * f$ belongs to $\mathscr{E}_M$, where $M = \max(n, N)$. It shows also that the operator T defined by $Tf = \mu * f$ has the property that its restriction to each subspace $\mathscr{E}_n$ is continuous relative to the norm q_n. Hence [1, p. 62] T is continuous from $\mathscr{E}$ into itself. In this way the converse part of Theorem 2 is established.

The direct part also runs much as before. The sequence (h_ν) is constructed again and will this time be bounded in $\mathscr{E}$. The limiting measure μ exists, but it remains to show that μ satisfies (2.6) for some n. This will follow as soon as it is shown that the h_ν lie in some $\mathscr{E}_n$, where n is fixed independent of ν, and remain bounded in $\mathscr{E}_n$. This does not follow directly from the boundedness of (h_ν) in $\mathscr{E}$ by virtue of [2, p. 8, Proposition 6] since $\mathscr{E}$ is not a *strict* inductive limit. Nevertheless the desired result can be proved as follows.

LEMMA. *Let B be a bounded subset of $\mathscr{E}$. There exists an integer n such that $B \subset \mathscr{E}_n$ and B is bounded relative to the norm q_n.*

Proof. The dual of $\mathscr{E}$ may be identified with the space $\mathscr{B}$ of measurable functions φ on R_+ which satisfy

$$r_n(\varphi) = \operatorname*{ess\,sup}_{t>0} |e^{nt}\varphi(t)| < +\infty$$

for all n, the linear form associated with such a φ is given by (3.2). Since B is bounded in $\mathscr{E}$, the quantity

$$Q(\varphi) = \sup_{h\in B} \left| \int_0^\infty h(t)\varphi(t)\,dt \right|$$

is finite for each φ in $\mathscr{B}$. Now $\mathscr{B}$, equipped with the seminorms $r_n (n = 1, 2, \cdots)$, is a Fréchet space. Further Q is a seminorm on $\mathscr{B}$ which is plainly lower semicontinuous, this last since Q is expressly defined as the upper envelope of continuous seminorms. It follows from this that Q is in fact continuous on $\mathscr{B}$. This signifies precisely that there is an integer n and a number C such that

$$Q(\varphi) \leqq C \cdot r_n(\varphi)$$

for all φ in $\mathscr{B}$; C is independent of φ. Thus

$$\left| \int_0^\infty h(t)\varphi(t)\,dt \right| \leqq C \cdot \operatorname*{ess\,sup}_{t>0} \left| e^{nt}\varphi(t) \right|$$

holds for all φ in $\mathscr{B}$ and all h in B. From this it is an easy deduction that

$$\int_0^\infty e^{-nt} |h(t)|\,dt \leqq C$$

for all h in B, which is the result stated.

This lemma permits the proof of Theorem 2 to be effected.

5. Factor Operators on $\mathscr{E}$. By a factor operator we shall mean one which is defined via a factor function for the Laplace transformation. The factor function $F(p)$ is assumed to be defined on some half-plane $\Re p \geqq n$, where n may depend on F, and to have the property that, for each f in $\mathscr{E}$ the function $F(p) \cdot \hat{f}(p)$ concides on some right-hand half-plane with the transform $\hat{g}(p)$ of some g in $\mathscr{E}$. This g, whose existence is postulated, is then unique. The corresponding factor operator T is then defined by $Tf = g$.

Such a factor operator T plainly satisfies (O_1), but continuity of T is not at all obvious. The relation

$$Tf * g = f * Tg\,, \tag{5.1}$$

which plays a crucial rôle on the above proofs, has hitherto been deduced from (O_1) by means of continuity. In the case of a factor operator, (5.1) is verifiable right from the start due to basic properties of the Laplace transformation. This fact permits us to deduce continuity of T and thus renders possible an appeal to Theorem 2.

As we shall now see, continuity of T will follow if (5.1) is known to hold for all f and for all g of a quite restricted class, say G. For this purpose we use again the generalised closed graph theorem. According to this, in order to show that T is continuous it will suffice to show that: if a directed family (f_i) converges to 0 in such a way that Tf_i converges to a limit, say f, then f is necessarily 0. However, we have seen in §4 that convolution is continuous in each factor, so that $Tf_i \to g$ implies $Tf_i * g \to f * g$ for each g in $\mathscr{E}$. Assuming that g belongs to G, (5.1) permits this to be written $f_i * Tg \to f * g$. Since $f_i \to 0$, the left member tends to 0. Hence $f * g = 0$ for all g in G. If this holds, even for quite small classes G, it follows that $f = 0$.

In this way we see from Theorem 2 that the factor functions F are precisely those which are themselves Laplace transforms of measures μ satisfying (2.6) for some n.

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THE FIVE-POINT DIFFERENCE EQUATION WITH PERIODIC COEFFICIENTS

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The five-point difference equation described in § 1 has most of the important second order partial difference equations as special cases and as limiting forms of these the more important partial differential equations of the second order. In the present paper all coefficients are assumed periodic in the same one of the two independent variables. The purpose of the paper is the study of the form of the general solution as affected by the periodic character of the coefficients. This study centers around the roots of the characteristic equation and so-called semi-periodic solutions. The reader is referred to the theorem of § 5 for a precise statement of results.

1. General discussion. Let us be given the five-point equation

$$k_1(i,j)y(i-1,j)+k_2(i,j)y(i+1,j)+k_3(i,j)y(i,j-1)+k_4(i,j)y(i,j+1)+k_5(i,j)y(i,j)=0 \tag{1}$$

where k_1, k_2, k_3, k_4 and k_5 are defined for integral values of i and j over the rectangle $1\leq i\leq n\omega-1$, $1\leq j\leq\omega-1$ where $n>1$ and $\omega>1$ are integers. This rectangle will be called the *defining rectangle* and will be denoted by R. We assume moreover that

$$k_\nu(i+\omega,j)=k_\nu(i,j)\ , \qquad \nu=1,2,3,4,5 \tag{2}$$

and that neither, k_1, k_2, k_3, nor k_4 is zero at any point of R.

A *solution* of (1) is a function of (i,j) defined at points of R and at the border points $(i=0,\ j=1,2,\cdots,\omega-1)$, $(i=n\omega, j=1,2,\cdots,\omega-1)$, $(j=0,\ i=1,2,\cdots,n\omega-1)$, $(j=\omega.\ i=1,2,\cdots,n\omega-1)$ and which satisfies (1) at all points of R. Notice that this second set of points, namely R plus the border points, form a lattice which is rectangular except that its corner points are missing. It will be referred to as the rectangle S.

A *fundamental domain* is a set of points of S such that there exists one and only one solution taking on prescribed arbitrary values at each point of the set.

All fundamental domains[1] contain the same number of points. We denote this number by L. For the rectangle S

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[1] For a detailed discussion see T. Fort, Amer. Math. Monthly, **62**, (1955), 161.

$$L=2\omega(n+1)-4 .$$

We note that all sets of L points belonging to S do not form a fundamental domain.

A *fundamental system of solutions* consists of L solutions which are linearly independent over a fundamental domain.

If $y_1(i,j), y_2(i,j), \cdots, y_L(i,j)$ are a fundamental system of solutions then any solution $y(i,j)$ can be written

$$y(i,j)=\sum_{\mu=0}^{L} c_\mu y_\mu(i,j)$$

where the c's are constants.

We choose the following fundamental domain for S namely points where

$$\begin{aligned}
&j=0, && i=1,\ 2,\ \cdots,\ n\omega-1\ ;\\
&j=1, && i=0,\ \omega,\ \omega+1,\ \cdots,\ 2\omega-1,\ 2\omega,\ 3\omega,\ \cdots,\ n\omega\ ;\\
&j=2, && i=0,\ \omega,\ 2\omega,\ \cdots,\ n\omega\ ;\\
&\cdots && \cdots\\
&j=\omega-1, && i=0,\ \omega,\ 2\omega,\ \cdots,\ n\omega\ ;\\
&j=\omega, && i=\omega,\ 2\omega,\ \cdots,\ (n-1)\omega\ .
\end{aligned}$$

We shall refer to this particular fundamental domain as D. The domain D is pictured in figure 1 with $\omega=4$, $n=3$. The points of D have no accompanying numeral.

To prove D a fundamental domain simply assign values for y at each of the points of D and fill in the rectangle by means of (1). This can be done in a variety of ways. For example, the value of y at the points of S not in D can be determined in the order indicated by the accompanying numerals in Figure 1.

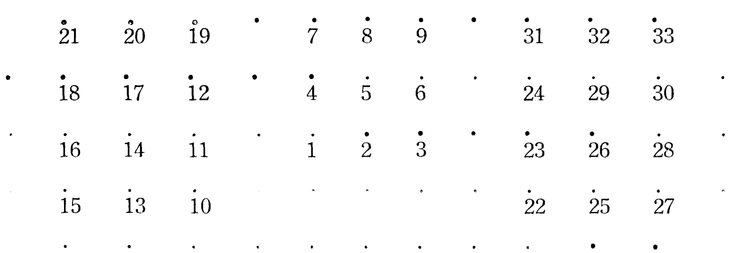

Figure 1.

We now define all coefficients to the right of $i=n\omega-1$ by formula (2). We then define every solution $y(i,j)$ at the points $(n\omega, 0)$, $(n\omega+1, 0)$, $\cdots$, $((n+1)\omega-1, 0)$ by the formula

$$y(n\omega+\nu, 0)=y(\nu+1, 0)\ , \qquad \nu=0, 1, \cdots, \omega-1\ ;$$

and at the points $((n+1)\omega, j)$, $j=1, 2, \cdots, \omega-1$ by the formula

$$y((n+1)\omega, j)=y(0, j) \quad \text{also} \quad y(n\omega, \omega)=y(\omega, \omega)\ .$$

This definition serves to determine y over a longer rectangle than S, $n\omega$ being replaced by $(n+1)\omega$, the rectangles being in every other way the same. We call this the rectangle T.

BASIC THEOREM. *If $y(i, j)$ is a solution over S then $y(i+\omega, j)$ is also a solution over S.*

This theorem follows immediately from the periodic character of the coefficients in (1).

THEOREM. *If $y_1(i, j)$, $y_2(i, j)$, $\cdots$, $y_L(i, j)$ are a fundamental system of solutions for S then so are $y_1(i+\omega, j)$, $y_2(i+\omega, j)$, $\cdots$, $y_L(i+\omega, j)$.*

This theorem follows from the fact that $y_1(i, j), y_2(i, j), \cdots, y_L(i, j)$ considered at the points of D constitute L sets of L constants linearly independent over D and that, due to the extension of each solution over T described above, $y_1(i+\omega, j), y_2(1+\omega, j), \cdots, y_L(i+\omega, j)$ at the points of D are precisely the same sets of constants as $y_1(i, j), y_2(i, j), \cdots, y_L(i, j)$ although the order may be different.

2. Semiperiodic solutions. We ask the question : Does there exist a solution of (1) not identically zero over S and satisfying the relation

$$y(i+\omega, j)=\rho y(i, j) \tag{3}$$

where $\rho\neq 0$ is constant? We, of course, except the case where either $(i+\omega, j)$ or (i, j) is a corner point of S since solutions are not defined at corner points.

Let us assume a solution $y_q(i, j)\neq 0$ satisfying (3) and work for necessary conditions. As previously, let $y_1(i, j)$, $y_2(i, j)$, $\cdots$, $y_1(i, j)$ be a fundamental system of solutions for S. Then so are $y_1(i+\omega, j)$, $y_2(i+\omega, j)$, $\cdots$, $y_L(i+\omega, j)$. Consequently

$$y_\nu(i+\omega, j)=\sum_{\mu=1}^{L} a_{\nu\mu}y_\mu(i, j)\ , \qquad \nu=1, \cdots, L\ ,$$

where $\det(a_{\nu\mu})\neq 0$. Moreover

$$y_q(i, j)=\sum_{\mu=1}^{L}\alpha_\mu y_\mu(i, j)$$

where not all the α's are zero. Then

$$y_q(i+\omega, j)=\sum_{\mu=1}^{L}\alpha_\mu y_\mu(i+\omega, j)=\sum_{\mu=1}^{L}\alpha_\mu \sum_{\nu=1}^{L} a_{\mu\nu}y_\nu(i, j)$$

$$=\sum_{\nu=1}^{L}\left[\sum_{\mu=1}^{L}\alpha_\mu a_{\mu\nu}\right]y_\nu(i, j) .$$

Also

$$y_q(i+\omega, j)=\rho y_q(i, j)=\rho\sum_{\nu=1}^{L}\alpha_\nu y_\nu(i, j) .$$

We can equate coefficients since $y_1(i, j), \cdots, y_L(i, j)$ are linearly independent over D. We get

$$\begin{array}{llll} (a_{11}-\rho)\alpha_1+ & a_{21}\alpha_2 & +\cdots\cdots+ & a_{L1}\alpha_L =0 \\ a_{12}\alpha_1 & +(a_{22}-\rho)\alpha_2 & +\cdots\cdots+ & a_{L2}\alpha_L =0 \\ \cdot & \cdot & \cdot & \cdot \\ a_{1L}\alpha_1 & +\quad a_{2L}\alpha_2 & +\cdots\cdots+ & (a_{LL}-\rho)\alpha_L=0 . \end{array}$$

But the α's are not all zero. Hence

$$(4) \qquad \begin{vmatrix} (a_{11}-\rho) & a_{21} & \cdots & a_{L1} \\ a_{12} & (a_{22}-\rho) & \cdots & a_{L2} \\ \cdot & \cdot & \cdot & \cdot \\ a_{1L} & a_{2L} & \cdots & (a_{LL}-\rho) \end{vmatrix}=0 .$$

This condition is not only necessary but it is also sufficient as is seen by retracing steps.

Equation (4) is the *characteristic equation* for the problem and its roots are the *characteristic values.*

THEOREM. *The characteristic equation is independent of the particular fundamental system of solutions chosen.*

Consider a second fundamental system, $y_1^{(2)}(i, j)$, $y_2^{(2)}(i, j)$, $\cdots$, $y_L^{(2)}(i, j)$. Then

$$y_\nu^{(2)}(i+\omega, j)=\sum_{\mu=1}^{L} b_{\nu\mu}y_\nu^{(2)}(i, j) , \qquad \nu=1, \cdots, L .$$

The characteristic equation is

$$\begin{vmatrix} (b_{11}-\rho) & b_{21} & \cdots & b_{L1} \\ b_{12} & (b_{22}-\rho) & \cdots & b_{L2} \\ \cdot & \cdot & \cdot & \cdot \\ b_{1L} & b_{2L} & \cdots & (b_{LL}-\rho) \end{vmatrix}=0 .$$

But

$$y_\nu^{(2)}(i,j)=\sum_{\mu=1}^{L} h_{\nu\mu} y_\mu(i,j)\ , \qquad \nu=1,\cdots,L\ ,$$

where $\det(h_{\nu\mu})\neq 0$. Hence

$$y_\nu^{(2)}(i+\omega,j)=\sum_{\mu=1}^{L} b_{\nu\mu} y_\mu^{(2)}(i,j)=\sum_{\mu=1}^{L} b_{\nu\mu}\sum_{\eta=1}^{L} h_{\mu\eta} y_\mu(i,j)$$

$$=\sum_{\eta=1}^{L}\left[\sum_{\mu=1}^{L} b_{\nu\mu} h_{\mu\eta}\right] y_\eta(i,j)\ .$$

On the other hand

$$y_\nu^{(2)}(i+\omega,j)=\sum_{\mu=1}^{L} h_{\nu\mu} y_\mu(i+\omega,j)=\sum_{\mu=1}^{L} h_{\nu\mu}\sum_{\eta=1}^{L} a_{\mu\eta} y_\eta(i,j)$$

$$=\sum_{\eta=1}^{L}\left[\sum_{\mu=1}^{L} h_{\nu\mu} a_{\mu\eta}\right] y_\eta(i,j)\ .$$

We can equate coefficients, as already explained, because $y_1, y_2, \cdots, y_L$ are linearly independent. We have

$$\sum_{\mu=1}^{L} b_{\nu\mu} h_{\mu\eta}=\sum_{\mu=1}^{L} h_{\nu\mu} a_{\mu\eta}\ , \quad \eta=1,\cdots,L\ ;\quad \nu=1,\cdots,L\ . \tag{5}$$

Now let us form the products

$$\begin{vmatrix} h_{11} & \cdots & h_{1L} \\ \cdot & \cdots & \cdot \\ h_{L1} & \cdots & h_{LL}\end{vmatrix}\cdot\begin{vmatrix} (a_{11}-\rho) & \cdots & a_{1L} \\ \cdot & \cdots & \cdot \\ a_{L1} & \cdots & (a_{LL}-\rho)\end{vmatrix}$$

and

$$\begin{vmatrix} (b_{11}-\rho) & \cdots & b_{1L} \\ \cdot & \cdots & \cdot \\ b_{L1} & \cdots & (b_{LL}-\rho)\end{vmatrix}\cdot\begin{vmatrix} h_{11} & \cdots & h_{1L} \\ \cdot & \cdots & \cdot \\ h_{L1} & \cdots & h_{LL}\end{vmatrix}\ .$$

If we perform the indicated multiplication and then use (5) we get identical determinants. This establishes the theorem.

THEOREM. *No characteristic value is zero.*

This theorem follows from the fact that $\det(a_{ij})\neq 0$.

If it were zero then $y_1(i+\omega,j), y_2(i+\omega,j), \cdots, y_L(i+\omega,j)$ would be linearly independent over a fundamental domain which they are not.

3. Roots distinct. Let the roots of the characteristic equation be $\rho_1, \rho_2, \cdots, \rho_L$ and assume that no two are equal. Let corresponding semiperiodic solutions be $y_1(i, j), \cdots, y_L(i, j)$, that is $y_\nu(i+\omega, j)=\rho_\nu(i, j)$, $\nu=1, 2, \cdots, L$ and assume, as we can, that no one of these is identically zero.

THEOREM. *The solutions $y_1, \cdots, y_L$ constitute a fundamental system of solutions.*

To prove this theorem we assume first $y_1, \cdots, y_{L-k}$ are linearly dependent over D but that $y_1, \cdots, y_{L-k-1}$ are linearly independent over D. Then

$$\sum_{\nu=0}^{L-k} \mu_\nu y_\nu(i, j)=0 \tag{6}$$

over D with at least one $\mu \neq 0$. Replace i by $(i+\omega)$. Then

$$\sum_{\nu=0}^{L-k} \mu_\nu y\ (i+\omega, j)=0 \tag{7}$$

over D. This is true because solutions linearly dependent over D are linearly dependent over all of T.

From (7)

$$\sum_{\nu=0}^{L-k} \mu_\nu \rho_\nu y_\nu(i, j)=0\ . \tag{8}$$

But $\mu_L - k \neq 0$ else $y_1, \cdots, y_{L-k-1}$ would be linearly dependent. Eliminate y_{L-k} between (6) and (8). We get

$$\sum_{\nu=1}^{L-k-1} \mu_\nu(\rho_\nu - \rho_{L-k}) y_\nu(i, j)=0\ .$$

The only way that this can be true with the linear independence of $y_1, \cdots, y_{L-k-1}$ is that

$$\mu_1(\rho_1-\rho_{L-k})=\mu_2(\rho_2-\rho_{L-k})=\cdots=\mu_{L-k-1}(\rho_{L-k-1}-\rho_{L-k})=0\ .$$

If $\mu_1=\mu_2=\cdots=\mu_{L-k-1}=0$, then $\mu_{L-k} y_{L-k}(i, j)\equiv 0$. This is not the case since $\mu_{L-k}\neq 0$ and $y_{L-k}(i, j)\not\equiv 0$ over D. Consequently ρ_{L-k} must equal some other ρ. This contradicts our simple root hypothesis. Hence, $y_1, \cdots, y_L$, are linearly independent over D and the theorem is proved.

4. Multiple roots; special discussion. Let us assume that ρ_1 is a double root of the characteristic equation but that all other roots are simple. Let $y_1(i, j)$ be as before; namely $y_1(i+\omega, j)=\rho_1 y_1(i, j)\not\equiv 0$ and let $\tilde{y}_2(i, j), \tilde{y}_3(i, j), \cdots, \tilde{y}_L(i, j)$ be so chosen that $y_1, \tilde{y}_2, \tilde{y}_3, \cdots, \tilde{y}_L$ form a fundamental system. We have the relations

$$(9)\qquad y_1(i+\omega, j)=\rho_1 y_1(i, j)\ ,$$

$$\tilde{y}_\nu(i+\omega, j)=c_{\nu 1}y_1(i, j)+\sum_{\mu=2}^{L} c_{\nu\mu}\tilde{y}_\mu(i, j)\ , \qquad \nu=2, \cdots, L\ .$$

The characteristic equation is

$$(10)\qquad \begin{vmatrix} (\rho_1-\rho) & 0 & 0 & \cdots & 0 \\ c_{21} & (c_{22}-\rho) & c_{23} & \cdots & c_{2L} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{L1} & c_{L2} & c_{L3} & \cdots & (c_{LL}-\rho) \end{vmatrix}=0\ .$$

Since ρ_1 is a double root of (10),

$$\begin{vmatrix} (c_{22}-\rho_1) & c_{23} & \cdots & c_{2L} \\ \cdot & \cdot & \cdot & \cdot \\ c_{L2} & c_{L3} & \cdots & (c_{LL}-\rho_1) \end{vmatrix}=0.$$

Hence from (9) the solutions $\tilde{y}_\nu(i+\omega, j)-c_{\nu 1}y_1(i, j)-\rho_1\tilde{y}_\nu(i, j)$, $\nu=2, \cdots, L$ are linearly dependent.

This means that

$$\sum_{\nu=2}^{L} C_\nu \tilde{y}_\nu(i+\omega, j)=y_1(i, j)\sum_{\nu=2}^{L} C_\nu c_{\nu 1}+\rho_1\sum_{\nu=2}^{L} C_\nu \tilde{y}_\nu(i, j)\ .$$

Let $\sum_{\nu=2}^{L} C_\nu c_{\nu 1}=\kappa$ and $Y_2(i, j)=\sum_{\nu=2}^{L} C_\nu \tilde{y}_\nu(i, j)$. We note that $Y_2(i, j)\not\equiv 0$ since $y_1, \tilde{y}_z, \cdots, \tilde{y}_L$ are linearly independent over D. Then

$$(11)\qquad Y_2(i, +\omega, j)=\rho_1 Y_2(i, j)+\kappa y_1(i, j)\ .$$

This is a difference equation in Y_2 as a function of i with difference interval ω. We shall solve[2] for $Y_2(i+\mu\omega, j)$. Let $U(i+\mu\omega, j)$ be a solution of the difference equation

$$(12)\qquad U(i+\omega, j)=\rho_1 U(i, j)\ .$$

Then $U(i+\mu\omega, j)=\rho_1^\mu U(i, j)$. Moreover $U(i, j)$ is arbitrary so we assume it different from zero. Then

$$(13)\qquad Y_2(i+\mu\omega, j)=U(i+\mu\omega, j)\left[\sum_{\nu=0}^{\mu-1}\frac{\kappa y_1(i+\nu\omega, j)}{U(i+(\nu+1)\omega, j)}+c_1(i, j)\right].$$

We note that $y_1(i+\nu\omega, j)=\rho_1^\nu y_1(i, j)$. With this in mind (13) yields

$$Y_2(i+\mu\omega, j)=\left[\frac{\kappa}{\rho_1}\mu y_1(i, j)+U(i, j)\,c_1(i,j)\right]\rho_1^\mu$$

[2] T. Fort, *Finite differences and difference equations in the real domain.* Clarendon, 1948, p. 117.

We rewrite this[3]

(14) $$Y_2(i+\mu\omega, j)=\left[\frac{k}{\rho_1}\mu y_1(i, j)+Y_2(i, j)\right]\rho_1^{\mu} .$$

This is an interesting form for $Y_2(i+\mu\omega, j)$. We note particularly the μ in the first term of the bracket.

THEOREM. *The solution $y_1(i, j)$, $Y_2(i, j)$, $y_3(i, j)$, $\cdots$, $y_L(i, j)$ form a fundamental system*

To prove this theorem assume the contrary, namely linear dependence:

(15) $$c_1y_1(i, j)+c_2Y_2(i, j)+c_3y_3(i, j)+\cdots+c_Ly_L(i, j)=0 .$$

Then increasing i by ω yields

(16) $$c_1\rho_1y_1(i, j)+c_2\kappa y_1(i, j)+c_2\rho_1Y_2(i, j)+c_3\rho_3y_3(i, j)+\cdots+c_L\rho_Ly_L(i, j)=0 .$$

Now c_2 is not zero else $y_1, y_3, \cdots, y_L$ would be linearly dependent which they are not. We eliminate $Y_2(i, j)$ from (15) and (16). We get

$$c_2\kappa y_1(i, j)+c_3(\rho_3-\rho_1)y_3(i, j)+\cdots+c_L(\rho_L-\rho_1)y_L(i, j)=0 .$$

But $c_3, \cdots, c_L$ are not all zero. If they were we would have $y_1(i, j)$ and $Y_2(i, j)$ linearly dependent. They are not since $Y_2(i. j)$ is linearly dependent upon $\tilde{y}_3(i, j), \cdots, \tilde{y}_L(i, j)$ and by hypothesis $y_1(i, j)$ is not. It results that ρ_1 must equal at least one of $\rho_3, \cdots, \rho_L$. This contradicts our hypothesis.

We now assume ρ_1 a triple root but that other roots are distinct.

We consider $y_1(i, j)$ and $Y_2(i, j)$ of the double root discussion and note that they are not linearly dependent. We then define $\tilde{y}_3(i, j), \cdots, \tilde{y}_L(i, j)$ so that $y_1(i, j), Y_2(i, j), \tilde{y}_3(i, j), \cdots, \tilde{y}_L(i, j)$ form a fundamental system. The characteristic equation takes the form

$$\begin{vmatrix} (\rho_1-\rho) & 0 & 0 & 0 & \cdots & 0 \\ c_{21} & (\rho_1-\rho) & 0 & 0 & \cdots & 0 \\ c_{31} & c_{32} & (c_{33}-\rho) & c_{34} & \cdots & c_{3L} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{L1} & c_{L2} & c_{L3} & c_{L4} & \cdots & (c_{LL}-\rho) \end{vmatrix}=0 .$$

Since ρ_1 is a triple root of this equation we have

[3] The convention used in (13) is $\sum_{\nu=0}^{-1} f(\nu)=0$.

$$\tag{17} \begin{vmatrix} (c_{33}-\rho_1) & c_{34} & \cdots & c_{3L} \\ \cdot & \cdot & \cdot & \cdot \\ c_{L3} & c_{L4} & \cdots & (c_{LL}-\rho_1) \end{vmatrix} = 0$$

It follows from (17) that $\tilde{y}_\nu(i+\omega, j) - c_{\nu 1}y_1(i, j) - c_{\nu 2}Y_2(i, j) - \rho_1\tilde{y}_\nu(i, j)$, $\nu=3, \cdots, L$ are linearly dependent. Let constants determining the linear dependence be $\gamma_3, \gamma_4, \cdots, \gamma_L$. Then let $\kappa_\nu = \sum_{\mu=3}^{L} \gamma_\mu c_{\mu\nu}$, $\nu=1, 2$. Let

$$Y_3(i, j) = \sum_{\mu=3}^{L} \gamma_\mu \tilde{y}_\mu(i, j) .$$

Then

$$Y_3(i+\omega, j) = \kappa_1 y_1(i, j) + \kappa_2 Y_2(i, j) + \rho_1 Y_3(i, j) .$$

This is a difference equation in Y_3 as a function of i with difference interval ω. We solve precisely as we solved (11).

As previously we choose $U(i, j)$ not equal to zero then

$$Y_3(i+\mu\omega, j) = U(i+\mu\omega, j)\left[\sum_{\nu=1}^{\mu-1} \frac{\kappa_1 y_1(i+\nu\omega, j) + \kappa_2 Y_2(i+\nu\omega, j)}{U(i+(\nu+1)\omega, j)} + c(i, j)\right] .$$

Now substitute $y_1(i+\nu\omega, j) = \rho_1^\nu y_1(i, j)$ and

$$Y_2(i+\nu\omega, j) = \rho_1^\nu\left[\frac{\kappa}{\rho_1^\nu} y_1(i, j) + Y_2(i, j)\right] .$$

we get

$$Y_3(i+\mu\omega, j) = \rho_1^{\mu-1}\mu\kappa y_1(i, j) + \kappa\kappa_2\rho_1^{\mu-2}\frac{\mu(\mu-1)}{3\,!}y_1(i, j)$$
$$+ \kappa_2\rho_1^{\mu-1}\mu Y_2(i, j) + \rho_1^\mu Y_3(i, j) .$$

5. **Multiple roots ; general discussion.** The work that we have just done is easily generalized. Details are ommitted but can be readily supplied by one who has read § 4.

THEOREM. *If ρ_1 is an α_1-fold root of the characteristic equation there exist solutions of* (1) *which we call* $Y_1^{(1)}(i, j)$, $Y_2^{(1)}(i, j)$, $\cdots$, $Y_{\alpha_1}^{(1)}(i, j)$, *were*

$$Y_1^{(1)}(i+\mu\omega, j) = \rho_1^\mu Y_1^{(1)}(i, j) ,$$
$$Y_2^{(1)}(i+\mu\omega, j) = \rho_1^\mu[c_1^{(1)}\mu Y_1^{(1)}(i, j) + Y_2^{(1)}(i, j)] ,$$
$$Y_3^{(1)}(i+\mu\omega, j) = \rho_1^\mu\left[\left\{c_1^{(2)}\mu + c_2^{(2)}\frac{\mu(\mu-1)}{2\,!}\right\}Y_1^{(1)}(i, j) + c_3^{(2)}\mu Y_2^{(1)}(i, j) + Y_3(i, j)\right] ,$$

. . . .

$$
\begin{aligned}
&Y^{(1)}_{\alpha_1}(i+\mu\omega, j)\\
&= \rho_1^{\mu}\Big[\Big\{c_1^{(\alpha_1)}\mu + c_2^{(\alpha_1)}\frac{\mu(\mu-1)}{2\,!} + \cdots + c^{(\alpha_1)}_{\alpha_1-1}\frac{\mu(\mu-1)\cdots(\mu-\alpha_1+2)}{(\alpha_1-1)\,!}Y_1^{(1)}(i, j)\Big\}\\
&\quad + \Big\{c^{(\alpha_1)}_{\alpha_1}\mu + \cdots + c^{(\alpha_1)}_{2\alpha_1-3}\frac{\mu(\mu-1)\cdots(\mu-\alpha_1+3)}{(\alpha_1-2)\,!}\Big\}Y_2^{(1)}(i, j) + \cdots\\
&\quad + c^{(\alpha_1)}_{(\alpha_1^2-\alpha_1)/2}\,\mu\, Y^{(1)}_{\alpha_1-1}(i, j) + Y^{(1)}_{\alpha_1}(i, j)\Big].
\end{aligned}
$$

If the roots are ρ_1 *of order* α_1, ρ_2 *of order* α_2, $\cdots$, ρ_t *of order* α_t; *then the solutions* $Y_1^{(1)}, Y_2^{(1)}, \cdots, Y^{(2)}_{\alpha_1}, Y_1^{(1)}, \cdots, Y^{(2)}_{\alpha_2}, \cdots, Y_1^{(t)}, \cdots, Y^{(t)}_{\alpha_t}$ *form a fundamental system of solutions.*

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ON LINEAR SYSTEMS WITH INTEGRAL VALUED SOLUTIONS

I. HELLER

1. Introduction. We consider a system of linear equations and inequalities in k variables

$$(1.1) \qquad Ax=b\,, \quad x \geqq o\,,$$

where the matrix A has r rows, k columns, and rank less than k.

Assuming the system consistent, the solution set is a convex polyhedron P in k-space. A solution x^0 that satisfies k independent relations of (1.1) as equations, is a vertex of P, and conversely. Such solution is generally called basic or extremal, and is equivalently defined by the property, that the columns of A corresponding to nonzero coordinates of x^0 are independent. Basic solutions are of particular interest in problems where a linear functional is extremised over P, the extremum then being assumed at a vertex or at all points of a positive dimensional face F of P, that is, the convex hull of the vertices of F. In such problems the interest is often restricted to the integral valued basic solutions as the only ones that have meaning in the application. Now given P, any vertex of P can appear as solution of an extremum problem for some linear functional, and a question of interest is: when, that is for which systems (1.1), are all the vertices of P integral valued.

Directing the attention to the system

$$(1.2) \qquad Ax=b\,,$$

we may, slightly generalizing, respectively specializing, carry over the definition and the question:

(1.3) DEFINITION. A solution x^0 of (1.2) is *basic*, when its nonzero coordinates correspond to linearly independent columns of A.

(1.4) QUESTION. Which systems (1.2) have all their basic solutions integral valued?

Obviously (1.4) is not equivalent to the same question for systems (1.1); the basic solutions of (1.2) contain those of (1.1); but they may also contain others, namely such with negative integral coordinates. Hence (1.4) asks more and will therefore yield a smaller family of

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systems as answer.

A further specialization in the same direction is obtained, when the attention is restricted to the matrix A above and the question varied as follows:

QUESTION. Which matrices A have the property that

(1.5) whenever b is such that (1.2) has an integral solution (that is whenever b belongs to the integral span of A), then all basic solutions of (1.2) are integral?

The subject of this note is precisely the question above, which will receive a partial answer.

We note first that (1.5) is equivalent to

(1.6) If a column of A is a linear combination of a set of independent columns of A, then the coefficients in the combination are integers.

The proof is nearly obvious: If d is a column of A, d is certainly in the integral span of A; hence, when A satisfies (1.5), the basic solutions of $Ax=d$ are integral, which is precisely (1.6). Conversely, if A satisfies (1.6), let x^0 be some (not necessarily basic) integral and y^0 an arbitrary basic solution of (1.2); let B and C be the set of columns of A corresponding to nonzero coordinates of x^0 and y^0 respectively, that is,

$$b=L(B)=M(C)\,,$$

where L, M denote linear combinations. Extending C in A to a basis, say C^*, for the span of A, and substituting in $L(B)$ for each column of B its (certainly integral) representation in C^*, yields an integral representation of b in C^*, which representation, because of uniqueness, is identical with $M(C)$.

Next we observe that (1.6) is equivalent to

(1.7) THE DANTZING PROPERTY. If a column of A is a linear combination of a set of independent columns of A, then the coefficients in the combination are 1, -1, or 0.

To see that (1.6) implies (1.7): a representation of a column d where a column c enters with coefficient $\alpha \neq 0$, yields a representation of c where d enters with coefficient $1/\alpha$.

After these remarks the question can be rephrased as: which matrices A satisfy (1.7)?

Recent investigations on the subject comprise the following.

In the so-called Transportation Problem, there appears a matrix D, which G. Dantzig [1] showed to have the property (1.7). This fact was used by T. C. Koopmans and Dantzig to prove the existence of integral solutions to the mentioned problem, and by Dantzig [1] to establish a simplified computational procedure for solving the problem.

The mentioned matrix D appears partitioned into an upper and a lower submatrix, and the columns of D consist of all possible vectors having a single 1 in each of the two submatrices and zeros everywhere else. If e_ν denotes the νth unit vector, then

$$D=\{e_i+e_j\} \qquad (i=1, 2, \cdots, m;\ j=m+1, \cdots, m+n=r) \tag{1.8}$$

Later C. Tompkins and the author [2] showed the property (1.7) to hold for a somewhat larger class of matrices:

If

$$u_1, u_2, \cdots, u_m, \quad v_1, v_2, \cdots, v_n$$

is a set of linearly independent vectors in r-dimensional vector space $(r \geqq m+n)$, then the set

$$T=\{\pm u_i,\ \pm v_j,\ \pm(u_i+v_j),\ (u_i-u_{i^*}),\ (v_j-v_{j^*})\} \tag{1.9}$$

$$(i,\ i^*=1, 2, \cdots, m;\ j, j^*=1, 2, \cdots, n)$$

has property (1.7).

Finally A. J. Hoffman and J. Krushall [5] showed property (1.7) to hold for several classes of incidence matrices associated with partially ordered sets.

The property (1.7) will be referred to as *Dantzig property* throughout this note. The term *unimodular property* has also been proposed and used [5]. This term seems quite appropriate for the case of incidence matrices, as in [5], where nonsingular submatrices then represent unimodular transformations; in the general case it is the transition from one basis in the matrix to another that is a unimodular transformation.

2. Unification of prior results. This is achieved by a few trivial observations.

First, since the Dantzig property does not depend on the order in which the columns of A are arranged, it is convenient to interpret A simply as a set of vectors.

Second, the Dantzig Property is invariant under nonsingular linear transformations, hence if A has the property, so does the image of A under a nonsingular linear transformation.

Third, in (1.9) the partition of the set of vectors into two sets $\{u_i\}$ and $\{v_j\}$ is rather artificial. If, for instance, we substitute $-w_j$ for v_j, (1.9) becomes

$$T=\{\pm u_i,\ \pm w_j,\ \pm(u_i-w_j),\ (u_i-u_{i^*}),\ (w_{j^*}-w_j)\}$$

which shows that T can be simply described by

(2.1) $$T=\{\pm x_i,\ (x_i-x_j)\} \qquad (i\neq j;\ i,j=1,2,\cdots,r),$$

or

(2.2) $$T=\{x_i-x_j\} \qquad (i\neq j;\ i,\ j=0,1,\cdots,r),$$

where x_0 denotes the null vector, and $x_1, x_2, \cdots, x_r$ are linearly independent vectors.

In the last formulation T is the set of differences of the x_i. Since differences are invariant under translations, the x_i in (2.2) may also be specified as a set of $r+1$ vectors whose affine span (all linear combinations with coefficients sum equal 1) is of dimension r; in other words, the x_i are the vertices of an r-simplex. This reduces the result (1.9) of [2] to the simple statement:

(2.3) *The set of edges (that is, one-dimensional faces, taken in both orientations and interpreted as vectors) of a simplex has the Dantzig property.*

In this form the statement is nearly obvious. Clearly, a basis B among the edges:

(i) contains all the $r+1$ vertices (otherwise the vectors of B would be among the edges of a lower-dimensional simplex, and hence not a basis for the span of all edges),

(ii) is connected (otherwise the vectors of B would be among the edges of two simplices of s and $r+1-s$ vertices, so that $\dim B\leqq s-1+r-s=r-1$),

(iii) is free of cycles (the vectors of a cycle being linearly dependent).

Hence B is a tree containing all vertices and r oriented segments. Any edge not in B closes a chain in B, which proves the statement.

Using the statement (2.3) one can show the Dantzig property to hold for a series of incidence matrices (incidence matrices are defined here simply as having only 0's and ± 1's as entries), some of which can be identified with matrices exhibited in [5]. Let E be Euclidean n-space, S an n-simplex in E, T the set of edges of S and B a maximal independent subset of T, hence a basis in S. If B is taken as the basis for the coordinate system, the representation of T is the set of columns of an incidence matrix with Dantzig property.

It is worthwhile to follow this somewhat closer. Since choosing a basis among the vectors of T amounts to choosing, in the net of vertices and edges of S, a tree containing all $n+1$ vertices and n oriented segments, the construction leads to as many essentially different incidence matrices as there are graphically different trees of $n+1$ vertices (note that permutation of columns or rows in a matrix preserves the Dantzig property, so that matrices obtained from each other by such permutations may be considered as equivalent; by *essentially different* we then mean *not equivalent*).

We point out two particular choices.

(i) The star consisting of all edges radiating from a given vertex and oriented from this vertex to the remaining vertices. This yields the set T of (2.1) with the x_i as unit vectors.

(ii) The oriented chain obtained by numbering the vertices from 0 to n and taking the set of oriented edges

$$x_1 - x_0,\ x_2 - x_1,\ \cdots,\ x_n - x_{n-1}\ .$$

If these vectors are taken as basis in the listed order, then the representation of all edges in this basis is the set of all columns that have a consecutive string of 1's or (-1)'s, and 0's everywhere else. This is a result of [5].

Obviously the transition from one basis to another is a unimodular transformation.

3. **Maximal Dantzig sets.** Since with a set D each subset of D has the Dantzig property, or briefly is a *Dantzig set*, the interest lies in determining maximal Dantzig sets.

Obviously a maximal Dantzig set contains with each vector x also $-x$. Further, it should contain, but we agree to exclude, the null vector.

(3.1) *A set T consisting of the edges of a simplex is a Dantzig set which is maximal for its dimension* (*in the sense that there is no Dantzig set of the same dimension properly containing T*).

Proof. We have to show that when a vector x not belonging to T is adjoined to T, the new set does not have the Dantzig property. In the representation (2.1) with the x_i as basis vectors, x will have at least two coordinates of the same sign (both $=1$ or both $=-1$), since all other possibilities are already in T. Say

$$x = x_1 + x_2 + L(x_3,\ \cdots,\ x_n)\ ,$$

where L denotes linear combination. But then

$$x = (x_1 - x_2) + 2x_2 + L(x_3,\ \cdots,\ x_n)\ ,$$

that is, the representation of x in the basis

$$x_1 - x_2,\ x_2,\ x_3,\ \cdots,\ x_n$$

does not satisfy the Dantzig property, since the coefficient of x_2 equals $2 \neq 0,\ \pm 1$.

The question whether every maximal Dantzig set is the set of edges of a simplex will obtain a negative answer by an example. We first note that in order to test whether a Dantzig set D can be extended to contain an additional vector b without losing the Dantzig property, it is sufficient to test the representation of b in every basis of D. That is:

(3.2) *Let D be a Dantzig set, b a vector not in D, and C the union of D and $\{b\}$. Then C has the Dantzig property if and only if the coordinates of b with respect to every basis in D consist of 0's and ± 1's.*

To see (indirectly) that the condition is sufficient, let d be a vector of C, B a basis in C, and let the representation of d in B have a coefficient $\neq 0, \pm 1$. Then obviously $d \neq b$, b is in B, and the coefficient of b is not 0:

$$d = \lambda_1 b + \lambda_2 b_2 + \cdots + \lambda_n b_n \quad (\lambda_1 \neq 0;\ \text{some } \lambda_i \neq 0,\ \pm 1)$$

But then the representation of b in the basis $\{d, b_2, \cdots, b_n\}$ contradicts the condition. This proves (3.2), since the necessity of the condition is obvious.

Further we formulate a necessary consistency condition for the Dantzig property which will be helpful in the sequel. Let b and d be two vectors in a Dantzig set D, and C a basis in D. Comparing the representations of b and d in C, we consider those vectors of C (if any) that enter with nonzero coefficients in both representations; say these are $c_1, c_2, \cdots, c_s$, so that

$$b = \beta_1 c_1 + \beta_2 c_2 + \cdots + \beta_s c_s + \cdots; \quad d = \gamma_1 c_1 + \gamma_2 c_2 + \cdots + \gamma_s c_s + \cdots$$

$$(\beta_i \neq 0 \neq \gamma_i\colon\ i = 1, 2, \cdots, s)$$

Obviously $\beta_i = {}_i\gamma \epsilon_i$; $\epsilon_i = \pm 1$. However, we confirm that ϵ_i remains constant, that is

(3.3) $$\beta_i = \epsilon \gamma_i \qquad (i = 1, 2, \cdots, s)$$

where $\epsilon = \text{constant} = \pm 1$.

Proof (indirect). Assume

$$b = c_1 + c_2 + \cdots, \quad d = c_1 - c_2 + \cdots .$$

Replacing c_1 by d yields a new basis in which b is represented by $b = d + 2c_2 + \cdots$, contradicting the Dantzig Property. This proves (3.3), which excludes "mixed incidences" (and permits to assign an "incidence number" 0, 1, -1 to every pair of vectors, with respect to a given basis in D).

Finally we give the example:

(3.4) *Let e_1, e_2, e_3, e_4 be independent vectors and*

$$A=\{\pm e_1,\ \pm e_2,\ \pm e_3,\ \pm e_4,\ \pm(e_1+e_2+e_3+e_4)\ ,$$

$$\pm(e_1+e_2),\ \pm(e_2+e_3),\ \pm(e_3+e_4),\ \pm(e_4+e_1)\}\ .$$

Then A is a maximal Dantzig set which is not the set of edges of a simplex.

To see that A has the Dantzig property, we note that the subset A^*, obtained from A by deleting $\pm(e_1+e_2+e_3+e_4)$, consists of (not all) edges of the simplex S given by the vertices

$$o,\ e_1,\ -e_2,\ e_3,\ -e_4\ .$$

Hence A^* is a Dantzig set. The deleted vector is represented with coefficients 0, ± 1 in every basis of A^*, as seen by direct verification. By (3.2) this implies that A has the Dantzig property.

To see that A is maximal, assume a vector h can be adjoined to A without disturbing the Dantzig property. If h is expressed in the basis e_1, e_2, e_3, e_4, then the nonzero coefficients are all equal, otherwise h would have "mixed incidence" with $d=e_1,+e_2+e_3+e_4$ in that basis and contradict (3.3). This leaves for h the following possibilities:

$$\pm h=e_1+e_3$$

$$\pm h=e_2+e_4$$

$\pm h=e_1+e_2+e_3=d-e_4$ and the equivalents.

However, each of these possibilities contradicts the Dantzig property, since, after adequate choice of bases, we obtain:

$$e_1+e_3=(e_1+e_2)+(e_2+e_3)-2e_2$$

$$e_2+e_4=(e_1+e_2)+(e_1+e_4)-2e_1$$

$$e_1+e_2+e_3=(e_1+e_4)+e_2+(e_3+e_4)-2e_4\ .$$

Finally A has 18 elements and therefore is not the set of 20 edges of a simplex (of dimension 4).

4. The two theorems in this section are prepared by the following lemma:

(4.1) *The image D' of a Dantzig set D under a projection, along a subspace N spanned by vectors of D, is a Dantzig set.*

Proof. Let D be in a vectorspace V, both of dimension n,

N the span of $\{d_1, d_2, \cdots, d_k\} \subset D$, ($k < n$; for $k=n$ the lemma is trivial),
M the range of the projection (some complement of N in V),
$\{b'_1, b'_2, \cdots, b'_s\}$ a basis (for M) in D' (hence $k+s=n$),
$\{b_1, b_2, \cdots, b_s\}$ some set of originals in D (that is b'_i is image of b_i),
b' an arbitrary vector in D',
b an original of b' in D, and
$b' = \beta_1 b'_1 + \beta_2 b'_2 + \cdots + \beta_s b'_s$.

Clearly the set $B = \{d_1, d_2, \cdots, d_k, b_1, b_2, \cdots, b_s\}$ is a basis (for V) in D (a nontrivial representation of o could not have all its nonzero coefficients attached to the d_i alone, since these are independent; on the other hand, nonzero coefficients of the b_i would imply dependence for the b'_i). Therefore b is representable in B:

$$b = \gamma_1 d_1 + \cdots + \gamma_k d_k + \beta_1 b_1 + \cdots + \beta_s b_s ,$$

where all coefficients, and hence in particular the β_i, are 0, ± 1, which proves the lemma.

(4.2) THEOREM. *A Dantzig set of dimension n contains at most $n(n+1)$ elements* (not counting the nullvector); *that is, if it contains $n(n+1)$ elements, then it is maximal.*

The proof is by induction on the dimension n. For $n=1$ the theorem is obvious. Assuming it holds for dimensions $< n$, we prove it to hold for $n (n \geqq 2)$.

Let D be a Dantzig set of dimension $n \geqq 2$ containing at least $n(n+1)$ elements. We may assume that D contains with each vector also its negative (otherwise we extend D to that effect, since adjoining the negatives does not remove the Dantzig property).

After choosing a basis $B = \{b_1, b_2, \cdots, b_n\}$ in D, D is projected along b_1 on the span of $\{b_2, b_3, \cdots, b_n\}$. Then the image D' of D is of dimension $\leqq n-1$, has the Dantzig property (by Lemma 4.1), and, excluding the nullvector, has at most $n(n-1)$ elements (by the induction's assumption that the theorem holds for dimensions $< n$).

We prove that D has at most, and hence exactly, $n(n+1)$ elements, in showing that the number of nonzero elements cannot be reduced, by the projection, by more than $2n = (n+1)n - n(n-1)$; this will be shown in two steps, namely:

(i) that a vector in D' is image of at most two originals in D, and
(ii) that the set of nonzero vectors with double originals consists of a linearly independent set and its negatives.

If distinct vectors x and y of D have the same nonzero image, then, with respect to the basis B, they coincide in all but their first coordinates. Further they cannot both have nonzero values for the first coordinate, since these would then have to be 1 and -1 and contradict the consistency condition (3.3). Therefore the first coordinate of the two vectors is 0 and ± 1 respectively. This implies that no three vectors can have the same nonzero image. If the image is 0, the only two originals are $\pm b_1$. Hence

(4.3) a vector x' in D' is the image of at most two vectors x and y in D; if $x \neq y$ and $x'=y' \neq 0$, then $x=x'$ and $y=x' \pm b_1$ (if $x'=0$, then $x=\pm b_1$, $y=\mp b_1$)

Denoting by D^* the set obtained from D' after removal of the null vector, let E^* be the set of vectors in D^* that have double originals in D. Since D contains with each vector also its negative, so does E^*. Furthermore E^* is also in D. If x' is in E^*, then its originals are

$$x' \text{ and } y=x'+\epsilon b_1 \qquad (\epsilon=\pm 1)$$

while the originals of $-x'$ are $-x'$ and $-y=-x'-\epsilon b_1$.
From the pair $-x'$, x' we choose one vector, call it d' so, that its originals are d' and $d=d'+b_1$. Making this choice from each such pair in E^*, we obtain the set $F^*=\{d_1', d_2', \cdots, d_s'\}$, where certainly $d_i' \neq \pm d_j'$ for $i \neq j$, and d_i' and $d_i=d_i'+b_1$ are the originals of d_i' in D.

An indirect proof will establish that the vectors of F^* are linearly independent. Obviously a linear relation between them must involve at least 3 vectors, say the first 3, with nonzero coefficients (which implies in particular that the assertion is true when F^* contains less than 3 vectors). We consider separately each of the two following possibilities

(i) $$d_1'=\pm(d_2'+d_3')+L(d_4', \cdots, d_t')$$

(ii) $$d_1'=d_2'-d_3'+L(d_4', \cdots, d_t'),$$

where L denotes a linear combination with nonzero coefficients throughout. We assume to have chosen, among all existing linear relations, the one that involves the smallest number of vectors. Then the vectors appearing on the right hand side, that is $d_2', d_3', d_4', \cdots, d_t'$, are linearly independent. Therefore each of the following two sets in D is also linearly independent:

(a) $$b_1, d_2, d_3, d_4', d_5', \cdots, d_t'$$

(b) $$b_1, d_2', d_3, d_4', d_5', \cdots, d_t'.$$

We now obtain,

in case (i): $d'_1 = \pm(-2b_1 + d_2 + d_3) + L(d'_4, \cdots, d'_t)$

in case (ii): $d_1 = 2b_1 + d'_2 - d_3 + L(d'_4, \cdots, d'_t)$,

hence in either case a contradiction to the Dantzig property (note that all vectors are in D).

This completes the proof that the vectors of F^* are linearly independent, which implies, because of $\dim D^* \leq n-1$, that F^* contains at most $n-1$ vectors. Hence E^* contains at most $2(n-1)$ vectors.

Now, since E^* consists of all nonnull vectors with double originals and the null vector has two originals (namely $\pm b_1$), it follows that the number of vectors in D exceeds the number of vectors in D^* by at most $2n$. Since D^*, as a Dantzig set of dimension $\leq n-1$, contains at most $n(n-1)$ vectors, it follows that D contains at most $n(n-1)+2n = n(n+1)$ vectors.

This completes the proof of Theorem (4.2), and, in addition yields the following conclusions, which will be used in the proof of next theorem.

From the assumption that D contains at least $n(n+1)$ vectors it now follows that

(4.4) D contains exactly $n(n+1)$ vectors
D^* contains exactly $n(n-1)$ vectors
F^* contains exactly $n-1$ vectors,

and hence

(4.5) $F^* = \{d'_1, d'_2, \cdots, d'_{n-1}\}$ is a basis in D^*.

(4.6) THEOREM. *If a Dantzig set D of dimension n contains $n(n+1)$ vectors (not counting the null vector), then D is the set of edges of an n-simplex.*

Proof. We will construct a basic $H = \{h_1, h_2, \cdots, h_n\}$ in D, such that every element of D which is not in H, is a difference of two elements of H. The mechanism that governs the construction is based on the obvious geometrical picture (assuming the theorem true).

We take over the projection, notation and facts from the proof of theorem (4.2); the assumptions made in that proof contain the assumptions of the present theorem as special case (note, that the induction's hypothesis made there, is now a true statement).

For ease of writing we renumber the vectors of F^* in (4.5) to

(4.7) $F^* = \{d'_2, d'_3, \cdots, d'_n\}$, and first show that

(4.8) *the representation of an element of D^* in the basis F^* has at most two nonzero coefficients.*

Proof (indirect). Let x' be in D^*, and

$$x' = \epsilon_2 d_2' + \epsilon_3 d_3' + \epsilon_4 d_4' + L(d_5', \cdots, d_n'); \qquad \epsilon_i = \pm 1.$$

We distinguish whether x' is, or is not, in D.

(i) x' is in D: We use the fact, that two of the ϵ_i are equal, say $\epsilon_2 = \epsilon_3 = 1$ (if $= -1$, we take $-x'$), and consider the basis in D (see page 1356):

$$b_1,\ d_2,\ d_3,\ d_4',\ d_5',\ \cdots,\ d_n'\ .$$

Then

$$x' = -2b_1 + d_2 + d_3 + \epsilon_4 d_4' + L\ ,$$

which contradicts the Dantzig property.

(ii) x' is not in D: Then its original $x = x' + \epsilon b_1$ ($\epsilon = \pm 1$) is in D, and we distinguish whether all three ϵ_i are equal or not. In the first case we may assume all $\epsilon_i = 1$ (otherwise we take $-x$), and obtain, after adequate choice of basis

$$x = (\epsilon - 3)b_1 + d_2 + d_3 + d_4 + L\ ,$$

where $\epsilon - 3 = -2$ or -4 contradicts the Dantzig property. In the second case, ϵ and one of the ϵ_i, say ϵ_2, have opposite sign. Then a contradiction is obtained by the coefficient of b_1 in the representation

$$x = (\epsilon - \epsilon_2)b_1 + \epsilon_2 d_2 + \epsilon_3 d_3' + \epsilon_4 d_4' + L\ .$$

This completes the proof of (4.8), and furthermore establishes the more specific assertions (i) and (ii) of the following statement:

(4.9) (i) *If x' of $(D^* - E^*)$ is in D, then $x' = d_\mu' - d_\nu'$,*
(ii) *If y' of $(D^* - E^*)$ is not in D, then $\pm y' = d_\mu' + d_\nu'$,*
(iii) *Conversely, for any two distinct d_μ' and d_ν' of F^*, either $\pm x'$ of* (i) *or $\pm y'$ of* (ii), *but not both, are in $(D^* - E^*)$.*

Part (iii) follows from the fact that D^* has $n(n-1)$ elements and the observation that the sum and the difference of d_μ' and d_ν' cannot both belong to the Dantzig set D^* because of the consistency condition (3.3).

By means of (4.9) F^* can be divided in (at most two) classes, by putting two distinct vectors of F^* into the same class when their difference is in D'.

We first prove that this is an equivalence relation. Reflexivity and symmetry are obvious. Transitivity is shown indirectly. Let only the first two of the following three differences be in D'

$$d'_i - d'_j,\ d'_j - d'_k,\ d'_k - d'_i\ .$$

Then in particular $d'_k - d'_i \neq o$, and hence by (4.9 iii), $d'_k + d'_i = d'$ is in D^*. But then

$$d' = (d'_i - d'_j) - (d'_j - d'_k) + 2d_j$$

violates the Dantzig property of D^*.

To see that there are at most two classes, we assume that d'_i, d'_j, d'_k belong to three distinct classes, which by (4.9 iii) implies that the sum of any two of the three vectors is in D^*. Then the representation

$$(d'_i + d'_k) = (d'_i + d'_j) + (d'_j + d'_k) - 2d'_j$$

violates the Dantzig property of D^*. This establishes that F^* decomposes in two classes

$$\mathrm{I} = \{d'_2,\ d'_3,\ \cdots,\ d'_k\}$$

$$\mathrm{II} = \{d'_{k+1},\ d'_{k+2},\ \cdots,\ d'_n\}$$

(where II may be empty), such that

(4.10) (i) the difference of two distinct vectors of the same class is in D^*

(ii) the (positive and negative) sum of two vectors of distinct classes is in D^*

(iii) the representations (i) and (ii) comprise all vectors of D^* which are not in E^*

We are now ready to construct the basis $H = \{h_1, h_2, \cdots, h_n\}$ of D, setting

(4.11) $$h_1 = b_1;\quad h_i = d'_i + b_1\ (2 \leqq i \leqq k);\quad h_j = -d'_j\ (k < j \leqq n)\ .$$

That $h_i = d'_i + b_1 = d_i$ is in D, follows from the construction of the d'_i on page 1359.

To verify that every x of D is represented by either $x = \pm h_\nu$ or $x = h_\mu - h_\nu$ we consider the projection x' of x, so that $x = x' + \alpha b_1$ where α may be one of the values 0, 1, -1. We may disregard $\alpha = -1$ (which amounts to consider only one vector of each pair x, $-x$), and distinguish the following cases:

(a) $x' = o$

(b) $x' \neq o$ and $\alpha = 0$

(c) $x' \neq o$ and $\alpha = 1$.

(a) implies $x = b_1 = h_1$.

(b) implies $x = x'$, that is, x is in D^*; we distinguish (b1) x is in E^*, (b2) x is in $D^* - E^*$.

(b1) implies $\pm x = d'_\nu$; hence, according to whether d'_ν belongs to class I or II, we have either $\pm x = h_\nu - b_1 = h_\nu - h_1$ or $\pm x = -h_\nu$.

(b2) and (4.9 i) imply $x = d'_\mu - d'_\nu$, where the last two vectors are in the same class because of (4.10); hence either $x = h_\mu - h_\nu$ or $x = -(h_\mu - h_\nu)$.

(c) implies $x = x' + b_1$; we distinguish: (c1) x' is in E^*, (c2) x' is in $D^* - E^*$ and in D, (c3) x' is in $D^* - E^*$ and is not in D.

(c1) implies $x = \pm d'_\nu + b_1$; the negative sign would yield mixed incidence of x and $d_\nu = d'_\nu + b_1$ and hence contradict (3.3); this leaves only $x = d'_\nu + b_1$; hence either $x = h_\nu$ or $x = b_1 - h_\nu = h_1 - h_\nu$.

(c2) cannot occur, since $x' \neq x$ and x' in D imply that x' has two distinct originals in D and therefore x' is in E^*.

(c3), (4.9 ii) and (4.10) imply $x' = \pm(d'_i + d'_j)$; hence $x = \pm(d'_i + d'_j) + b_1$; the negative sign would yield $x = -d_i - d_j + 3b_1$ violating the Dantzig property. This leaves only

$$x = d'_i + d'_j + b_1 = h_i - h_j .$$

This completes the proof of Theorem (4.6).

5. Open questions. While the set of edges of a simplex, which we may briefly call "difference set", is maximal in the sense of statement (3.1), it is, by Theorems (4.2) and (4.6), also maximal in the sense that it contains the largest number of elements for its dimension. Obviously the class of all difference sets of a given dimension can be obtained from a single one of its members by nonsingular linear transformations, and we may consider the set

(5.1) $$D = \{e_i - e_j\} \quad (i \neq j;\ i, j = 0, 1, \cdots, n;\ e_0 = o;\ e_i = i\text{th unit vector})$$

as a canonical representative of the class.

In regard to computational aspects we refer to [3].

For dimensions $n \geqq 4$ the example (3.4) establishes the existence of other maximal Dantzig sets of necessarily less than $n(n+1)$ elements. A classification of these sets has not been attempted, yet would certainly constitute the next natural step. The problem may be formulated as follows: Determine, for each dimension n, a complete (obviously finite) set of representatives $D_1, D_2, \cdots, D_k$ $(k = k(n))$ of maximal Dantzig sets, in the sense that

(i) two distinct D_i are not related by a linear transformation

(ii) every maximal Dantzig set of dimension n is the image of some D_i under a linear transformation.

6. Interpretations. Geometrically, the statement (3.1) and Theorem (4.6) solve the following problem: Given a set S of $n(n+1)/2$ (free) vectors $\neq o$ in Euclidean space, such that S is of dimension n, and does not contain the negative of any of its vectors; what is a necessary and sufficient condition that S may be so arranged in space as to form a simplex? Statement (3.1) gives the Dantzig property as obvious necessary condition, while Theorem (4.6) proves that it is also sufficient.

The considerations of this note were carried on in vector space in order to assure the benefit of intuition from the geometric picture. It is clear, however, that the study of Dantzig sets belongs properly to group theory; from the number field underlying the vector space only the integers are used, which amounts to actually restricting the considerations to an Abelian group. To interpret the results in terms of this structure, let G be a free Abelian group, and S a set of rank n, in G. The Dantzig property for S is, by § 1, precisely the condition that every set of n linearly independent elements of S span the same group as S. In particular; if S spans G, the Dantzig property means that every set of n linearly independent elements of S is a basis for G. The translation of statement (3.1) and Theorems (4.2) and (4.6) is immediate (compare [4]).

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ADDITION THEOREMS FOR SOLUTIONS OF THE WAVE EQUATION IN PARABOLIC COORDINATES

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1. Introduction. The wave equation

$$\Delta U + k^2 U = 0$$

admits solutions of the form

$$U_{\kappa,\mu} = A_{\kappa,\mu}(\xi) B_{\kappa,\mu}(\eta) C_{\kappa,\mu}(\phi)$$

if the coordinate system is such that separation of variables is possible. ξ, η and ϕ are the three independent variables, and κ and μ represent arbitrary complex parameters. In general $U_{\kappa,\mu}$ will not be regular and one-valued over the whole space, but will be so for special values of κ and μ. Let ξ', η' and ϕ' be functions of ξ, η, and ϕ resulting from a translation or rotation of the coordinate system; then a relation which expresses $U_{\kappa,\mu}(\xi', \eta', \phi')$ as a summation of terms of the form $U_{\kappa,\mu}(\xi, \eta, \phi)$ is called an addition theorem.

Addition theorems for cylindrical and spherical coordinate systems are well known. These are the addition theorems for Bessel and Hankel functions, Legendre polynomials, spherical harmonics, Mathieu functions and spheroidal wave functions (see Meixner and Schäfke [5] and Erdélyi [2]).

It is proposed to derive such addition theorems for those functions of the paraboloid of revolution which are regular and one-valued in the whole space. As will be seen subsequently, these restrictions are not always necessary. That such theorems might exist can be inferred from the invariance of ΔU under rotations and translations of space, and from the fact that the family of solutions that are everywhere regular and one-valued will be mapped onto itself by motions of space.

It is possible to derive several of these theorems by using known addition theorems. For example, it is possible to derive linear relations between the functions of the paraboloid of revolution and spherical harmonics. Since an addition theorem under a rotation of coordinates is known for the latter functions, it is possible to derive one for the functions of the paraboloid of revolution.

2. The functions of the paraboloid of revolution. The introduction

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of the parabolic coordinates

$$x=2\sqrt{\xi\eta}\cos\phi$$

$$y=2\sqrt{\xi\eta}\sin\phi$$

$$z=\xi-\eta$$

into the wave equation

$$\Delta U+k^2U=0$$

leads to the equation

$$\frac{1}{2(\xi+\eta)}\left\{\frac{\partial}{\partial\xi}2\xi\frac{\partial U}{\partial\xi}+\frac{\partial}{\partial\eta}2\eta\frac{\partial U}{\partial\eta}+\frac{\xi+\eta}{2\xi\eta}\frac{\partial^2U}{\partial\phi^2}\right\}+k^2U=0\ .$$

The method of separation of variables then shows, that the solution U can be expressed in terms of functions of the type

$$U=f_1(\xi)f_2(\eta)e^{-i\mu\phi}\ .$$

In the notation of Buchholz [1], these can be represented by

$$f_1(\xi)=m_\chi^\mu(-2ik\xi)=(-2ik\xi)^{\mu/2}e^{ik\xi}\frac{{}_1F_1\left(\frac{1+\mu}{2}-\chi;\ 1+\mu;\ -2ik\xi\right)}{\Gamma(1+\mu)}$$

and

$$f_1(\xi)=w_\chi^\mu(-2ik\xi)=\frac{\pi}{\sin\pi\mu}\left[\frac{m_\chi^{-\mu}(-2ik\xi)}{\Gamma\left(\frac{1+\mu}{2}-\chi\right)}-\frac{m_\chi^{\mu}(-2ik\xi)}{\Gamma\left(\frac{1-\mu}{2}-\chi\right)}\right].$$

In case μ is an integer, $w_\chi^\mu(-2ik\xi)$ must be derived by a limit process from the above definition. Similarly

$$f_2(\eta)=m_\chi^\mu(2ik\eta)=(2ik\eta)^{\mu/2}e^{-ik\eta}\frac{{}_1F_1\left(\frac{1+\mu}{2}-\chi;\ 1+\mu;\ 2ik\eta\right)}{\Gamma(1+\mu)}$$

and

$$f_2(\eta)=w_\chi^\mu(2ik\eta)=\frac{\pi}{\sin\pi\mu}\left[\frac{m_\chi^{-\mu}(2ik\eta)}{\Gamma\left(\frac{1+\mu}{2}-\chi\right)}-\frac{m_\chi^{\mu}(2ik\eta)}{\Gamma\left(\frac{1-\mu}{2}-\chi\right)}\right].$$

When μ is an integer the function $m_\chi^\mu(z)$ is regular and single-valued over the entire space; $w_\chi^\mu(z)$ in general is neither single-valued nor regular.

For the case $\chi = n + \frac{1+\mu}{2}$ the function $m_\chi^\mu(z)$ can be expressed in terms of the more familiar Laguerre polynomials

$$m^\mu_{n+\frac{1+\mu}{2}}(z) = \frac{n!}{\Gamma(n+\mu+1)} z^{\mu/2} e^{-z/2} L_n^\mu(z) .$$

However, the more general notation introduced by Buchholz in his book on confluent hypergeometric functions will be used throughout this article.

The generating function for the functions

$$\Omega_n^\mu(P) = \frac{\Gamma(1+n+\mu)}{n!} m^\mu_{n+\frac{1+\mu}{2}}(-2ik\xi) m^\mu_{n+\frac{1+\mu}{2}}(2ik\eta) e^{-i\mu\phi} \qquad n=0, 1, 2, \cdots$$

is known as the Hardy-Hille expansion (for proof and additional reference see [1].) For the sake of completeness, it will be stated as a theorem.

THEOREM 1. *For* $|t| < 1$, $\mu \neq -1, -2, \cdots$

$$(1) \qquad G_\mu(P, t) = \sum_{n=0}^\infty \Omega_n^\mu(P)(-t)^n = \frac{\exp\left[ik(\xi-\eta)\frac{1-t}{1+t}\right] J_\mu\left(\frac{4k\sqrt{\xi\eta t}}{1+t}\right) e^{-i\mu\phi}}{t^{\mu/2}(1+t)} .$$

The case in which μ is a negative integer must be treated with some care. From the limit relationship [1]

$$\lim_{\mu\to -m} m^\mu_{n+\frac{1+\mu}{2}}(-2ik\xi) m^\mu_{n+\frac{1+\mu}{2}}(2ik\eta)$$

$$= \begin{cases} \left[\frac{n!}{(n-m)!}\right]^2 m^m_{n+\frac{1-m}{2}}(-2ik\xi) m^m_{n+\frac{1-m}{2}}(2ik\eta) , & n \geq m \\ 0 , & n < m \end{cases}$$

it follows that

$$\lim_{\mu\to -m} G_\mu(P, t) = (-t)^m G_m(P, t)\, e^{2im\phi} .$$

A relationship between the spherical wave functions and the parabolic functions can now be established. The Fourier expansions of a plane wave in cylindrical and spherical coordinates respectively are [4]

$$\exp(ik[z\cos\Psi + \rho\cos\phi\sin\Psi]) = \sum_{m=0}^\infty i^m \varepsilon_m J_m(k\rho\sin\Psi)\, e^{ikz\cos\Psi} \cos m\phi ,$$

$$e^{ikr\cos\gamma} = \sqrt{\frac{\pi}{2kr}} \sum_0^\infty (2n+1) i^n J_{n+1/2}(kr) P_n(\cos\gamma) ,$$

$$\cos \gamma = \cos \theta \cos \Psi + \sin \theta \sin \Psi \cos \phi \ ,$$

$$P_n(\cos \gamma) = \sum_{m=0}^{n} \varepsilon_m \frac{(n-m)!}{(n+m)!} P_n^m(\cos \theta) P_n^m(\cos \Psi) \cos m\phi \ .$$

Comparison of coefficients of $\cos m\phi$ leads to

$$\exp(ikz \cos \Psi) J_m(k\rho \sin \Psi)$$
$$= \sum_{n=|m|}^{\infty} i^{n-m}(2n+1) \frac{(n-m)!}{(n+m)!} j_n(kr) P_n^m(\cos \theta) P_n^m(\cos \Psi) \ ,$$
$$m = 0, \pm 1, \pm 2, \cdots \ ,$$

where

$$j_n(kr) = \sqrt{\frac{\pi}{2kr}} J_{n+1/2}(kr) \ .$$

If we substitute $\frac{1-t}{1+t}$ for $\cos \Psi$ here, introduce parabolic coordinates, and then use Theorem 1, we obtain an expression for $G_n(P, t)$ in terms of spherical harmonics:

$$(2) \quad G_m(P, t) = \sum_{n=m}^{\infty} i^{n-m}(2n+1) \frac{(n-m)!}{(n+m)!} j_n(kr) P_n^m(\cos \theta) \frac{P_n^m\left(\frac{1-t}{1+t}\right) e^{-im\phi}}{t^{m/2}(1+t)} \ ,$$

$$r = \xi + \eta \ , \qquad \cos \theta = \frac{\xi - \eta}{\xi + \eta} \ .$$

The right-hand side of (2) can be expanded in a power series in t by using

$$\frac{P_n^m\left(\frac{1-t}{1+t}\right)}{t^{m/2}(1+t)} = \frac{(-)^m \frac{(n+m)!}{(n-m)!} {}_2F_1\left(m-n, m+n+1; 1+m; \frac{t}{1+t}\right)}{m!(1+t)^{m+1}} \ ,$$

The left-hand side of (2) has been defined as a power series in t by equation (1). Comparing coefficients of equal powers of t in this series leads to

$$\Omega_s^m(P) = \sum_{n=m}^{\infty} a(n; m, s) j_n(kr) P_n^m(\cos \theta) e^{-im\phi} \ ,$$

(3)

$$a(n; m, s) = \frac{i^{n+m}(2n+1)}{m!} \sum_{r=0}^{s} \frac{(-)^r (m-n)_{(r)} (m+n+1)_{(r)} (r+m+1)_{(s-r)}}{(m+1)_{(r)} (s-r)! \, r!} \ ,$$
$$m = 0, 1, 2, \cdots \ .$$

That the above series converges everywhere follows from the fact that $a(n;\, m,\, s)P_n^m(\cos\theta)$ behaves like a power of n for large n, but $j_n(kr)$ is $O\left(\frac{1}{n!}\right)$.

In order to find the inverse to the above relationship, the variable t is replaced by $\frac{w}{1-w}$ in (2). From the resulting power series expansion it now follows that

$$(4)\quad \sum_{s=0}^{l}(-)^s\frac{l![(m+l)!]^2}{(l-s)!(m+s)!}\Omega_s^m(P)$$
$$=\sum_{n=l+m}^{\infty} i^{n+m+2l}(2n+1)\frac{(n-m)!}{(n+m)!}b(n;\, m,\, l)j_n(kr)P_n^m(\cos\theta)e^{-im\phi}\,,$$

where

$$b(n;\, m,\, l)=\frac{(n+m+l)!}{(n-m-l)!}\,,\qquad m=0,\, 1,\, 2,\, \cdots.$$

The following vectors and matrices can now be defined:

$$a_l(m)=\sum_{s=0}^{l}(-)^{s+l}\frac{l![(m+l)!]^2}{(l-s)!(m+s)!}\Omega_s^m(P)\,,$$

$$A(m)=\begin{pmatrix} a_0(m)\\ a_1(m)\\ a_2(m)\\ \cdot\\ \cdot\\ \cdot \end{pmatrix},$$

$$\beta_n^{(m)}=i^{n+m}(2n+1)\frac{(n-m)!}{(n+m)!}j_n(kr)P_n^m(\cos\theta)e^{-im\phi}\,,$$

$$B(m)=\begin{pmatrix} \beta_m(m)\\ \beta_{m+1}(m)\\ \beta_{m+2}(m)\\ \cdot\\ \cdot\\ \cdot \end{pmatrix},$$

$$C(m)=\begin{pmatrix} b(m;\, m,\, 0) & b(m+1;\, m,\, 0) & b(m+2;\, m,\, 0) & \cdots\\ 0 & b(m+1;\, m,\, 1) & b(m+2;\, m,\, 1) & \cdots\\ 0 & 0 & b(m+2;\, m,\, 2) & \cdots\\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{pmatrix}.$$

With this notation the system of equations represented by (4) can be written as

$$A(m)=C(m)B(m)\,, \qquad m=0,\,1,\,\cdots\,. \tag{5}$$

In order to express the spherical functions in terms of parabolic functions it is necessary to invert the system (5). The inverse of the matrix $C(m)$ is given by

$$C^{-1}(m)=\begin{pmatrix} \gamma(m;\,m,\,0) & \gamma(m;\,m,\,1) & \gamma(m;\,m,\,2) & \cdots \\ 0 & \gamma(m+1;\,m,\,1) & \gamma(m+1;\,m,\,2) & \cdots \\ 0 & 0 & \gamma(m+2;\,m,\,2) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

where

$$\gamma(n;\,m,\,l)=\frac{(-)^{n+m+l}(2n+1)}{(m-n+l)!(m+l+n+1)!}\,.$$

To prove the assertion that this matrix is really the inverse of $C(m)$, it must be shown that

$$\sum_{i=j}^{k}\gamma(m+j;\,m,\,i)b(m+k;\,m,\,i)=\delta_{jk}\,.$$

We have

$$\sum_{i=j}^{k}\gamma(m+j;\,m,\,i)b(m+k;\,m,\,i)$$

$$=\sum_{i=j}^{k}\frac{(-)^{i+j}(2m+2j+1)(2m+k+i)!}{(i-j)!(2m+j+i+1)!(k-i)!}$$

$$=\frac{(2m+k+j)!}{(k-j)!(2m+2j)!}\,{}_2F_1(j-k,\,2m+k+j+1;\,2m+2j+2;\,1)$$

$$=\frac{(2m+k+j)!}{(k-j)!(2m+2j)!}\,\frac{\Gamma(2m+2j+2)\Gamma(1)}{\Gamma(2m+k+j+2)\Gamma(1+j-k)}=\begin{cases}0, & k\geq 1+j\\ 1, & k=j\,.\end{cases}$$

Use of the inverse matrix allows one to write

$$j_n(kr)P_n^m(\cos\theta)e^{-im\phi} \tag{6}$$

$$=\frac{(n+m)!}{(n-m)!}\sum_{j=n-m}^{\infty}\frac{i^{n+m}[(m+j)!]^2}{(j-n+m)!(m+n+j+1)!}\sum_{s=0}^{j}(-)^s\frac{j!}{(j-s)!(m+s)!}\,\Omega_s^m(P)\,.$$

One can now state

THEOREM 2. *For* $m=0,\,1,\,2,\,\cdots$

$$\Omega_s^m(P) = \sum_{n=m}^{\infty} a(n;\, m,\, s) j_n(kr) P_n^m(\cos\theta) e^{-im\phi}\,,$$

$$a(n;\, m,\, s) = \frac{i^{n+m}(2n+1)}{m!} \sum_{r=0}^{s} \frac{(-)^r (m-n)_{(r)}(n+m+1)_{(r)}(r+m+1)_{(s-r)}}{(m+1)_{(r)}(s-r)!\, r!}\,,$$

$$j_n(kr)P_n^m(\cos\theta)\, e^{-im\phi}$$

$$= \frac{(n+m)!}{(n-m)!} \sum_{j=n-m}^{\infty} \frac{i^{n+m}[(m+j)!]^2}{(j-n+m)!(m+n+j+1)!} \sum_{s=0}^{j} (-)^s \frac{j!}{(j-s)!\,(m+s)!} \Omega_s^m(P)\,.$$

It is not permissible to interchange the two summations in (6) because the coefficient of the inner summation is $O(1/j)$. Although the series does not converge absolutely it can be shown to converge conditionally. The inverse Laplace transform of the Kummer function is given by [2]

$${}_2F_1(-\sigma;\, 1+m;\, -2ik\xi) = \frac{m!(-2ik\xi)^{-m}}{2\pi i} \int_C \frac{\exp\left[-2ik\xi z\left(1-\frac{1}{z}\right)^{\sigma}\right]}{z^{m+1}}\, dz$$

where C is a circle enclosing the origin and $z=1$. If $\Omega_s^m(P)$ is expressed in terms of Kummer functions, then (6) can be rewritten as

$$j_n(kr)P_n^m(\cos\theta)e^{-im\phi} = \sum_{j=n-m}^{\infty} \frac{(n+m)!\, i^{n+m} e^{-m\phi}[(m+j)!]^2 (2k\sqrt{\xi\eta})^{-m} e^{ik(\xi-\eta)}}{(n-m)!\qquad (j-n+m)!(m+n+j+1)!}$$

$$\cdot \frac{1}{(2\pi i)^2} \int_C \int_{C'} \frac{e^{2ik(\eta\zeta-\xi z)}}{(z\zeta)^{m+1}} \left[\frac{1}{z} + \frac{1}{\zeta} - \frac{1}{z\zeta}\right]^j dz\, d\zeta\,.$$

On sufficiently large circles the quantity $\left[\frac{1}{z} + \frac{1}{\zeta} - \frac{1}{z\zeta}\right]$ becomes sufficiently small so that an interchange of summation and integrations is permissible and the series converges. One then obtains the double integral

$$j_n(kr)P_n^m(\cos\theta)e^{-im\phi} = \frac{(n+m)!\ e^{-im\phi} e^{ik(\xi-\eta)} i^{n+m} n!\ n!}{(n-m)!\ (2k\sqrt{\xi\eta})^{m+1}(2n+1)!}$$

$$\cdot \frac{1}{(2\pi i)^2} \int_C \int_{C'} \frac{e^{2ik(\eta\zeta-\xi z)}}{(\zeta z)^{m+1}} \left[\frac{1}{z} + \frac{1}{\zeta} - \frac{1}{z\zeta}\right]^{n-m} {}_2F_1\Big(n+1,\, n+1;\, 2n+2;$$

$$\cdot \frac{1}{\zeta} + \frac{1}{z} - \frac{1}{\zeta z}\Big) dz\, d\zeta\,.$$

As consequences of Theorem 2 and the integral relations [4]

$$\int_0^{\pi} P_n^m(\cos\theta) P_{n'}^m(\cos\theta) \sin\theta\, d\theta = \frac{2(n+m)!}{(2n+1)(n-m)!} \delta_{n,n'}$$

$$\int_0^\pi \frac{[P_n^m(\cos\theta)]^2\,d\theta}{\sin\theta} = \frac{(n+m)!}{m(n-m)!}$$

one can state the following.

COROLLARY 1.

$$\int_0^\pi [\Omega_s^m(P)]^2 \sin\theta\,d\theta = \sum_{n=m}^{\infty} [a(n;\,m,\,s) j_n(kr) e^{-im\phi}]^2 \frac{2(n+m)!}{(2n+1)(n-m)!}\ ,$$

$$\int_0^\pi \Omega_s^m(P) P_n^m(\cos\theta)\sin\theta\,d\theta = a(n;\,m,\,s) j_n(kr)\frac{2(n+m)!}{(2n+1)(n-m)!} e^{-im\phi}$$

$$\int_0^\pi \frac{\Omega_s^m(P)P_n^m(\cos\theta)\,d\theta}{\sin\theta} = \sum_{n=m}^{\infty} a(n;\,m,\,s) j_n(kr) e^{-im\phi}\frac{(n+m)!}{m(n-m)!}$$

$$\int_0^\pi \Omega_s^m(P)\Omega_\sigma^m(P)\sin\theta\,d\theta = \sum_{n=m}^{\infty} a(n;\,m,\,s)a(n;\,m,\,\sigma) j_n^2(kr) e^{-2im\phi}\frac{2(n+m)!}{(2n+1)(n-m)!}\ ,$$

3. The addition theorem resulting from a translation of the axes along the axis of symmetry.

Since z is the axis of symmetry one can introduce the translated coordinates

$$x' = x\ , \qquad y' = y\ , \qquad z' = z - \xi_0\ .$$

It follows from Theorem 1 that

$$(7)\quad G_\mu(P,\,t) = \frac{\exp\left[ikz\frac{1-t}{1+t}\right] J_\mu\left(\frac{2kp\sqrt{t}}{1+t}\right) e^{-i\mu\phi}}{t^{\mu/2}(1+t)} = \exp\left[ik\xi_0\frac{1-t}{1+t}\right] G_\mu(P',\,t)\ .$$

In particular, for $\mu = \eta = 0$, $\xi = \xi_0$ Theorem 1 yields

$$\exp\left[ik\xi_0\frac{1-t}{1+t}\right] = (1+t)\sum_{n=0}^{\infty} m_{n+1/2}^0(-2ik\xi_0)(-t)^n\ .$$

Using this expression in (7), expanding and multiplying the power series in t and comparing coefficients, we obtain the following.

THEOREM 3.

$$\Omega_n^\mu(P) = \sum_{j=0}^{n} [m_{n+1/2-j}^0(-2ik\xi_0) + m_{n-1/2-j}^0(-2ik\xi_0)(\delta_{nj}-1)]\Omega_j^\mu(P')\ ,$$

$$\mu \neq -1,\ -2,\ \cdots\ ; \quad n = 0,\ 1,\ 2,\ \cdots\ .$$

The case in which μ is a negative integer can be handled as a limiting case of Theorem 3. By differentiating both sides with respect to ξ_0 at $\xi_0=0$ one obtains the following.

COROLLARY 2.

$$\frac{d}{d(2ik\xi_0)}\Omega_n^{\mu}(P')\Big|_{\xi_0=0}=-\sum_{n=0}^{n}\Omega_j^{\mu}(P)\left(1-\frac{\delta_{jn}}{2}\right).$$

In particular for $\eta=0$ one obtains from the above

$$\frac{1}{n!}\Gamma(1+\mu+n)\frac{d}{d(2ik\xi)}m^{\mu}_{n+(1+\mu)/2}(-2ik\xi)$$
$$=\frac{1}{4ik\xi n!}\mu m^{\mu}_{n+(1+\mu)/2}(-2ik\xi)\Gamma(1+\mu+n)$$
$$+\sum_{j=0}^{n}\frac{1}{j!}\Gamma(1+\mu+j)m^{\mu}_{j+(1+\mu)/2}(-2ik\xi)\left(1-\frac{\delta_{jn}}{2}\right).$$

It is possible to define a vector

$$V^{\mu}(P)=\begin{pmatrix}\Omega_0^{\mu}(P)\\ \Omega_1^{\mu}(P)\\ \Omega_2^{\mu}(P)\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}$$

and a matrix

$$T(\xi_0)=\begin{pmatrix}a_{00} & 0 & 0 & \cdots\\ a_{10} & a_{11} & 0 & \cdots\\ a_{20} & a_{21} & a_{22} & \cdots\\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{pmatrix},$$

where

$$a_{nj}=\begin{cases}[m^0_{n+1/2-j}(-2ik\xi_0)+m^0_{n-1/2-j}(-2ik\xi_0)(\delta_{nj}-1)]\,, & n\geqq j\\ 0\,, & n<j\,,\end{cases}$$

such that Theorem 3 can be restated as follows.

THEOREM 3′.

$$V^{\mu}(P)=T(\xi_0)V^{\mu}(P')\qquad \mu\neq-1,\ -2,\ -3,\ \cdots.$$

4. The addition theorem resulting from a translation of axes perpendicular to the axis of symmetry.

The translation can be assumed to be in the x-direction without loss of generality. Introducing the new coordinates

$$x=x'-\delta\ ,\quad y=y'\ ,\quad z=z'\ ,$$

$$R=\sqrt{\rho^2+\delta^2-2\rho\delta\cos\phi'}\ ,$$

$$e^{2i(\phi-\phi')}=\frac{\rho-\delta e^{-i\phi'}}{\rho-\delta e^{i\phi'}}\ ,$$

$$P=(x,\ y,\ z)\ ,$$

$$P'=(x',\ y',\ z')\ ,$$

one obtains from Theorem 1

$$G_\mu(P,\ t)=\frac{\exp\left[ikz\frac{1-t}{1+t}\right]J_\mu\left(\frac{2kR\sqrt{t}}{1+t}\right)e^{-i\mu\phi}}{t^{\mu/2}(1+t)}\ .$$

Under the condition $\rho>\delta$ one can take advantage of the addition theorem for the Bessel functions

$$J_\mu(kR)e^{-i\mu\phi}=\sum_{-\infty}^{\infty}J_n(k\delta)J_{n+\mu}(kr)e^{-i(n+\mu)\phi'}$$

and obtain

$$\begin{aligned} G_\mu(P,\ t)&=\sum_{-\infty}^{\infty}J_n\left(\frac{2k\delta\sqrt{t}}{1+t}\right)t^{n/2}\frac{\exp\left[ikz\frac{1-t}{1+t}\right]J_{n+\mu}\left(\frac{2k\rho\sqrt{t}}{1+t}\right)e^{-i(n+\mu)\phi'}}{t^{(\mu+n)/2}(1+t)} \\ &=\sum_{-\infty}^{\infty}J_n\left(\frac{2k\delta\sqrt{t}}{1+t}\right)t^{n/2}G_{\mu+n}(P',\ t) \qquad \mu\neq\pm1,\ \pm2,\ \cdots\ . \end{aligned} \tag{8}$$

The case where μ is an integer must be handled as a limiting case. To determine the addition theorem one must expand both sides in powers of t and compare coefficients. Using

$$t^{-n/2}J_n\left(\frac{2k\delta\sqrt{t}}{1+t}\right)=\sum_{s=0}^{\infty}g_{s,n}t^s\ ,$$

$$g_{s,n}=\sum_{r=0}^{s}\frac{(k\delta)^{2s-2r+n}(-)^s(2s-2r+n)_{(r)}}{(s-r)!r!(n+s-r)!}\ ,$$

one obtains the following.

THEOREM 4.

$$(-)^s\Omega_s^{\mu}(P)=\sum_{n=1}^{s}\sum_{j=0}^{s} g_{s-j,n}\Omega_j^{\mu+n}(P')+\sum_{n=0}^{\infty}(-)^n\sum_{j=0}^{s} g_{s-j,n}\Omega_j^{\mu-n}(P') ,$$

for $\mu\neq\pm 1, \pm 2, \cdots$. *For* $\mu=m$, *with* m *a positive integer,*

$$(-)^s\Omega_s^{m}(P)=\sum_{j=0}^{s}\sum_{n=0}^{s} g_{s-j,n}\Omega_j^{n+m}(P')+\sum_{j=0}^{s}\sum_{n=m}^{j+m} g_{s-j,n}(-)^n\Omega_{j+m-n}^{n-m}(P')e^{2i(n-m)\phi'} .$$

For $\mu=-m$

$$\lim_{\mu\to -m}\Omega_n^{\mu}(P)=\begin{cases}\Omega_{n-m}^{m}e^{2im\phi} , & n\geq m\\ 0 , & n<m .\end{cases}$$

Another method by which such addition theorems can be derived is to take advantage of a theorem by Friedman [3], which is an addition theorem for spherical harmonics under translations of the coordinate system. This theorem in combination with Theorem 2 will yield an addition theorem, but in a very cumbersome form. Conversely the theorem for spherical harmonics could be derived by using Theorems 2 and 4.

A similar plan will be used in the next section. The addition theorem for spherical harmonics under rotations of the coordinate system in combination with Theorem 2 yields the corresponding theorem for parabolic functions.

5. The addition theorem resulting from a rotation of coordinates.

Since a rotation about the axis of symmetry, namely the z-axis, yields trivial results, a rotation about the y-axis will be used without loss of generality. Let

$$\begin{aligned} z&=z'\cos\Psi-x'\sin\Psi\\ x&=x'\cos\Psi+z'\sin\Psi \qquad (9)\\ y&=y' . \end{aligned}$$

Under this rotation the following addition theorem holds for the spherical harmonics [2]:

$$P_n^m(\cos\theta)e^{-im\phi}=\sum_{l=-n}^{n} g_l\frac{(n-|l|)!}{(n+|l|)!}S_{2n}^{n+m,n+l}(\Psi)P_n^{|l|}(\cos\theta')e^{-il\phi'} ,$$

where

$$S_{2n}^{n+m,n+l}(\Psi)=(-)^{n+m}\binom{n-m}{n+l}\left(\cos\frac{\Psi}{2}\right)^{-m-l}\left(i\sin\frac{\Psi}{2}\right)^{m-l}$$
$$\cdot {}_2F_1\left(-n-l, n-l+1; 1-m-l; \cos^2\frac{\Psi}{2}\right)$$

for $m+l\leq 0$, and

$$S_{2n}^{n+m,n+l}(\Psi)=-\binom{n+m}{n-l}\left(\cos\frac{\Psi}{2}\right)^{m+l}\left(-i\sin\frac{\Psi}{2}\right)^{l-m}$$
$$\cdot {}_2F_1\left(l-n,\ n+l+1;\ 1+m+l;\ \cos^2\frac{\Psi}{2}\right)$$

for $m+l>0$, and where

$$g_l=\begin{cases}1\,, & l\geq 0\\ (-1)^l\,, & l\leq 0\,.\end{cases}$$

Using the above in conjunction with Theorem 2 one can state the full addition theorem.

THEOREM 5. *Under a rotation of coordinates* (9) *the following statement holds*:

$$\Omega_s^m(P)=\sum_{n=m}^{\infty}a(n;\ m,\ s)\sum_{l=-n}^{n}g_l\,S_{2n}^{n+m,n+l}(\Psi)\sum_{j=n-|l|}^{\infty}\frac{i^{n+|l|}[(j+|l|)!]^2}{(j-n+|l|)!(j+n+|l|+1)!}$$
$$\cdot\sum_{s=0}^{j}(-)^s\frac{j!}{(j-s)!(m+s)!}\,\Omega_s^{|l|}(P')e^{i(|l|-l)\phi'}\,.$$

6. **The infinitesimal transformations.** It is possible to restate the addition theorems for infinitesimal transformations. The theorem for a translation along the z-axis can be rewritten from Theorem 3:

$$a_{n,j}=[m^0_{n+1/2-j}(-2ik\xi_0)+m^0_{n-1/2-j}(-2ik\xi_0)(\delta_{nj}-1)]\,,\qquad n\geq j\,,$$

where

$$m_k^0(z)=e^{-z/2}\,{}_1F_1\left(\frac{1}{2}-k;\ 1;\ z\right).$$

For small values of ξ_0, namely $d\xi_0$, it follows that

$$a_{n,j}=\begin{cases}\delta_{nj}+2ikd\xi_0\left(1-\dfrac{\delta_{nj}}{2}\right), & n\geq j\\ 0\,, & n<j\end{cases}$$

and that

$$T(d\xi_0)=I+ikd\xi_0\begin{pmatrix}1&0&0&\cdots\\2&1&0&\cdots\\2&2&1&\cdots\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{pmatrix},\qquad (10)$$

where I is the identity matrix.

THEOREM 3″. *Consider an infinitesimal translation along the z-axis such that*

$$x'=x\ ,\qquad y'=y\ ,\qquad z'=z-d\xi_0\ .$$

Then

$$V^{\mu}(P)=T(d\xi_0)V^{\mu}(P')\ ,\qquad\qquad \mu\neq-1,\ -2,\ \cdots\ ,$$

where $T(d\xi_0)$ *is given by* (10) *and* $V^{\mu}(P)$ *is as defined in Theorem* 3′.

Similarly one can find the addition theorem for translations in the x-direction from expression (8):

$$G_{\mu}(P,\ t)=\sum_{-\infty}^{\infty}J_n\left(\frac{2k\delta\sqrt{t}}{1+t}\right)t^{n/2}G_{\mu+n}(P',\ t)\ .$$

For a differential translation $d\delta$ this expression reduces to

$$G_{\mu}(P,\ t)=G_{\mu}(P',\ t)+\frac{kd\delta}{1+t}\left[tG_{\mu+1}(P',\ t)-G_{\mu-1}(P',\ t)\right]$$

from which it is possible to state

THEOREM 4′. *For an infinitesimal translation of coordinates given by*

$$x=x'-d\delta\ ,\qquad y=y'\ ,\qquad z=z'$$

the following holds:

$$\Omega_n^{\mu}(P)=\Omega_n^{\mu}(P')-kd\delta\left\{\sum_{l=0}^{n}\Omega_l^{\mu-1}(P')+\sum_{l=1}^{n-1}\Omega_l^{\mu+1}(P')\right\}\ ,\quad \mu\neq 0,\ -1,\ -2,\ \cdots\ .$$

For negative integral values of μ one can use limit processes.

To derive the analogous theorem for a rotation of coordinates it is first necessary to derive the addition theorem for the spherical harmonics. This can be done conveniently by starting with the following definition of the spherical harmonics [2]:

$$D_1^{n-m}(D_2+iD_3)^m\frac{1}{r}=\frac{(-)^{n-m}(n-m)!}{r^{n+1}}P_n^m(\cos\theta)e^{\pm im\phi}\ , \tag{11}$$

where

$$D_1=\frac{d}{dz}\ ,\quad D_2=\frac{d}{dx}\ ,\quad D_3=\frac{d}{dy}\ .$$

Under the rotation

$$x' = z \sin \Psi + x \cos \Psi$$

$$y' = y$$

$$z' = z \cos \Psi - \sin \Psi$$

these differential operators are also transformed:

$$D_1 = D_1' \cos \Psi + D_2' \sin \Psi$$

$$D_2 = -D_1' \sin \Psi + D_2' \cos \Psi$$

$$D_3 = D_3' .$$

Let

$$D_2 - iD_3 = Q , \qquad D_2 + iD_3 = \overline{Q} .$$

Then it follows that

$$D_1^{n-m} Q^m = \left[D_1' \cos \Psi + \frac{1}{2} \sin \Psi (Q' + \overline{Q}') \right]^{n-m} \left[-D_1' \sin \Psi + \frac{1}{2} \cos \Psi (Q' + \overline{Q}') + \frac{1}{2} (Q' - \overline{Q}') \right]^m . \tag{12}$$

The existence of the operational equivalence

$$Q\overline{Q} \frac{1}{r} \equiv -D_1^2 \frac{1}{r}$$

follows from

$$(D_1^2 + Q\overline{Q}) \frac{1}{r} \equiv \Delta \frac{1}{r} = 0 .$$

If Ψ is taken to be a differential angle $d\Psi$ in (12), then one obtains from (11)

$$e^{-im\phi} P_n^m(\cos \theta) = e^{-im\phi'} P_n^m(\cos \theta') - \frac{d\Psi}{2} [e^{-i(m+1)\phi'} P_n^{m+1}(\cos \theta') - (n+m)(n-m+1) e^{-i(m-1)\phi'} P_n^{m-1}(\cos \theta')] . \tag{13}$$

Equation (2) written in the form

$$G_m(P, t) = \sum_{n=m}^{\infty} i^{n+m} (2n+1) j_n(kr) P_n^m(\cos \theta) e^{-im\phi} \frac{{}_2F_1\left(m-n, m+n+1; m+1; \frac{t}{1+t}\right)}{m! (1+t)}$$

combined with (13) yields

$$G_m(P, t)=G_m(P', t)-\frac{d\Psi}{2}\sum_{n=m}^{\infty} i^{n+m}(2n+1)j_n(kr)P_n^{m+1}(\cos\theta')e^{-i(m+1)\phi'} \cdot \frac{{}_2F_1\left(m-n,\ m+n+1;\ m+1;\ \frac{t}{1+t}\right)}{m!\,(1+t)^m}$$

(14)

$$+\frac{d\Psi}{2}\sum_{n=m}^{\infty} i^{n+m}(2n+1)j_n(kr)P_n^{m-1}(\cos\theta')e^{-i(m-1)\phi'} \cdot \frac{(n+m)(n-m+1)\,{}_2F_1\left(m-n,\ m+n+1;\ m+1;\ \frac{t}{1+t}\right)}{m!\,(1+t)^m}\,.$$

In order to be able to rewrite the above as generating functions one can make use of the differentiation formulas [2]

$$\frac{d}{dz}[z^{m+1}(1-z)^{m+1}{}_2F_1(m-n+1,\ m+n+2;\ m+2;\ z)]$$
$$=(m+1)z^m(1-z)^m{}_2F_1(m-n,\ m+n+1;\ m+1;\ z)\ ,$$
$$\frac{d}{dz}[{}_2F_1(m-n-1,\ m+n;\ m;\ z)]$$
$$=\frac{-(n+m)(n-m+1)}{m}\,{}_2F_1(m-n,\ m+n+1;\ m+1;\ z)\ .$$

Using these in (14) one obtains

$$G_m(P, t)=G_m(P', t)+\frac{id\Psi}{2}\left\{\frac{(1+t)^{m+2}}{t^m}\frac{d}{dt}\left[\left(\frac{t}{1+t}\right)^{m+1}G_{m+1}(P', t)\right]\right.$$
$$\left.-(1+t)^{2-m}\frac{d}{dt}[(1+t)^{m-1}G_{m-1}(P', t)]\right\}$$

from which one derives

$$G_m(P, t)=G_m(P', t)+\frac{id\Psi}{2}\left[(m+1)G_{m+1}(P', t)+t(1+t)\frac{d}{dt}G_{m+1}(P', t)\right.$$
$$\left.-(m-1)G_{m-1}(P', t)-(1+t)\frac{d}{dt}G_{m-1}(P', t)\right].$$

One can now state the following.

THEOREM 5′. *Under the infinitesimal rotation*

$$x'=x+zd\Psi\ , \qquad y'=y\ , \qquad z'=z-xd\Psi$$

one has the formula

$$\Omega_n^m(P)=\Omega_n^m(P')+\frac{id\Psi}{2}\,[(m+1+n)\Omega_n^{m+1}(P')-(n-1)\Omega_{n-1}^{m+1}(P')$$
$$-(m+n-1)\Omega_n^{m-1}(P')+(n+1)\Omega_{n+1}^{m-1}(P')]\ .$$

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THE COEFFICIENT REGIONS OF STARLIKE FUNCTIONS

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1. The coefficient regions of schlicht functions have been studied at some length by Schaeffer, Schiffer, and Spencer [2, 3]. Properties of these coefficient regions are obtained only with difficulty, and in particular the actual coefficient regions can be computed only with a great deal of labor [2]. In fact, the computations necessary to determine the coefficient region of (a_2, a_3, a_4) probably would be prohibitive.

The class of starlike functions is of course much simpler in behavior. Since $f(z)=z+a_2z^2+a_3z^3+\cdots$ is starlike if and only if $zf'(z)/f(z)$ has a positive real part in $|z|<1$, one might say that everything is known about such functions. However, in practice, our rather complete knowledge about functions with positive real part proves difficult to apply back to the class of starlike functions. This is easily seen to be true by noting the number of papers on starlike functions which appear every year.

In an earlier paper, the writer presented a new variational method in the class of starlike functions. It is the purpose of this paper to apply this variational method to find the coefficient regions for starlike functions.

Let S^* be the class of all normalized functions $f(z)=z+a_2z^2+a_3z^3+\cdots$, schlicht and starlike in the unit circle. Let $\boldsymbol{V}_n^*$ be the $(2n-2)$ dimensional region composed of all points $(a_2, a_3, \cdots, a_n)$ belonging to the functions of S^*. Since the class of functions $p(z)$ with $p(0)=1$, regular and having a positive real part in $|z|<1$, is a compact family, so is S^*. Thus $\boldsymbol{V}_n^*$ is a closed domain (i.e., the closure of a domain).

We will study $\boldsymbol{V}_n^*$ by determining its cross sections with $a_2, a_3, \cdots, a_{n-1}$ held fixed. In § 2, a simple proof of the fact that each such cross section is convex is given. It is then shown that any point on the boundary of this cross section must lie on a particular circle, and thus that the cross section itself is a circle. The actual equations for the region $\boldsymbol{V}_n^*$ can be determined for each n by means of a simple recursion, but the calculation becomes tedious after the first few n.

2. For fixed $a_2, a_3, \cdots, a_{n-1}$, let $\boldsymbol{C}_n^*=\boldsymbol{C}_n^*(a_2, \cdots, a_{n-1})$ be the two dimensional cross section of $\boldsymbol{V}_n^*$ in which a_n varies.

LEMMA 1. *$\boldsymbol{C}_n^*$ is a closed, convex set.*

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Proof. C_n^* is certainly closed, since it is a cross section of the closed set V_n^*. To show that it is convex, we introduce a new variation.

If $f(z)$ and $g(z)$ belong to S^*, define for any ε, $0 \leq \varepsilon \leq 1$,

$$h_\varepsilon(z) = f(z)^{1-\varepsilon} g(z)^\varepsilon . \tag{1}$$

Here, appropriate branches of the powers are chosen so that $h_\varepsilon(z)$ is regular at the origin and has a series expansion $z + \cdots$ there. Taking the logrithmatic derivative of (1), we have,

$$\frac{zh_\varepsilon'(z)}{h_\varepsilon(z)} = (1-\varepsilon)\frac{zf'(z)}{f(z)} + \varepsilon\frac{zg'(z)}{g(z)} .$$

Therefore, if f and g are in S^*, so is $h_\varepsilon(z)$, for all ε between 0 and 1.

If $f(z)$ and $g(z)$ are any two functions of S^* belonging to C_n^*, say, $f(z) = f_0(z) + a_n z^n + \cdots$, $g(z) = f_0(z) + b_n z^n + \cdots$, where $f_0(z) = z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$, then by direct computation from (1),

$$h_\varepsilon(z) = f^{1-\varepsilon} g^\varepsilon = f_0\left(1 + \frac{a_n z^n}{f_0} + \cdots\right)^{1-\varepsilon}\left(1 + \frac{b_n z^n}{f_0}\right)^\varepsilon$$

$$= f_0 + [a_n - \varepsilon(a_n - b_n)]z^n + \cdots ,$$

and so, as ε goes from 0 to 1, the n-th coefficient of $h_\varepsilon(z)$ moves along the line between a_n and b_n. Therefore this entire line segment is contained in C_n^*, and the lemma is proved.

3. In an earlier paper [1], the writer showed by use of a variational method in the class of starlike functions, that any function $f(z)$ in S^* which maximizes $\Re\left\{\sum_{\nu=2}^{n} \lambda_\nu a_\nu\right\}$ must be of the form

$$f(z) = \frac{z}{\prod_{\nu=1}^{m}(1-\kappa_\nu z)^{\mu_\nu}} , \quad \mu_\nu \geq 0, \ \sum_{\nu=1}^{m} \mu_\nu = 2, \ m \leq n-1 \tag{2}$$

and that $f(z)$ must satisfy the differential equation

$$\frac{zf'(z)}{f(z)} R(z) = Q(z) , \tag{3}$$

where

$$\begin{cases} R(z) = \sum_{\nu=2}^{n}\left[\lambda_\nu \sum_{\mu=1}^{\nu-1} \frac{a_\mu}{z^{\nu-\mu}} - \lambda_\nu^* \sum_{\mu=1}^{\nu-1} a_\mu^* z^{\nu-\mu}\right] , \\ Q(z) = \sum_{\nu=2}^{n}\left[\lambda_\nu \sum_{\mu=1}^{\nu-1} \frac{\mu a_\mu}{z^{\nu-\mu}} + (\nu-1)\lambda_\nu a_\nu + \lambda_n^* \sum_{\mu=1}^{\nu-1} \mu a_\mu^* a^{\nu-\mu}\right] . \end{cases} \tag{4}$$

(Here, and throughout the paper, an asterisk attached to a value indicates the complex conjugate of that value.) The function $R(z)$ has m zeros on $|z|=1$ corresponding to the m poles of $f'(z)/f(z)$. The function $Q(z)$ has m zeros on $|z|=1$ corresponding to the tips of the m slits (where $f'(z)=0$). The functions $R(z)$ and $Q(z)$ have $2n-m-2$ additional zeros in common.

In order to study the coefficient regions, we will determine the nature of $\boldsymbol{C}_n^*$ $(a_2, \cdots, a_{n-1})$. Since $\boldsymbol{C}_n^*$ is convex, as shown above, the boundary points of $\boldsymbol{C}_n^*$ can be determined by finding a function which maximizes $\Re\{\lambda_n a_n\}$ for fixed $a_2, a_3, \cdots, a_{n-1}$ and for each $\lambda_n=e^{i\theta}$. If $f(z)$ maximizes $\Re\{\lambda_n a_n\}$, then it also maximizes $\Re\left\{\sum_{\nu=2}^{n} \lambda_\nu a_\nu\right\}$ where $\lambda_2, \lambda_3, \cdots, \lambda_{n-1}$ are a set of Lagrange multipliers which are determined by the fact that $a_2, \cdots, a_{n-1}$ must take on the prescribed values.

The desired results are obtained by use of $2n-m-2$ zeros which $R(z)$ and $Q(z)$ in (4) have in common. To this end, we obtain the GCD of $R(z)$ and $Q(z)$. The Euclidean algorithm is used in a simple form. That is, having two polynomials of the same degree.

$$p_1(z)=\alpha_0+\alpha_1 z+\cdots+\alpha_n z^n ,$$

$$p_2(z)=\beta_0+\beta_1 z+\cdots+\beta_n z^n ,$$

two new polynomials of lesser degree are obtained by the process

$$\text{(5)}\qquad \begin{cases} q_1(z)=\dfrac{1}{z}[\beta_0 p_1(z)-\alpha_0 p_2(z)] , \\ q_2(z)=\beta_n p_1(z)-\alpha_n p_2(z) . \end{cases}$$

This scheme is started by taking $Q(z)-R(z)$ and multiplying through by an appropriate power of z (the functions $Q(z)$ and $R(z)$ have no zeros at $z=0$ or $z=\infty$). From (4) this gives a polynomial

$$R_1(z)=\alpha_{0,1}+\alpha_{1,1}z+\cdots+\alpha_{n-2,1}z^{n-2}+\beta^*_{n-2,1}z^{n-1}+\cdots+\beta^*_{0,1}z^{2n-3}$$

where

$$\text{(6)}\qquad \begin{cases} \alpha_{\nu,1}=(\nu+1)\lambda_n a_{\nu+2}+\nu\lambda_{n-1}a_{\nu+1}+\cdots+\lambda_{n-\nu}a_2 , \\ \beta_{\nu,1}=(\nu+2)\lambda_n a_{\nu+1}+(\nu+1)\lambda_{n-1}a_\nu+\cdots+2\lambda_{n-\nu} . \end{cases}$$

In a similar fashion, taking $Q(z)+R(z)$ we obtain

$$Q_1(z)=\beta_{0,1}+\beta_{1,1}z+\cdots+\beta_{n-2,1}z^{n-2}+\alpha^*_{n-2,1}z^{n-1}+\cdots+\alpha^*_{0,1}z^{2n-3} .$$

The coefficients of $Q_1(z)$ are exactly the conjugates of the coefficients of $R_1(z)$ in reverse order. This is easily seen from (4), except that it must be noted that for the extremal $f(z)$, the center term $\sum_{\nu=2}^{n}(\nu-1)\lambda_\nu a_\nu$ is a purely real number, (see [3])

The polynomials $R_1(z)$ and $Q_1(z)$ have in common the same $2n-m-2$ zeros that $R(z)$ and $Q(z)$ have in common, and each has in addition $m-1$ other zeros. The latter zeros are distinct in $R_1(z)$ and $Q_1(z)$ since any common zero of $R_1(z)$ and $Q_1(z)$ must be a common zero of $R(z)$ and $Q(z)$.

This process may then be continued, combining $R_1(z)$ and $Q_1(z)$ as in the scheme (5) to produce two new polynomials $R_2(z)$ and $Q_2(z)$, each one lower in degree. It is easily seen from (5) that the relationship between the coefficients of $R_1(z)$ and $Q_1(z)$ will be preserved in the reduced polynomial. Thus, as this scheme is continued, pairs of polynomials $R_k(z)$ and $Q_k(z)$ of degree $2n-k-2$ will be produced. The coefficients of $Q_k(z)$ will be the conjugates of the coefficients of $R_k(z)$, in reverse order. $R_k(z)$ and $Q_k(z)$ will have in common the $2n-m-2$ zeros that $R(z)$ and $Q(z)$ have in common, and $m-k$ others, not in common. The process will terminate with $R_m(z)$ and $Q_n(z)$, for these two will then be identical up to a constant factor.

Because of the relationship between the coefficients, we need to determine only $R_k(z)$ for each k. The corresponding $Q_k(z)$ can be computed as needed.

LEMMA 2. *For $1 \leq k \leq m$, the polynomial $R_k(z)$ is of the form*

$$R_k(z)=\alpha_{0,k}+\alpha_{1,k}z+\cdots+\alpha_{n-k-1,k}z^{n-k-1}+\cdots+\beta^*_{n-k-1,k}z^{n-1}$$
$$+\beta^*_{n-k-2,k}z^n+\cdots+\beta^*_{0,k}z^{2n-k-2},$$

with

$$\alpha_{\mu,k}=\lambda_n A_{\mu+1,k}+\lambda_{n-1}A_{\mu,k}+\cdots+\lambda_{n-\mu}A_{1,k},$$
$$\beta_{\mu,k}=\lambda_n B_{\mu+1,k}+\lambda_{n-1}B_{\mu,k}+\cdots+\lambda_{n-\mu}B_{1,k}.$$

Here, each $A_{j,k}$ and each $B_{j,k}$ is a polynomial in the a_ν and their conjugates (independent of the λ_ν), and the $A_{j,k}$ and $B_{j,k}$ satisfy the recursion relations

$$(7)\qquad \begin{cases} A_{j,\nu+1}=B_{1,\nu}A_{j+1,\nu}-A_{1,\nu}B_{j+1,\nu}, \\ B_{j,\nu+1}=B^*_{1,\nu}B_{j,\nu}-A^*_{1,\nu}A_{j,\nu}. \end{cases}$$

Proof. We first remark that the coefficients of $R_k(z)$ belonging to powers of z between z^{n-k-1} and z^{n-1} are of no interest to us here. From

(6) we see that the form of the coefficients is as asserted in the lemma for $k=1$. Suppose now that the form is correct for $k=\nu$. Then using the scheme (5) (removing a common factor of λ_n) we can compute

$$\alpha_{\mu,\nu+1}=\frac{1}{\lambda_n}\beta_{0,\nu}\alpha_{\mu+1,\nu}-\frac{1}{\lambda_n}\alpha_{0,\nu}\beta_{\mu+1,\nu}$$

$$=B_{1,\nu}[\lambda_n A_{\mu+2,\nu}+\cdots+\lambda_{n-\mu}A_{2,\nu}+\lambda_{n-\mu-1}A_{1,\nu}]$$

$$-A_{1,\nu}[\lambda_n B_{\mu+2,\nu}+\cdots+\lambda_{n-\mu}B_{2,\nu}+\lambda_{n-\mu-1}B_{1,\nu}] .$$

Thus, $\alpha_{\mu,\nu+1}$ has the form asserted in the lemma, with the $A_{j,\nu+1}$ determined by the recursion formula (7). The other recursion formula and the remainder of the lemma is proved in an exactly similar fashion.

LEMMA 3. *For each j, k, $0\leq j\leq n-k-1$, $1\leq k\leq m$, the $A_{j,k}$ and $B_{j,k}$ of Lemma 2 satisfy the following*:

(i) $A_{j,k}$ *is a polynomial in* $a_2, a_3, \cdots, a_{j+k}$ *and* $a_2^*, a_3^*, \cdots, a_{k-1}^*$.

(ii) $B_{j,k}$ *is a polynomial in* $a_2, a_3, \cdots, a_{j+k-1}$ *and* $a_2^*, a_3^*, \cdots, a_k^*$.

(iii) $B_{1,k}$ *is real for any choice of* $a_2, a_3, \cdots, a_k$.

(iv) $$A_{j,k}=(j+k-1)B_{1,1}B_{1,2}\cdots B_{1,k-1}a_{j+k}-A_{1,1}B_{1,2}B_{1,3}\cdots B_{1,k-1}B_{j+k-1,1}$$
$$-A_{1,2}B_{1,3}B_{1,4}\cdots B_{1,k-1}B_{j+k-2,2}-\cdots-A_{1,k-2}B_{1,k-1}B_{j+2,k-2}$$
$$-A_{1,k-1}B_{j+1,k-1}.$$

(v) *For any* ν, $1\leq\nu<k$

$$B_{1,k}=B_{1,\nu}^2B_{1,\nu+1}B_{1,\nu+2}\cdots B_{1,k-1}-|A_{1,k-1}|^2-B_{1,k-1}|A_{1,k-2}|^2$$
$$-B_{1,k-1}B_{1,k-2}|A_{1,k-3}|^2-\cdots$$
$$-B_{1,k-1}B_{1,k-2}\cdots B_{1,\nu+1}|A_{1,\nu}|^2 .$$

Proof. From (6) we see that

$$(8)\qquad \begin{cases} A_{j,1}=ja_{j+1} , \\ B_{j,1}=(j+1)a_j, \quad (B_{1,1}=2) , \end{cases}$$

hence properties (i), (ii), and (iii) of the lemma hold true for $k=1$. Using the recursion formulas (7), properties (i) and (ii) can be verified inductively for all $k\leq m$. Property (iii) is obvious from (7) since $B_{1,k}=|B_{1,k-1}|^2-|A_{1,k-1}|^2$.

Property (iv) is clearly true for $k=1$ from (8). It also can be verified simply by induction on k.

Finally, property (v) is clearly true by (7) and (iii) for $\nu=k-1$ and any k, $1 < k \leqq m$. It can then be proved in general by backward induction on ν. Thus, from (7)

$$B_{1,\nu} = B_{1,\nu-1}^2 - |A_{1,\nu-1}|^2$$

and substituting this for one of the $B_{1,\nu}$ factors in the first term of (v), the corresponding formula for $\nu-1$ is obtained.

4. The reduction process given above must lead to $R_{m+1}(z) \equiv 0$ since $R_m(z)$ and $Q_m(z)$ have all of their roots in common. Therefore the extremal function $f(z)$, maximizing $\Re\left\{\sum_{\nu=2}^{n} \lambda_\nu a_\nu\right\}$, must have $|A_{1,m}| = |B_{1,m}|$ because of (7). We may now prove.

THEOREM 1. *Let* $(a_2, a_3, \cdots, a_{n-1}) \in V_{n-1}^*$. *If* $(a_2, a_3, \cdots, a_{n-1})$ *is an interior point of* V_{n-1}^* *then* $C_n^*(a_2, \cdots, a_{n-1})$ *is a circular disc determined by* $|A_{1,n-1}| = B_{1,n-1}$; *furthermore* $|A_{1,k}| < B_{1,k}$ *for* $k < n-1$. *If* $(a_2, \cdots, a_{n-1})$ *is a boundary point of* V_{n-1}^* *then* $C_n^*(a_2, \cdots, a_{n-1})$ *consists of a single point.*[1]

Proof. Note that the statement of this theorem makes the tacit assumption that $B_{1,k}$ (which is real by Lemma 3) is always non-negative. This of course will be true by (7) if we merely prove $|A_{1,k}| \leqq |B_{1,k}|$ for all k.

Given $(a_2, \cdots, a_{n-1})$ in V_{n-1}^*, Lemma 1 shows that the cross section C_n^* is convex. Hence, given any point a_n on the boundary of C_n^*, there is a line of support for C_n^* passing through this point, and therefore a λ_n such that the function (or functions) belonging to this point satisfy (2) and (3). The reduction process described above then leads to $|A_{1,m}| = |B_{1,m}|$ for some m, $1 \leqq m \leqq n-1$.

We now procede to prove the first half of the theorem by induction. If $n=2$, then m must be $n-1=1$, and hence the function corresponding to each boundary point of C_2^* must satisfy $|A_{1,1}| = B_{1,1}$, or, using the values from (8), there is some θ such that $a_2 = 2e^{i\theta}$. Therefore each boundary point of C_2^* is a point of this circle and hence C_2^* consists of the disc $|a_2| \leqq 2$. However, a_2 is an interior point of C_2^* if and only if $|A_{1,1}| < B_{1,1}$.

Now suppose $(a_2, \cdots, a_{n-1})$ is an interior point of V_{n-1}^*. Then a_ν is an interior point of $C_\nu^*(a_2, \cdots, a_{\nu-1})$ for $\nu = 2, \cdots, n-1$, and hence by the inductive hypothesis $|A_{1,\nu}| < B_{1,\nu}$ for $\nu = 1, 2, \cdots, n-2$. Therefore $m = n-1$ and each boundary point of C_n^* must, from (iv) of Lemma 3, satisfy

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[1] Professor G. Pólya has shown the writer that the fact that the cross sections are circular discs can easily be proved with the help of the Carathéodory theory for functions with positive real part. The exact expressions for these cross sections found from (9), (8), and (7) do not seem to be obtainable from the Carathéodory theory in any simple way however.

$$a_n = \frac{A_{1,1}B_{n-1,1}}{(n-1)B_{1,1}} + \frac{A_{1,2}B_{n-2,2}}{(n-1)B_{1,1}B_{1,2}} + \cdots + \frac{A_{1,n-2}B_{2,n-2}}{(n-1)B_{1,1}B_{1,2}\cdots B_{1,n-2}} \tag{9}$$

$$+ e^{i\theta}\frac{B_{1,n-1}}{(n-1)B_{1,1}B_{1,2}\cdots B_{1,n-2}}$$

$$= C_n + e^{i\theta}R_n ,$$

for some θ, $0 \leqq \theta \leqq 2\pi$. Then expressions C_n and R_n are rational functions of the a_ν and their conjugates and are defined by (9). In particular R_n is real and positive since $B_{1,n-1} = |B_{1,n-2}|^2 - |A_{1,n-2}|^2 > 0$.

From (9), each a_n on the boundary of $\boldsymbol{C}_n^*$ must lie on the circle with center C_n and radius R_n. This means that $\boldsymbol{C}_n^*$ is itself this circle. Thus if a_n is interior point of $\boldsymbol{C}_n^*$, we must have $|A_{1,n-1}| < B_{1,n-1}$. By induction, the first half of the theorem is proved.

Now suppose that $(a_2, \cdots, a_{n-1})$ is a boundary point of $\boldsymbol{V}_{n-1}^*$. Then there is a unique smallest $\nu \leqq n-1$ such that a_μ is an interior point of $\boldsymbol{C}_\mu^*(a_2, \cdots, a_{\mu-1})$ for $\mu = 2, \cdots, \nu-1$ and a_ν is a boundary point of $\boldsymbol{C}_\nu^*(a_2, \cdots, a_{\nu-1})$. But then $|A_{1,\nu-1}| = B_{1,\nu-1} > 0$, $|A_{1,\mu}| < B_{1,\mu}$ for $\mu < \nu - 1$ (and in particular $B_{1,\mu} > 0$ for $\mu = 1, 2, \cdots, \nu-1$), and $B_{1,\nu} = 0$. Choose a sequence of interior points $\{(a_2^{(j)}, \cdots, a_{n-1}^{(j)})\}$ of $\boldsymbol{V}_{n-1}^*$ which approach $(a_2, \cdots, a_{n-1})$. For each such point, $a_n^{(j)}$ is contained in a circle (9) of center $C_n^{(j)}$ and radius $R_n^{(j)}$. Now C_n is a rational function of the coefficients and their conjugates. Hence as $j \to \infty$, $C_n^{(j)}$ must approach some limit, finite or infinite. However this limit cannot be infinite since C_n is always bounded (indeed $|C_n| \leqq n$ because $|a_n| \leqq n$ for starlike functions). Thus the limiting value C_n must exist and be finite. On the other hand, the radius $R_n^{(j)} \to 0$, since by (v) of Lemma 3

$$R_n^{(j)} = \frac{B_{1,n-1}^{(j)}}{(n-1)B_{1,1}^{(j)} \cdots B_{1,n-2}^{(j)}} \leqq \frac{B_{1,\nu}^{(j)^2} B_{1,\nu+1}^{(j)} \cdots B_{1,n-2}^{(j)}}{(n-1)B_{1,1}^{(j)} \cdots B_{1,n-2}^{(j)}}$$

$$= \frac{B_{1,\nu}^{(j)}}{(n-1)B_{1,1}^{(j)} \cdots B_{1,\nu-1}^{(j)}} \to 0 .$$

Therefore, the cross section $\boldsymbol{C}_n^*(a_2, \cdots, a_{n-1})$ consists of the single point $C_n = \lim_{j\to\infty} C_n^{(j)}$. This completes the proof of the theorem.

5. With the help of the above theorem, we may now describe something of the nature of the coefficient region $\boldsymbol{V}_n^*$. The region $\boldsymbol{V}_n^*$ is $(2n-2)$-dimensional and its boundary is a $(2n-3)$-dimensional manifold. This manifold, however may be decomposed into $n-1$ parts. That is, the boundary of $\boldsymbol{V}_n^*$ is composed of $\Pi_n^{(1)}, \Pi_n^{(2)}, \cdots, \Pi_n^{(n-1)}$, where $\Pi_n^{(\nu)}$ is a $(2\nu-1)$-dimensional manifold lying on the surface of $\boldsymbol{V}_n^*$ and such

that $(a_2, a_3, \cdots, a_n)$ is in $\prod_n^{(\nu)}$ if and only if $(a_2, \cdots, a_\nu)$ is an interior point of V_ν^* and $(a_2, \cdots, a_{\nu+1})$ is a boundary point of $V_{\nu+1}^*$.

For example, from (9) we can explicitly calculate the first few cross sections C_2^*, C_3^*, C_4^*. The boundaries of these cross sections are given by

$$a_2 = 2e^{i\theta}, \tag{10}$$

$$a_3 = \frac{3a_2^2}{4} + e^{i\theta}\frac{4-|a_2|^2}{4}, \tag{11}$$

$$a_4 = \frac{4a_2a_3}{6} + \frac{(4a_3-3a_2^2)(6a_2-2a_2^*a_3)}{6(4-|a_2|^2)} + e^{i\theta}\frac{(4-|a_2|^2)^2-|4a_3-3a_2^2|^2}{6(4-|a_2|^2)}. \tag{12}$$

Taking for example V_4^*, the 5-dimensional manifold $\prod_4^{(3)}$ is defined by (10), (11), and (12) as a_2 varies in the interior of the disc (10), a_3 varies in the interior of the disc (11), and θ varies from 0 to 2π. The 3-dimensional manifold $\prod_4^{(2)}$ is determined by (10), (11), and

$$a_4 = \frac{4a_2a_3}{6} + e^{i\theta}\frac{6a_2-2a_2^*a_3}{6}$$

as a_2 varies in the interior of the disc (10) and θ varies from 0 to 2π. Finally, the 1-dimensional manifold $\prod_4^{(1)}$ is determined by $a_2=2e^{i\theta}$, $a_3=3e^{2i\theta}$, $a_4=4e^{3i\theta}$ and θ varies from 0 to 2π.

As a final remark, we may note that the coefficient regions V_n^* become quite "thin" as n becomes large. In fact, using (v) of Lemma 3

$$R_n = \frac{B_{1,n-1}}{(n-1)B_{1,1}\cdots B_{1,n-2}} \leq \frac{B_{1,1}^2B_{1,2}\cdots B_{1,n-2}}{(n-1)B_{1,1}\cdots B_{1,n-2}} = \frac{2}{n-1},$$

and hence the radius of any cross section C_n^* is less than or equal to $2/(n-1)$. This estimate in sharp since it is attained for $a_2=a_3=\cdots=a_{n-1}=0$, the functions being

$$f(z)=z(1-e^{i\theta}z^{n-1})^{-2/(n-1)}.$$

Since a function $f(z)$ is convex if and only if the function $zf'(z)$ is starlike, the structure of the coefficient regions for convex functions can be determined directly from the structure of the coefficient regions of starlike functions.

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ESTIMATES FOR THE EIGENVALUES OF INFINITE MATRICES

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1. Introduction. In most of the self-adjoint differential eigenvalue problems occurring in mathematical physics we are concerned with finding the extremal values of the quotient of two integro-differential quadratic forms in a certain space of admissible functions. By setting up a suitable basis in this space the problem can be reduced to that of finding the extremal values of a quotient of the form $(\alpha X, X)/(\beta X, X)$, where α and β are infinite symmetric matrices and X is a vector. The ordinary Rayleigh-Ritz method of approximating the solutions of the latter problem is to replace the infinite matrices $\alpha=(a_{ij})_1^\infty$ and $\beta=(b_{ij})_1^\infty$ by their finite sections $\alpha^n=(a_{ij})_1^n$ and $\beta^n=(b_{ij})_1^n$. The extremal values of the quotient $(\alpha^n X^n, X^n)/(\beta^n X^n, X^n)$, where X^n is an n dimensional vector, are the roots λ of the equation

$$\det(\alpha^n-\lambda\beta^n)=0\ , \tag{1}$$

and these are taken as approximations to the first n solutions of the original problem. If the roots of (1) are denoted by λ_k^n with $\lambda_1^n \geqq \lambda_2^n \geqq \cdots \geqq \lambda_n^n$, then for any fixed k, λ_k^n increases monotonically with n and its limit as $n\to\infty$ is the kth eigenvalue of the original problem. It should be stated here that the quotient of integro-differential quadratic forms in the original problem is taken as the reciprocal of the usual Rayleigh quotient so that the eigenvalues are all bounded.

If we let

$$\lambda_k=\lim_{n\to\infty}\lambda_k^n\ , \tag{2}$$

then the problem arises of estimating the difference $\lambda_k-\lambda_k^n$.

We shall consider this problem under certain assumptions with regard to the matrices α and β. These assumptions are that α and β are both positive definite, that the matrix $(b_{ij})_{n+1}^\infty$ has a positive lower bound independent of n, that the matrix $(a_{ij})_{n+1}^\infty$ has an upper bound which tends towards zero as $n\to\infty$, and that

$$\lim_{n\to\infty}\sum_{i=1}^n\sum_{j=n+1}^\infty a_{ij}^2=0\ , \qquad \lim_{n\to\infty}\sum_{i=1}^n\sum_{j=n+1}^\infty b_{ij}^2=0\ .$$

2. The simplest case, which we take up first, is that in which β

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is the unit matrix. Let $X_k^{(n)}$ be the orthonormal eigenvectors corresponding to the eigenvalues λ_k^n as defined above. Let numbers ε_n and ρ_n be defined by

$$\varepsilon_n \geqq \left(\sum_{i=1}^{n} \sum_{j=n+1}^{\infty} a_{ij}^2 \right)^{1/2}, \tag{3}$$

$$\rho_n \geqq \sup_{x_i} \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij} x_i x_j \Big/ \sum_{i=n+1}^{\infty} x_i^2 . \tag{4}$$

In general the exact values of the right-hand members of (3) and (4) will not be available, and for this reason we define ε_n and ρ_n as merely upper bounds for these quantities. The more closely these upper bounds can be estimated, the better will be the subsequent estimates of the eigenvalues. For the effectiveness of the method it is necessary that the values of ε_n and ρ_n can be made arbitrarily small for n sufficiently large. One method of defining ρ_n is to take it as an upper bound for $\left(\sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} a_{ij}^2 \right)^{1/2}$ in those cases where the latter series converges. A different method is given in the example of § 6.

We shall adopt the convention that, if X is a vector, $(x_i)_1^\infty$, then X^n stands for the n-dimensional vector $(x_i)_1^n$. Let $k \leq n < N$. By the minimax principle,

$$\lambda_k^N = \min_{U_i} \max_{X} \frac{(\alpha^N X^N, X^N)}{(X^N, X^N)}, \qquad (X^N, U_i^N) = 0,\ i = 1, 2, \cdots, k-1 . \tag{5}$$

Choose the vector U_i so that its first n components are equal respectively to those of $X_i^{(n)}$ and its remaining components are zero. Let

$$X = (x_i)_1^\infty, \quad y_1 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}, \quad y_2 = (x_{n+1}^2 + x_{n+2}^2 + \cdots + x_N^2)^{1/2} .$$

Then

$$\lambda_k^N \leqq \max_X \frac{(\alpha^N X^N, X^N)}{(X^N, X^N)}, \qquad (X^n, X_i^{(n)}) = 0,\ i = 1, 2, \cdots, k-1$$

$$= \max_X \left[(\alpha^n X^n, X^n) + 2 \sum_{i=1}^{n} \sum_{j=n+1}^{N} a_{ij} x_i x_j + \sum_{i=n+1}^{N} \sum_{j=n+1}^{N} a_{ij} x_i x_j \right] \Big/ (y_1^2 + y_2^2)$$

$$(X^n, X_i^{(n)}) = 0,\ i = 1, 2, \cdots, k-1$$

$$\leqq \max_{y_i} \frac{\lambda_k^n y_1^2 + 2\varepsilon_n y_1 y_2 + \rho_n y_2^2}{y_1^2 + y_2^2} .$$

The last step is justified by use of the maximum principle for the first term of the numerator and the Schwarz inequality for the second term.

The quantity on the right side of this inequality is the larger root λ of the equation

$$\begin{vmatrix} \lambda_k^n - \lambda & \varepsilon_n \\ \varepsilon_n & \rho_n - \lambda \end{vmatrix} = 0 .$$

Hence,

$$\lambda_k^N \leq \frac{\lambda_k^n + \rho_n + \sqrt{(\lambda_k^n - \rho_n)^2 + 4\varepsilon_n^2}}{2} ,$$

and, since the right side is independent of N,

$$\lambda_k^n \leq \lambda_k \leq \frac{\lambda_k^n + \rho_n + \sqrt{(\lambda_k^n - \rho_n)^2 + 4\varepsilon_n^2}}{2} . \tag{6}$$

If $\rho_n < \lambda_k^n$, this inequality gives the simpler, but less precise, one

$$\lambda_k^n \leq \lambda_k \leq \lambda_k^n + \frac{\varepsilon_n^2}{\lambda_k^n - \rho_n} . \tag{6a}$$

The inequality (6) (or 6a) makes it possible to obtain arbitrarily close bounds for λ_k by taking n sufficiently large.

Better estimates for λ_k can be obtained if one makes full use of the available data, namely λ_k^n and $X_k^{(n)}$. With these it is possible to transform α into an equivalent matrix (one having the same eigenvalues) $\bar{\alpha} = (\bar{a}_{ij})$, where

$$\begin{aligned} &\bar{a}_{kk} = \lambda_k^n && (k = 1, 2, \cdots, n) , \\ &\bar{a}_{ij} = 0 && (i, j = 1, 2, \cdots, n;\ i \neq j) , \\ &\bar{a}_{ij} = a_{ij} && (i, j = n+1, n+2, \cdots) , \\ &\sum_{i=1}^n \bar{a}_{ij}^2 = \sum_{i=1}^n a_{ij}^2 && (j = n+1, n+2, \cdots) . \end{aligned}$$

The actual formula for $\bar{\alpha}$ is $\bar{\alpha} = \Gamma^{tr} \alpha \Gamma$ where $\Gamma = \begin{pmatrix} \Gamma^{(n)} & 0 \\ 0 & E \end{pmatrix}$, $\Gamma^{(n)} = (X_1^{(n)}, X_2^{(n)}, \cdots, X_n^{(n)})$ and the vectors $X_k^{(n)}$ are orthonormal.

Let

$$\varepsilon_{nk} \geq \left(\sum_{j=n+1}^{\infty} \bar{a}_{kj}^2 \right)^{1/2} \qquad (k = 1, 2, \cdots, n) . \tag{7}$$

If any one of the numbers ε_{nk} is equal to zero, then the corresponding eigenvalue λ_k^n of $\bar{\alpha}^n$ is actually an eigenvalue of $\bar{\alpha}$ and the kth row and column of $\bar{\alpha}$ can be deleted before proceeding with any further calculations. We may therefore assume without loss of generality that all the numbers ε_{nk} appearing in subsequent formulas are different from zero.

Apply (5) with α^N replaced by $\bar{\alpha}^N$ and with U_i equal to the vector whose ith component is 1 and whose remaining components are zero. This gives, with $y=(x_{n+1}^2+\cdots+x_N^2)^{1/2}$

$$(8)\qquad \lambda_k^N \leqq \frac{\lambda_k^n x_k^2+\lambda_{k+1}^n x_{k+1}^2+\cdots+\lambda_n^n x_n^2+2\sum_{i=k}^{n}\sum_{j=n+1}^{N}\bar{a}_{ij}x_i x_j+\sum_{i=n+1}^{N}\sum_{j=n+1}^{N}a_{ij}x_i x_j}{x_k^2+x_{k+1}^2+\cdots+x_N^2}$$

$$\leqq \frac{\lambda_k^n x_k^2+\cdots+\lambda_n^n x_n^2+2\sum_{i=k}^{n}\varepsilon_{ni}|x_i|y+\rho_n y^2}{x_k^2+\cdots+x_n^2+y^2} .$$

The maximum value of the quotient

$$\frac{\lambda_k^n x_k^2+\cdots+\lambda_n^n x_n^2+2\sum_{i=k}^{n}\varepsilon_{ni}x_i y+\rho_n y^2}{x_k^2+\cdots+x_n^2+y^2}$$

can be attained when the variables $x_k, \cdots, x_n, y$ are restricted to non-negative values. Hence λ_k^N cannot exceed the largest root λ of the equation

$$(9)\qquad \begin{vmatrix} \lambda_k^n-\lambda & 0 & \cdots & 0 & \varepsilon_{nk} \\ 0 & \lambda_{k+1}^n-\lambda & \cdots & 0 & \varepsilon_{n,k+1} \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \lambda_n^n-\lambda & \varepsilon_{nn} \\ \varepsilon_{nk} & \varepsilon_{n,k+1} & \cdots & \varepsilon_{nn} & \rho_n-\lambda \end{vmatrix}$$

$$=(\rho_n-\lambda)\prod_{i=k}^{n}(\lambda_i^n-\lambda)-\sum_{j=k}^{n}\frac{\varepsilon_{nj}^2\prod_{i=k}^{n}(\lambda_i^n-\lambda)}{\lambda_j^n-\lambda}=0 .$$

If a number r appears $m+1$ times in the set $\lambda_k^n, \lambda_{k+1}^n, \cdots, \lambda_n^n$, then this number is an m-fold root of (9). If $\mu_1>\mu_2>\cdots>\mu_l$ are the distinct values in the set $\lambda_k^n, \lambda_{k+1}^n, \cdots, \lambda_n^n$, then (9) also has roots $r_1, r_2, \cdots, r_{l+1}$, where $r_1<\mu_1<r_2<\mu_2<\cdots<\mu_l<r_{l+1}$. The latter roots are all the roots of the equation

$$(9a)\qquad \lambda-\rho_n=\sum_{j=k}^{n}\frac{\varepsilon_{nj}^2}{\lambda-\lambda_j^n} .$$

3. As a simple example illustrating the estimates of the last section, let us take the problem of finding the eigenvalues Λ defined by

$$y''=-\Lambda(1+x)y , \qquad (0<x<1) ,$$

$$y(0)=y(1)=0 .$$

The reciprocals of these will be the extremal values $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ of the quotient

$$Q(y)=\int_0^1 (1+x)y^2\,dx \Big/ \int_0^1 y'^2\,dx$$

in the space $\mathscr{F}$ consisting of all functions $y(x)$ with sectionally continuous first derivatives and with $y(0)=y(1)=0$. As a basis for this space we take

$$\varphi_n(x)=\frac{\sqrt{2}\,\sin n\pi x}{n\pi} \qquad (n=1, 2, \cdots)$$

and let

$$a_{ij}=\int_0^1 (1+x)\varphi_i\varphi_j\,dx=\begin{cases} \dfrac{3}{2i^2\pi^2} & \text{if } i=j\,, \\ \dfrac{4[(-1)^{i-j}-1]}{\pi^4(i^2-j^2)^2} & \text{if } i\neq j\,, \end{cases}$$

$$b_{ij}=\int_0^1 \varphi_i'\varphi_j'\,dx=\delta_{ij}\,.$$

If $y=\sum_{i=1}^{\infty} x_i\varphi_i$, then

$$Q(y)=\frac{(\alpha X, X)}{(\beta X, X)}\,,$$

where $\alpha=(a_{ij})_1^\infty$, $\beta=(b_{ij})_1^\infty$, $X=(x_i)_1^\infty$, so the problem is reduced to one of the type for which the estimates of the last section apply.

Let $n=3$. The equation for λ_1^3, λ_2^3, λ_3^3 is

$$\begin{vmatrix} \dfrac{3}{2\pi^2}-\lambda & -\dfrac{8}{9\pi^4} & 0 \\ -\dfrac{8}{9\pi^4} & \dfrac{3}{8\pi^2}-\lambda & -\dfrac{8}{25\pi^4} \\ 0 & -\dfrac{8}{25\pi^4} & \dfrac{3}{18\pi^2}-\lambda \end{vmatrix}=0\,.$$

The eigenvalues and eigenvectors are:

$$\lambda_1^3=.1527\ 0819\,, \qquad X_1^{(3)}=(.99684,\ -.07935,\ .00192)\,,$$

$$\lambda_2^3=.0377\ 8273\,, \qquad X_2^{(3)}=(.07869,\ .98480,\ -.15482)\,,$$

$$\lambda_3^3=.0163\ 7316\,, \qquad X_3^{(3)}=(.01040,\ .15449,\ .98794)\,.$$

We make the following estimates

$$\sum_{j=4}^{\infty} a_{1j}^2 = \frac{64}{\pi^8} \sum_{\sigma=2}^{\infty} \frac{1}{(4\sigma^2-1)^4}$$

$$< \frac{64}{\pi^8}\left[\frac{1}{15^4} + \frac{1}{35^4} + \frac{1}{63^4} + \sum_{\sigma=5}^{\infty} \frac{1}{(3\sigma^2)^4}\right]$$

$$< \frac{64}{\pi^8}\left[\frac{1}{15^4} + \frac{1}{35^4} + \frac{1}{63^4} + \frac{1}{81}\int_4^{\infty} \frac{dx}{x^8}\right] = 1.389 \times 10^{-7} ,$$

$$\sum_{j=4}^{\infty} a_{2j}^2 = \frac{64}{\pi^8} \sum_{\sigma=2}^{\infty} \frac{1}{[(2\sigma+1)^2-4]^4}$$

$$< \frac{64}{\pi^8}\left[\frac{1}{21^4} + \frac{1}{45^4} + \frac{1}{77^4} + \frac{1}{256}\int_4^{\infty} \frac{dx}{x^8}\right] = .368 \times 10^{-7} ,$$

$$\sum_{j=4}^{\infty} a_{3j}^2 = \frac{64}{\pi^8} \sum_{\sigma=2}^{\infty} \frac{1}{(4\sigma^2-9)^4}$$

$$< \frac{64}{\pi^8}\left[\frac{1}{7^4} + \frac{1}{27^4} + \frac{1}{55^4} + \frac{1}{81}\int_4^{\infty} \frac{dx}{x^8}\right] = 28.234 \times 10^{-7} ,$$

$$\sum_{j=4}^{\infty} a_{1j}a_{3j} = \frac{64}{\pi^8} \sum_{\sigma=2}^{\infty} \frac{1}{(4\sigma^2-1)^2(4\sigma^2-9)^2}$$

$$< \frac{64}{\pi^8}\left[\frac{1}{15^2\cdot 7^2} + \frac{1}{35^2\cdot 27^2} + \frac{1}{63^2\cdot 55^2} + \frac{1}{81}\int_4^{\infty} \frac{dx}{x^8}\right] = 6.206 \times 10^{-7} ,$$

$$\sum_{j=4}^{\infty} a_{1j}a_{2j} = \sum_{j=4}^{\infty} a_{2j}a_{3j} = 0 ,$$

$$\sum_{j=4}^{\infty} (a_{1j}^2 + a_{2j}^2 + a_{3j}^2) < 29.991 \times 10^{-7} = \varepsilon_3^2 ,$$

$$\sum_{i,j=1}^{\infty} a_{ij}^2 = \frac{9}{4\pi^4} \sum_{\sigma=4}^{\infty} \frac{1}{\sigma^4} + \frac{128}{\pi^8} \sum_{n=2}^{\infty} \sum_{\sigma=0}^{\infty} \frac{1}{[(2n+2\sigma+1)^2-4n^2]^4}$$

$$+ \frac{128}{\pi^8} \sum_{n=2}^{\infty} \sum_{\sigma=1}^{\infty} \frac{1}{[(2n+2\sigma)^2-(2n+1)^2]^4}$$

$$< \frac{9}{4\pi^4} \sum_{\sigma=4}^{\infty} \frac{1}{\sigma^4} + \frac{128}{\pi^8}\left\{ \sum_{n=2}^{\infty} \sum_{\sigma=0}^{\infty} \frac{1}{[4n(1+2\sigma)]^4}\right.$$

$$\left. + \sum_{n=2}^{\infty} \sum_{\sigma=1}^{\infty} \frac{1}{[2(2n+1)(2\sigma-1)]^4}\right\}$$

$$= \frac{9}{4\pi^4} \sum_{\sigma=4}^{\infty} \frac{1}{\sigma^4} + \frac{8}{\pi^8} \sum_{\sigma=0}^{\infty} \frac{1}{(1+2\sigma)^4}\left\{ \sum_{n=2}^{\infty} \frac{1}{(2n)^4} + \sum_{n=2}^{\infty} \frac{1}{(2n+1)^4}\right\}$$

$$= \sum_{\sigma=4}^{\infty} \frac{1}{\sigma^4}\left[\frac{9}{4\pi^4} + \frac{8}{\pi^8} \cdot \frac{\pi^4}{96}\right] = .00017\ 9117 = \rho_3^2 ,$$

$$\rho_3 = .013\ 3835 .$$

If the matrix α is transformed into the equivalent matrix $\bar{\alpha}$ in which the upper left hand 3×3 matrix is diagonalized, the formulas for the elements $\bar{a}_{ij}$ are (for $j \geqq 4$):

$$\bar{a}_{1j} = .99684\ a_{1j} - .07935\ a_{2j} + .00192\ a_{3j} ,$$

$$\bar{a}_{2j} = .07869\ a_{1j} + .98480\ a_{2j} - .15482\ a_{3j} ,$$

$$\bar{a}_{3j} = .01040\ a_{1j} + .15449\ a_{2j} + .98794\ a_{3j} .$$

Hence,

$$\sum_{j=4}^{\infty} \bar{a}_{1j}^2 < 1.395 \times 10^{-7} = \varepsilon_{31}^2 ,$$

$$\sum_{j=4}^{\infty} \bar{a}_{2j}^2 < 1.042 \times 10^{-7} = \varepsilon_{32}^2 ,$$

$$\sum_{j=4}^{\infty} \bar{a}_{3j}^2 < 27.630 \times 10^{-7} = \varepsilon_{33}^2 .$$

The first three extremal values of the quotient $Q(y)$ can now be estimated by either (6), (6a), or (9a). From (6) we get

$$.152\ 708 \leqq \lambda_1 \leqq .152\ 730 ,$$

$$.037\ 782 \leqq \lambda_2 \leqq .037\ 905 ,$$

$$.016\ 373 \leqq \lambda_3 \leqq .017\ 167 ;$$

whereas (9a) yields the following more precise estimates:

$$.152\ 7081 \leqq \lambda_1 \leqq .152\ 7092 ,$$

$$.037\ 7827 \leqq \lambda_2 \leqq .037\ 7871 ,$$

$$.016\ 3731 \leqq \lambda_3 \leqq .017\ 1139 .$$

4. **Returning to the general problem,** let us assume that, by a preliminary transformation, the matrices α and β are already diagonalized in the $n\times n$ upper left-hand corner; that is, that

$$a_{ii} = \lambda_i^n , \quad b_{ii} = 1 \qquad (i=1, 2, \cdots, n) ,$$

$$a_{ij} = b_{ij} = 0 \qquad (i, j=1, 2, \cdots, n;\ i \neq j) .$$

Let the bounds ρ_n and ε_{nk} be defined by (4) and (7) (with $\bar{a}_{kj}$ replaced by a_{kj}): In addition let bounds δ_{nk} and r_n be defined by

(10) $$\delta_{nk} \geqq \left(\sum_{j=n+1}^{\infty} b_{kj}^2 \right)^{1/2} \qquad (k=1, 2, \cdots, n) ,$$

(11) $$r_n \leqq \inf_{x_i} \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} b_{ij} x_i x_j \Big/ \sum_{i=n+1}^{\infty} x_i^2 .$$

We assume that all these bounds exist, that

(12) $$r_n > \sum_{k=1}^{n} \delta_{nk}^2 ,$$

and that $\varepsilon_{nk} + \delta_{nk} \neq 0$ $(k=1, 2, \cdots, n)$ (see remark following (7)).

By the minimax principle with $k \leqq n < N$,

$$\lambda_k^N = \min_{U_i} \max_{X} \frac{(\alpha^N X^N, X^N)}{(\beta^N X^N, X^N)} , \qquad (\beta^N X^N, U_i) = 0, \ i=1, 2, \cdots, k-1 .$$

Proceeding as before, let U_i be the vector whose ith component is 1 and whose remaining components are zero. Then

$$\lambda_k^N \leqq \max_{x_i} \frac{\lambda_k^n x_k^2 + \cdots + \lambda_n^n x_n^2 + 2 \sum_{i=k}^{n} \sum_{j=n+1}^{N} a_{ij} x_i x_j + \sum_{i=n+1}^{N} \sum_{j=n+1}^{N} a_{ij} x_i x_j}{x_k^2 + \cdots + x_n^2 + 2 \sum_{i=k}^{n} \sum_{j=n+1}^{N} b_{ij} x_i x_j + \sum_{i=n+1}^{N} \sum_{j=n+1}^{N} b_{ij} x_i x_j}$$

$$\leqq \max_{x_i} \frac{\lambda_k^n x_k^2 + \cdots + \lambda_n^n x_n^2 + 2 \sum_{i=k}^{n} \varepsilon_{ni} |x_i| y + \rho_n y^2}{x_k^2 + \cdots + x_n^2 - 2 \sum_{i=k}^{n} \delta_{ni} |x_i| y + r_n y^2} ,$$

where $y = (x_{n+1}^2 + x_{n+2}^2 + \cdots + x_N^2)^{1/2}$. The condition (12) is equivalent to the positive definiteness of the denominator of the last expression. Hence, λ_k^N and therefore λ_k, cannot exceed the largest root λ of the equation

(13) $$\begin{vmatrix} \lambda_k^n - \lambda & \cdots & 0 & \varepsilon_{nk} + \lambda \delta_{nk} \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & \lambda_n^n - \lambda & \varepsilon_{nn} + \lambda \delta_{nn} \\ \varepsilon_{nk} + \lambda \delta_{nk} & \cdots & \varepsilon_{nn} + \lambda \delta_{nn} & \rho_n - \lambda r_n \end{vmatrix}$$

$$= (\rho_n - \lambda r_n) \prod_{i=k}^{n} (\lambda_i^n - \lambda) - \sum_{j=k}^{n} (\varepsilon_{nj} + \lambda \delta_{nj})^2 \frac{\prod_{i=k}^{n} (\lambda_i^n - \lambda)}{\lambda_j^n - \lambda} = 0 ,$$

which is the same thing as the largest root of the equation

(13a) $$\lambda r_n - \rho_n = \sum_{j=k}^{n} \frac{(\varepsilon_{nj} + \lambda \delta_{nj})^2}{\lambda - \lambda_j^n} .$$

To analyze the location of the largest root of (13a), let

$$\varphi(\lambda)=\sum_{j=k}^{n}\frac{(\varepsilon_{nj}+\lambda\delta_{nj})^2}{\lambda-\lambda_j^n} .$$

Then

$$\varphi'(\lambda)=\sum_{j=k}^{n}\left[\frac{2\delta_{nj}(\varepsilon_{nj}+\lambda\delta_{nj})}{\lambda-\lambda_j^n}-\frac{(\varepsilon_{nj}+\lambda\delta_{nj})^2}{(\lambda-\lambda_j^n)^2}\right],$$

$$\varphi''(\lambda)=2\sum_{j=k}^{n}\frac{(\varepsilon_{nj}+\lambda_j^n\delta_{nj})^2}{(\lambda-\lambda_j^n)^3},$$

$$\lim_{\lambda\to\infty}\varphi'(\lambda)=\sum_{j=k}^{n}\delta_{nj}^2 .$$

For $\lambda>\lambda_k^n$, $\varphi''(\lambda)>0$, and therefore in this range the graph of $\varphi(\lambda)$ can intersect that of the function $r_n\lambda-\rho_n$ in at most two points. Since $\lim_{\lambda\to\lambda_k^n+}\varphi(\lambda)=+\infty$ and since, by (12), $r_n\lambda-\rho_n>\varphi(\lambda)$ for all λ sufficiently large, there must be exactly one point of intersection, that is, one root of (13) or (13a), in the range $\lambda>\lambda_k^n$. This root is the upper bound which we obtain for λ_k.

Let us now assume that

$$r_n\lambda_k^n-\rho_n\geqq a>0 \tag{14}$$

for all n sufficiently large, and that

$$\lim_{n\to\infty}\sum_{j=1}^{n}(\varepsilon_{nj}^2+\delta_{nj}^2)=0 . \tag{15}$$

Then, for any $\varepsilon>0$, and for n sufficiently large, $\varphi(\lambda_k^n+\varepsilon)<r_n(\lambda_k^n+\varepsilon)-\rho_n$ and so the largest root of (13) or (13a) is less than $\lambda_k^n+\varepsilon$. Therefore, (14) and (15) are sufficient to ensure that the method gives arbitrarily close bounds on λ_k, for any k, by taking n sufficiently large.

5. To illustrate the method of the last section let us consider the problem:

$$\frac{d}{dx}\left((1+x)\frac{dy}{dx}\right)=-\Lambda y \qquad (0<x<1) ,$$

$$y(0)=y(1)=0 .$$

The reciprocals of the eigenvalues Λ of this problem are the extremal values of the quotient

$$Q(y)=\int_0^1 y^2\,dx\Big/\int_0^1(1+x)y'^2\,dx$$

on the space of functions $y(x)$ with sectionally continuous first derivatives and with $y(0)=y(1)=0$. If $\{\varphi_n(x)\}_1^\infty$ is a basis in this space and

$$a_{ij}=\int_0^1\varphi_i\varphi_j\,dx\ ,\qquad b_{ij}=\int_0^1(1+x)\varphi_i'\varphi_j'\,dx\ ,$$

then the problem is reduced to that of finding the extremal values of the quotient $(\alpha X, X)/(\beta X, X)$, where $\alpha=(a_{ij})_1^\infty$, $\beta=(b_{ij})_1^\infty$.

Let the sequence $\{\varphi_n\}$ be defined as follows:

$$\varphi_i=\sum_{j=1}^{3}c_{ij}\sin j\pi x \qquad (i=1,\,2,\,3)\ ,$$

$$\varphi_i=\frac{\sqrt{2}\ \sin i\pi x}{i\pi} \qquad (i>3)\ ,$$

where the constants c_{ij} are chosen in such a way that

$$(b_{ij})_1^3=E\ ,$$

$$(a_{ij})_1^3=\begin{pmatrix} .0696\ 820 & 0 & 0\\ 0 & .0173\ 553 & 0\\ 0 & 0 & .0073\ 9145\end{pmatrix}.$$

The values of the constants c_{ij} are given by the table:

$i\diagdown j$	1	2	3
1	.3713655	.0378935	.0039777
2	−.0189824	.1828646	.0301791
3	.0007276	−.0197241	.1199722

We now apply the method of the last section with $n=2$. Since the matrix α is of diagonal form, ε_{21} and ε_{22} may be taken as zero and ρ_2 may be taken as the maximum of the elements a_{ii} $(i\geqq 3)$, namely $a_{33}=.0073\ 9145$.

For $i=1,\,2$ we have

$$\sum_{j=3}^{\infty}b_{ij}^2=\sum_{j=4}^{\infty}b_{ij}^2$$

$$=2\pi^2\sum_{j=4}^{\infty}\left(\int_0^1(1+x)(c_{i1}\cos\pi x+2c_{i2}\cos 2\pi x+3c_{i3}\cos 3\pi x)\cos j\pi x\,dx\right)^2$$

$$=2\pi^2\sum_{j=4}^{\infty}\left[c_{i1}^2\left(\int_0^1(1+x)\cos\pi x\cos j\pi x\,dx\right)^2+4c_{i2}^2\left(\int_0^1(1+x)\cos 2\pi x\cos j\pi x\,dx\right)^2\right.$$

$$+ 9c_{i3}^2\left(\int_0^1 (1+x)\cos 3\pi x \cos j\pi x\, dx\right)^2 + 6c_{i1}c_{i3}\left(\int_0^1 (1+x)\cos \pi x \cos j\pi x\, dx\right)$$

$$\times\left(\int_0^1 (1+x)\cos 3\pi x \cos j\pi x\, dx\right)\Big]$$

$$= \frac{8}{\pi^2}\Big[c_{i1}^2 \sum_{\sigma=2}^{\infty} \frac{(1+4\sigma^2)^2}{(4\sigma^2-1)^4} + 4c_{i2}^2 \sum_{\sigma=2}^{\infty} \frac{(4+(2\sigma+1)^2)^2}{((2\sigma+1)^2-4)^4}$$

$$+ 9c_{i3}^2 \sum_{\sigma=2}^{\infty} \frac{(9+4\sigma^2)^2}{(4\sigma^2-9)^4} + 6c_{i1}c_{i3} \sum_{\sigma=2}^{\infty} \frac{(1+4\sigma^2)(9+4\sigma^2)}{(4\sigma^2-1)^2(4\sigma^2-9)^2}\Big] .$$

We make the following estimates :

$$\sum_{\sigma=2}^{\infty} \frac{(1+4\sigma^2)^2}{(4\sigma^2-1)^4} < \frac{17^2}{15^4} + \frac{37^2}{35^4} + \frac{65^2}{63^4} + \frac{1}{15}\sum_{\sigma=5}^{\infty} \frac{1}{\sigma^4} = .00712722 ,$$

$$\sum_{\sigma=2}^{\infty} \frac{(4+(2\sigma+1)^2)^2}{((2\sigma+1)^2-4)^4} < \frac{29^2}{21^4} + \frac{53^2}{45^4} + \frac{85^2}{77^4} + \frac{5}{4}\sum_{\sigma=5}^{\infty} \frac{1}{(2\sigma+1)^4} = .00541918 ,$$

$$\sum_{\sigma=2}^{\infty} \frac{(9+4\sigma^2)^2}{(4\sigma^2-9)^4} < \frac{25^2}{7^4} + \frac{45^2}{27^4} + \frac{73^2}{55^4} + \frac{1}{8}\sum_{\sigma=5}^{\infty} \frac{1}{\sigma^4} = .26514737 ,$$

$$\sum_{\sigma=2}^{\infty} \frac{(1+4\sigma^2)(9+4\sigma^2)}{(4\sigma^2-1)^2(4\sigma^2-9)^2} < \frac{17\cdot 25}{15^2\cdot 7^2} + \frac{37\cdot 45}{35^2\cdot 27^2} + \frac{65\cdot 73}{63^2\cdot 55^2} + \frac{1}{8}\sum_{\sigma=5}^{\infty} \frac{1}{\sigma^4}$$

$$= .04125482 .$$

This gives

$$\sum_{j=3}^{\infty} b_{1j}^2 < .0011490 = \delta_{21}^2 ,$$

$$\sum_{j=3}^{\infty} b_{2j}^2 < .0023514 = \delta_{22}^2 .$$

To obtain a value for r_2 we let $F(x) = \sum_{i=3}^{N} x_i\varphi_i(x)$, where $(x_i)_3^N$ is any given vector. Then

$$\int_0^1 F'^2(x)\, dx = x_3^2\int_0^1 \varphi_3'^2 dx + \sum_{i=4}^{N} x_i^2$$

$$= .646936x_3^2 + \sum_{i=4}^{N} x_i^2 \geqq .646936 \sum_{i=3}^{N} x_i^2 ,$$

$$\int_0^1 (1+x)F'^2(x)\, dx = \sum_{i=3}^{N}\sum_{j=3}^{N} b_{ij}x_ix_j .$$

Hence,

$$\frac{\sum_{i=3}^{N}\sum_{j=3}^{N} b_{ij}x_ix_j}{\sum_{i=3}^{N} x_i^2} \geq \frac{\int_0^1 (1+x)F'^2(x)\,dx}{\int_0^1 F'^2(x)\,dx}(.646\ 936) \geq .646936 .$$

Since the bound on the right side is independent of N we may take

$$r_2 = .646936 .$$

The use of equation (13a) now gives the following results, where λ_1 and λ_2 are the reciprocals of the first two eigenvalues of the original problem :

$$.06968 \leq \lambda_1 \leq .06984 ,$$

$$.01735 \leq \lambda_2 \leq .01754 .$$

6. **In conclusion** we shall show how the method would work on the two dimensional problem of an oscillating square membrane of variable density ; namely,

$$u_{xx} + u_{yy} = -\Lambda g u \qquad \text{in} \quad R ,$$

$$u = 0 \qquad \text{on} \quad C ,$$

where R is the region $0 < x < 1$, $0 < y < 1$, C is the boundary of and g is a nonnegative function with the derivative g_{xy} sectionally continuous in $R+C$. The reciprocals of the eigenvalues Λ are the extremal values of the quotient

$$Q(u) = \int_0^1\int_0^1 gu^2\,dx\,dy \Big/ \int_0^1\int_0^1 (u_x^2 + u_y^2)\,dx\,dy$$

in the space of functions $u(x, y)$ with sectionally continuous first derivatives in $R+C$ and vanishing on C.

As a basis for this problem we take the functions

$$\frac{2 \sin m\pi x \sin n\pi y}{\pi(m^2+n^2)^{1/2}} , \qquad m, n = 1, 2, 3, \cdots ,$$

and arrange them in a sequence $\varphi_1, \varphi_2, \varphi_3, \cdots$ ordered according to the value of m^2+n^2; that is,

$$\varphi_i = \frac{2 \sin m_i\pi x \sin n_i\pi y}{\pi\sigma_i} , \qquad \sigma_i = (m_i^2 + n_i^2)^{1/2} ,$$

$$\sigma_1 \leq \sigma_2 \leq \sigma_3 \cdots .$$

As $N \to \infty$, $\sigma_N = O(\sqrt{N})$. Let

$$a_{ij}=\int_0^1\int_0^1 g\varphi_i\varphi_j\,dx\,dx\ ,$$

$$b_{ij}=\int_0^1\int_0^1\left(\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x}+\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}\right)dx\,dy=\delta_{ij}\ .$$

If $u=\sum_{i=1}^{\infty}x_i\varphi_i$, then

$$Q(u)=(\alpha X,\ X)/(\beta X,\ X)$$

where

$$\alpha=(a_{ij})_1^\infty\ ,\qquad \beta=(\delta_{ij})_1^\infty\ ,\qquad X=(x_i)_1^\infty\ .$$

In order to show that the method will give arbitrarily close estimates of the eigenvalues, we must show that the quantity defined in (4) can be determined and made arbitrarily small, and that $\sum_{i=1}^{n}\sum_{j=n+1}^{\infty}a_{ij}^2$ can be made arbitrarily small by taking n sufficiently large. The estimate ρ_n can be managed by noting that (4) is equivalent, in the present case, to

$$\rho_n\geq\sup_{v\in a_n}\int_0^1\int_0^1 gv^2\,dx\,dy\Big/\int_0^1\int_0^1(v_x^2+v_y^2)\,dx\,dy\ ,$$

where a_n is the set of admissible functions which are orthogonal to $\varphi_1,\varphi_2,\cdots,\varphi_n$. Let $g\leq M$ in R. Then we may define ρ_n by

(16) $$\rho_n=\sup_{v\in a_n}M\int_0^1\int_0^1 v^2\,dx\,dy\Big/\int_0^1\int_0^1(v_x^2+v_y^2)\,dx\,dy\ ,$$

and this gives

(17) $$\rho_n=\frac{M}{\pi^2\sigma_{n+1}^2}=O\left(\frac{1}{n}\right)$$

since the functions $\{\varphi_i\}$ are the extremal functions for the quotient in (16).

Next, the numbers a_{ij} satisfy

$$|a_{ij}|\leq\frac{C}{\sigma_i\sigma_j}\varDelta_{ij}\bar{\varDelta}_{ij}$$

where C is an absolute constant, and

$$\varDelta_{ij}=\begin{cases}\dfrac{1}{|m_i-m_j|} & \text{if}\quad m_i\neq m_j\ ,\\[2ex] 1 & \text{if}\quad m_i=m_j\ ,\end{cases}$$

$$\bar{\Delta}_{ij}=\begin{cases}\dfrac{1}{|n_i-n_j|} & \text{if} \quad n_i\neq n_j\ ,\\ 1 & \text{if} \quad n_i=n_j\ .\end{cases}$$

Hence, for $1\leq i\leq n$,

$$\sum_{j=n+1}^{\infty} a_{ij}^2 \leq \frac{C^2}{\sigma_i^2\sigma_{n+1}^2}\sum_{j=n+1}^{\infty}\Delta_{ij}^2\bar{\Delta}_{ij}^2\ ,$$

and

$$\sum_{j=n+1}^{\infty}\Delta_{ij}^2\bar{\Delta}_{ij}^2<\left(1+2\sum_{s=1}^{\infty}\frac{1}{s^2}\right)^2\ ,$$

so

$$\sum_{j=n+1}^{\infty} a_{ij}^2<\frac{C_1}{i(n+1)}\ .$$

Therefore,

$$\sum_{i=1}^{n}\sum_{j=n+1}^{\infty} a_{ij}^2<\frac{C_2\log n}{n} \qquad (n>1)\ ,$$

where C_1 and C_2 are absolute constants.

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PERTURBATION OF DIFFERENTIAL OPERATORS

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Introduction. N. Dunford, in a series of papers [3, 4, 5], has initiated the study of operators on Banach spaces that allow a representation analogous to the Jordan canonical form for operators on a finite dimensional vector space. Such operators he has called spectral operators. They include, of course, self-adjoint operators which have found such wide application to problems of analysis. J. Schwartz [9] has exhibited an interesting class of spectral operators which contains many classical ordinary differential operators. His chief tool was a perturbation theorem that guarantees that if T is a regular spectral operator with a discrete spectrum that converges to infinity sufficiently rapidly and B is a bounded operator, then $T+B$ is again a regular spectral operator. This result provides a tool for showing that second order differential operators with suitable boundary conditions are regular spectral but does not suffice for proving this property for differential operators of higher order. This paper refines the method of J. Schwartz to allow application also to differential operators of higher order by showing that under certain conditions a regular spectral operator T may be perturbed by an unbounded operator S with the result that $T+S$ is still regular spectral.

The paper is divided into three parts. The first part presents preliminary notions and lemmas to be used in part II where the principal theoretical tool is fashioned in Theorem 1. Its object is to set forth conditions under which an operators is spectral (see Definition 1). This problem is attacked in the following form. Suppose that T is known to be a spectral operator. Under what hypotheses on T and a perturbing operator S may it be said that the operator $T+S$ is spectral? An answer to this question is given in Theorem 1. This theorem is then applied in the third part to differential operators of even order with "separated" boundary conditions on a finite interval. First, the simple operator defined by means of the formal differential operator $\frac{d^{2\mu}}{dx^{2\mu}}$ and "separated" boundary conditions is shown to be spectral. Then, with the aid of Theorem 1, the perturbed operator

$$\frac{d^{2\mu}}{dx^{2\mu}}+\sum_{i=2}^{2\mu} Q_i \frac{d^{2\mu-i}}{dx^{2\mu-i}}$$

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where Q_i may be any bounded operator on $\mathscr{L}_2(0, 1)$ is seen to be spectral as well.

1. Preliminaries.

N. Dunford [3, p. 560] has laid down the following.

DEFINITION 1. Let X be Banach space and T a transformation on X to X. If $E(e)$ is an operator valued function of Borel sets in the complex plane and

(a) $E(e)E(g)=E(e \cap g)$, $E(e')=I-E(e)$, $TE(e)=E(e)T$,

(b) $E(e)x$ is completely additive in e for each $x \in X$,

(c) the spectrum of T, with domain and range restricted to $E(e)X$, is contained in the closure of e, and

(d) there exists a constant M such that for every Borel set e $||E(e)|| < M$, then $E(e)$ is called a *resolution of the identity* for T and T is called a *spectral operator*.

The preceding definition covers a wide class of operators. In what is to follow, attention is focussed on a very restricted subset consisting of the regular spectral operators. The meaning of the adjective regular is clarified as follows.

DEFINITION 2. An operator T is *regular* if the resolvent set $\rho(T) \neq \phi$ and if for some $\lambda \in \rho(T)$, $(T-\lambda)^{-1}$ is completely continuous. (To be abbreviated c.c.)

Note that the spectrum of the c.c. operator $R_\lambda(T) \equiv (T-\lambda)^{-1}$ consists of a sequence of isolated points converging to 0.

It follows by the spectral mapping theorem [12, p. 324 et seq.] that the spectrum of T consists of a sequence of points λ_n converging to ∞.

In the sequel, the condition

$$I = \sum_{k=1}^{\infty} E(\lambda_k)$$

shall sometimes be made in regard to the spectral measure of a regular spectral operator T. The above condition asserts that the spectral measure corresponding to the point at infinity is the null operator or $\mu=0$ is not an eigenvalue of T^{-1}. The existence of T^{-1} as a c.c. operator may be assumed without loss of generality in view of the following.

LEMMA 1. *If $\lambda_0 \in \rho(T)$ and $R_{\lambda_0}(T)$ is c.c., then $R_\lambda(T)$ is c.c. for all $\lambda \in \rho(T)$.*

Proof. The first resolvent identity [6, p. 99] states that for $\lambda_0 \in \rho(T)$

and $\lambda \in \rho(T)$

$$R_\lambda(T) = R_{\lambda_0}(T) + (\lambda - \lambda_0) R_\lambda(T) R_{\lambda_0}(T) . \tag{1}$$

The product of a bounded operator and a c.c. operator is c.c. and the sum of two c.c. operators is again c.c. Thus it is apparent from (1) that $R_\lambda(T)$ is c.c. for all $\lambda \in \rho(T)$.

LEMMA 2[1]. *If S is a closed operator and B is a bounded operator and $\mathscr{D}(S) \supset \mathscr{R}(B)$, then SB is bounded.*

Proof. SB is closed. For suppose that $x_n \to x$ and $SBx_n \to y$. Since B is continuous, $Bx_n \to Bx$. But since S is closed $S(Bx_n) \to S(Bx) = (SB)x = y$. Thus SB is a closed operator defined on all of X and therefore by virtue of the closed graph theorem [1, p. 41] it is bounded.

LEMMA 3. *Let J be a finite set of integers and suppose that B_n is a set of bounded operators and E_n a set of mutually orthogonal projections[2], both sets being indexed by J. Then*

$$\|\sum_{n \in J} E_n B_n\|^2 \leqq \sum_{n \in J} \|B_n\|^2 .$$

Proof. Let $f \in H$ and $\|f\| = 1$. It is an easy consequence of the Hermitian nature of E_n and Schwarz' Lemma that

$$\begin{aligned} \|\sum_{n \in J} E_n B_n f\|^2 &= \sum_{n \in J} \sum_{k \in J} (E_n B_n f,\ E_k B_k f) \\ &\leqq \sum_{n \in J} |(B_n f,\ E_n B_n f)| \leqq \sum_{n \in J} \|B_n f\|\ \|E_n B_n f\| \\ &\leqq \sum_{n \in J} \|B_n\|^2 . \end{aligned}$$

In the sequel, reference shall be made several times to the following.

CONDITION A. All but a finite number of the idempotents[2] $E(\lambda_k)$ associated with the points of the spectrum of T project onto a one-dimensional range and

$$I = \sum_{k=1}^{\infty} E(\lambda_k) .$$

For a regular spectral operator, the last statement is equivalent to the assertion that the range of

[1] If T is an operator then $\mathscr{D}(T)$ denotes its domain and $\mathscr{R}(T)$ its range.

[2] An *idempotent* is an operator E such that $E = E^2$. Idempotents E_1 and E_2 will be called *orthogonal* if $E_1 E_2 = 0$. If $E = E^*$, then E is a *projection*.

$$E_\infty = I - \sum_1^\infty E(\lambda_k)$$

consists only of the null vector.

CONDITION B. Let d_k denote the distance between λ_k and the rest of the spectrum of T. Then there exists a number $\tau > 0$ such that

$$\sum_{k=1}^\infty d_k^{-\tau} < \infty \; .$$

For use in the theorem to follow, it is necessary to define explicitly the concept of a fractional power for the special class of operators with which the theorem is concerned.

In this definition an application shall be made of a theorem of Lorch [8] and Mackey which asserts that if $E(e)$ is a uniformly bounded spectral measure, then there exists a nonsingular transformation of Hilbert space into itself such that $WE(e)W^{-1}$ is a Hermitian spectral measure.

Let T be a regular spectral operator on Hilbert space H which satisfies Condition A. Let $\mathscr{P}$ be the finite set of characteristic values λ for which the idempotents $E(\lambda)$ project onto ranges of multiple dimension. Let W be the automorphism of H into itself which carries the spectral measure $E(e)$ of T into the Hermitian spectral measure $E'(e) = WE(e)W^{-1}$ of $T' = WTW^{-1}$.

Since $E'(\mathscr{P}) \perp I - E'(\mathscr{P})$, the two projections effect a unique decomposition of H into a direct sum

$$H = H_1 \oplus H_2$$

where

$$H_1 = E'(\mathscr{P})H$$

and

$$H_2 = \{I - E'(\mathscr{P})\}H \, .$$

Now

$$T' = T_1' + T_2'$$

where

$$T_1' = T'E'(\mathscr{P})$$

and

$$T_2' = T'\{I - E'(\mathscr{P})\} \; .$$

Upon restricting the domain of T'_1 to H_1 and that of T'_2 to H_2 one is confronted by a finite dimensional operator $\hat{T}'_1$ and a normal operator $\hat{T}'_2$.

If $-1<\nu<1$, the function $f(\lambda)=\lambda^\nu$ of the complex variable λ is regular on the spectrum of $\hat{T}'_1$ provided $0\notin\sigma(T)$ (which is no essential limitation of generality) and $f(\lambda)$ is restricted to its principal value.

Then, following Dunford [4], one defines

$$(\hat{T}'_1)^\nu=\sum_{i=1}^{P}\sum_{m=0}^{\mu_i-1}\frac{(\hat{T}'_1-\lambda_i)^m}{m!}\{\nu(\nu-1)\cdots(\nu-m+1)\lambda_i^{\nu-m}\hat{E}'(\lambda_i)$$

where μ_i is the order of the pole λ_i or the resolvent and $\hat{E}'(\lambda_i)$ is the restriction of $E'(\lambda_i)$ to $E'(\mathscr{P})H$. Since T'_2 is normal one has the spectral decomposition

$$\hat{T}'_2=\sum_{k=1}^{\infty}\lambda_k\hat{E}'(\lambda_k)$$

and by the operational calculus for normal operators (cf. [10, pp. 48–51] for example)

$$(\hat{T}'_2)^\nu=\sum_{k=1}^{\infty}\lambda_k^\nu\hat{E}'(\lambda_k)$$

Now define $(T'_1)^\nu$ and $(T'_2)^\nu$ by the rules

$$f_1\in H_1\Longrightarrow(T'_1)^\nu f_1=(\hat{T}'_1)^\nu f_1\qquad(T'_2)^\nu f_1=0$$

$$f_2\in H_2\Longrightarrow(T'_1)^\nu f_2=0\qquad(T'_2)^\nu f_2=(\hat{T}'_2)^\nu f_2\,.$$

Then

$$(T')^\nu\equiv(T'_1)^\nu+(T'_2)^\nu$$

and finally,

$$T\equiv W^{-1}(T')^\nu W\,.$$

The proof of the perturbation theorem below strongly depends on the operational calculus for spectral operators developed by N. Dunford and explicitly adapted to the case at hand by J. Schwartz [9]. For the sake of ready reference the pertinent results are presented here.

If T is a regular operator with a finite set of characteristic numbers

$$\{\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n\}$$

which are multiple poles of the resolvent and

$$\sum_{k=1}^{\infty} E(\lambda_k)=I$$

and $f(\lambda)$ is a complex-valued function which is uniformly bounded on the spectrum of T and possesses the required derivatives, then

$$f(T)=\sum_{i=1}^{n}\sum_{j=0}^{\nu_i-1}\frac{f^{(j)}(\lambda_i)}{j!}(T-\lambda_i)^j E(\lambda_i)+\sum_{j=n+1}^{\infty} f(\lambda_i)E(\lambda_i)\,.$$

For such an $f(T)$ Dunford [5] has shown that the series defining it converges in the strong operator topology and that there exists a constant $K(T)$ such that

$$\|f(T)\|\leq K\cdot\max_{\lambda\in\sigma(T)}|f(\lambda)|\,.$$

On the basis of this result J. Schwartz [9] enunciates the following.

LEMMA. *If S is a regular spectral operator all but a finite set of whose eigenvalues λ_n are simple poles of the resolvent, and if S also satisfies*

$$\sum_{i=1}^{\infty} E(\lambda_i)=I\,,$$

then there exists an absolute constant K such that

$$\|(\lambda-S)^{-1}\|\leq K/\text{dist}\,(\lambda,\sigma(S))$$

for all λ not within a fixed radius ε of any multiple pole of the resolvent.

In the theorem below let it be understood that

$$\mathscr{D}(T+S)=\mathscr{D}(T)\cap\mathscr{D}(S)\,.$$

2. The perturbation theorem. The principal result of the present paper is the following.

THEOREM 1. *Let T be a regular spectral operator on Hilbert space H and suppose that it satisfies conditions A and B. Let S be such a closed operator that for some ν, $0<\nu<1$, $\mathscr{D}(S)\supset\mathscr{D}(T^\nu)$ and $\mathscr{D}(S^*)\supset\mathscr{D}(T^{*\nu})$. Moreover, suppose that for all but a finite set P of positive integers, for all*

$$\lambda\in C_n=\left\{\lambda\,\middle|\,|\lambda-\lambda_n|=\frac{1}{3}d_n\right\}\Longrightarrow\max_{\mu\in\sigma(T)}\frac{|\mu^\nu|}{|\lambda-\mu|}<\frac{M}{d_n^{\tau/2}}\,.$$

Under the above hypotheses, $T+S$ is again a regular spectral operator.

Proof. Since, for $\lambda \in \rho(T)$,

$$\mathscr{R}(R_\lambda(T)) = \mathscr{D}(T) \subset \mathscr{D}(T^\nu) \subseteqq \mathscr{D}(S)\,,$$

$SR_\lambda(T)$ is well defined and is, in fact, by Lemma 2, a bounded operator. By the same token $ST^{-\nu}$ is bounded. In order to show that $T+S$ is regular, it need merely be ascertained that $R_\lambda(T+S)$ is c.c. at one point $\lambda \in \rho(T)$ and for this purpose we examine the formula

$$(1) \qquad R_\lambda(T+S) = R_\lambda(T)\{I - SR_\lambda(T)\}^{-1}$$

which is valid for $\lambda \in \rho(T)$ provided only that $\{I-SR_\lambda(T)\}^{-1}$ exists. If, $\{I-SR_\lambda(T)\}^{-1}$ not only exists but is also bounded, then $R_\lambda(T+S)$ as the product of a c.c. and a bounded operator is itself c.c.

But the hypotheses of the theorem allow one to state that $\{I-SR_\lambda(T)\}^{-1} < 2$ for $\lambda \in C_n$ and all n sufficiently large. This is proved as follows

$$SR_\lambda(T) = ST^{-\nu}T^\nu\{\lambda I - T\}^{-1} = ST^{-\nu}\{(\lambda I - T)T^{-\nu}\}^{-1}\,.$$

By Dunford's operational calculus and the hypotheses of the theorem it is true that

$$\|\{(\lambda I - T)T^{-\nu}\}^{-1}\| \leqq M_2 \max_{\mu \in \sigma(T)} \frac{|\mu^\nu|}{|\lambda - \mu|} \leqq M \frac{1}{d_n^{7/2}}$$

Let $\|ST^{-\nu}\| = M_1$. Then

$$(2) \qquad \|SR_\lambda(T)\| \leqq \frac{M_1 M_2}{d_n^{7/2}} = \frac{M}{d_n^{7/2}}$$

and since in view of Condition B, $\lim d_n^{-7/2} = 0$, one has for all n sufficiently large $\|SR_\lambda(T)\| < 1/2$ while $\lambda \in C_n$. From this estimate follows the possibility of expanding

$$\{I - SR_\lambda(T)\}^{-1} = I + SR_\lambda(T) + (SR_\lambda(T))^2 + \cdots$$

in an absolutely convergent series so that

$$(3) \qquad \|\{I - SR_\lambda(T)\}^{-1}\| < \frac{1}{1 - 1/2} = 2\,.$$

It is immediate from the above that if λ lies outside the assemblage of circles C_k, then for each $\mu_k \in \sigma(T)$ we have

$$\frac{|\mu_k^\nu|}{|\lambda - \mu_k|} < \frac{|\mu_k^\nu|}{|\lambda' - \mu_k|}$$

where λ' is the intersection of the line connecting λ with μ_k and the circle C_k. From this, the above estimates follow a fortiori. Consequently, except for a finite set, all points of $\sigma(T+S)$ lie inside the circles C_k.

In order to show that the spectral measure $\{E'(\lambda'_k)\}$ of $T+S$ is uniformly bounded it is convenient to assume that the spectral measure $\{E(\lambda_k)\}$ of T consists of Hermitian idempotents, that is, that $E(\lambda_k)=E(\lambda_k)^*$. That this assumption may be made without sacrificing generality is due to the theorem of Lorch-Mackey. It must be verified that if T and S satisfy the conditions of the theorem so do $T'=WTW^{-1}$ and $S'=WSW^{-1}$.

(a) $\sigma(T')=\sigma(T)$. For suppose $\lambda \in \rho(T)$. Then $R_\lambda(T)$ is a bounded operator. But

$$W(\lambda I-T)^{-1}W^{-1}=(\lambda I-WTW^{-1})^{-1}\equiv(\lambda I-T')$$

is also bounded. Hence $\rho(T)=\rho(T')$ and the result follows on taking complements with respect to the extended complex plane.

(b) $$\dim WE(\lambda_k)W^{-1}=\dim WE(\lambda_k)\leqq \dim E(\lambda_k)\,.$$

However, since W is continuous, with a continuous inverse, it maps no nonzero vector into zero and thus, since $\dim E(\lambda_k)=1$ for almost all k, the same is true with regard to $WE(\lambda_k)W^{-1}$. Also

$$0=W\Big(I-\sum_{k=1}^{\infty}E(\lambda_k)\Big)W^{-1}=I-\sum_{k=1}^{\infty}WE_kW^{-1}$$

(c) $f\in \mathscr{D}(WT^{\nu}W^{-1}) \Longrightarrow W^{-1}f\in \mathscr{D}(T^{\nu}) \Longrightarrow W^{-1}f\in \mathscr{D}(S) \Longrightarrow f\in \mathscr{D}(S')$

and similarly for the adjoints.

In the remainder of the proof it shall, therefore, be supposed that the spectral family $E(\lambda_k)$ consists of Hermitian idempotents. For convenience, the primes introduced above shall be suppressed.

The proof of uniform boundedness rests on the formula

$$(4)\qquad R_\lambda(T+S)-R_\lambda(T)$$
$$=R_\lambda(T)SR_\lambda(T)+R_\lambda(T)SR_\lambda(T)SR_\lambda(T)\{I-SR_\lambda(T)\}^{-1}$$

which is easily obtained from (1) and the operator analogue of $1/(1-x)=1+x+x^2/(1-x)$, and on the basic relation

$$(5)\qquad E(\lambda_n)=\frac{1}{2\pi i}\oint_{C_n}R_\lambda(T)d\lambda\,.$$

Let J be a finite set of positive integers all of which are sufficiently large that is, $N\in J\Longrightarrow N>N_0$. The nature of $N_0(T,S)$ will be specified somewhat more precisely in the sequel. Then, on integrating both members of (4) one finds that

$$(6)\qquad \|\sum_{n\in J} E'_n - E(\lambda_n)\| \leq \left\| \sum_{n\in J} \frac{1}{2\pi i} \oint_{C_n} R_\lambda(T) S R_\lambda(T)\, d\lambda \right\|$$

$$+ \left\| \sum_{n\in J} \frac{1}{2\pi i} \oint_{C_n} R_\lambda(T) S R_\lambda(T) S R_\lambda(T) \{I - S R_\lambda(T)\}^{-1} d\lambda \right\|$$

where E'_n represents the spectral measure corresponding to that portion of the spectrum of $T+S$ which lies inside the circle C_n. In order to place bounds on the right member of (6) one employs a well established inequality for operator valued functions $A(\lambda)$ analytic on a contour C of length L [12, p. 324].

$$\left\| \oint_C A(\lambda) d\lambda \right\| \leq \max_{\lambda\in C} \|A(\lambda)\| \cdot L .$$

Applying this result to the second term of the right member of (6) one finds

$$\left\| \sum_{n\in J} \frac{1}{2\pi i} \oint_{C_n} R_\lambda(T) S R_\lambda(T) S R_\lambda(T) \{I - S R_\lambda(T)\}^{-1} d\lambda \right\|$$

$$\leq \sum_{n\in J} \frac{1}{2\pi} \|R_\lambda(T)\|\ \|S R_\lambda(T)\|^2 \|\{I - S R_\lambda(T)\}^{-1}\| \cdot \frac{1}{3} d_n .$$

Now using inequalities (2) and (3) to estimate $\|\{I - SR_\lambda(T)\}^{-1}\|$ and $\|SR_\lambda(T)\|$ and Lemma 3 of J. Schwartz reproduced above, one obtains for this term the bound

$$\frac{1}{2\pi} \sum_{n\in J} \frac{M^2}{d_n^\tau} \leq \frac{M^2}{2\pi} \sum_{n=1}^{\infty} \frac{1}{d_n^\tau} < \infty .$$

The term

$$\left\| \sum_{n\in J} \frac{1}{2\pi i} \oint_{C_n} R_\lambda(T) S R_\lambda(T) d\lambda \right\|$$

requires closer investigation. By employing the representation

$$R_\lambda(T) = \frac{E(\lambda_n)}{\lambda - \lambda_n} + R^0(\lambda_n) + A_n(\lambda - \lambda_n)$$

where $A_n(\lambda - \lambda_n)$ is a power series in $\lambda - \lambda_n$ without constant term and, applying the residue theorem, one finds that

$$\frac{1}{2\pi i} \oint_{C_n} R_\lambda(T) S R_\lambda(T) d\lambda = R^0(\lambda_n) S E(\lambda_n) + E(\lambda_n) S R^0(\lambda_n) .$$

It remains to find bounds for

$$\|\sum_{n\in J} E(\lambda_n)SR^0(\lambda_n)\|$$

and

$$\|\sum_{n\in J} R^0(\lambda_n)SE(\lambda_n)\| .$$

On observing that

$$\|\sum_{n\in J} R^0(\lambda_n)SE(\lambda_n)\|=\|\sum_{n\in J} E(\lambda_n)S^*R^0(\lambda_n)^*\|$$

and identifying $SR^0(\lambda_n)$ is one case and $S^*R^{0*}(\lambda_n)$ in the other with B_n of Lemma 3, one sees that the terms in question are bounded by

$$\sum_{n\in J} \|SR^0(\lambda_n)\|^2 .$$

It is not difficult to estimate $\|SR^0(\lambda_n)\|$. Again turning to the device

$$\|SR^0(\lambda_n)\|\leq\|ST^{-\nu}\| \ \|T^\nu R^0(\lambda_n)\|$$

and noting that

$$R^0(\lambda_n)=\lim_{\lambda\to\lambda_n} R_\lambda(T)-\frac{E(\lambda_n)}{\lambda-\lambda_n}$$

one has

$$R_0(\lambda_n)=\sum_{k\neq n}\frac{E(\lambda_k)}{\lambda_n-\lambda_k} .$$

(In this formula, in order to avoid notational complications, the effect of the finite set of multiple poles of the resolvent has been neglected.) One sees that $R^0(\lambda_n)=F(T)$, where $F(\lambda)$ is defined in the neighborhoods of the spectral points λ_k as follows:

$$F(\lambda)=\begin{cases}\dfrac{1}{\lambda_n-\lambda} & \lambda \text{ near } \lambda_k \ \ k\neq n\\ 0 & \lambda \text{ near } \lambda_n .\end{cases}$$

Consequently, $T^\nu P^0(\lambda_n)=G(T)$ with

$$G(\lambda)=\begin{cases}\dfrac{\lambda^\nu}{\lambda_n-\lambda} & \lambda \text{ near } \lambda_k \ \ k\neq n\\ 0 & \lambda \text{ near } \lambda_n\end{cases}$$

Now applying the bound arising from the operational calculus one obtains

$$||SR^0(\lambda_n)|| \leq ||ST^{-\nu}|| \; ||G(T)|| \leq M_1 M_2 \max_{\lambda \in \sigma(T)} |G(\lambda)| .$$

Let $\mu \in C_n$.

$$\left|\frac{\lambda^\nu}{\lambda_n - \lambda}\right| \leq \frac{|\lambda^\nu|}{\big||\lambda_n - \mu| - |\lambda - \mu|\big|} \leq \frac{|\lambda^\nu|}{|\lambda - \mu|\left|\dfrac{|\lambda_n - \mu|}{|\lambda - \mu|} - 1\right|}$$

$$\leq \frac{|\lambda^\nu|}{|\lambda - \mu|\left|\dfrac{\frac{1}{3}d_n}{d_n} - 1\right|} \leq \frac{3}{2}\frac{|\lambda^\nu|}{|\lambda - \mu|} .$$

Using the hypothesis made with regard to this function one finally has

$$||SR^0(\lambda_n)|| \leq \frac{3}{2} M_1 M_2 \frac{1}{d_n^{\tau/2}} = M' \frac{1}{d_n^{\tau/2}} .$$

Now one is prepared to state that

$$\left\|\sum_{n \in J} \frac{1}{2\pi i} \oint_{C_n} R_\lambda(T) S R_\lambda(T) d\lambda\right\|^2 \leq 2M^2 \sum_{n=1}^{\infty} \frac{1}{d_n^\tau} < \infty .$$

Thus

$$|| \sum_{n \in J} (E'_n - E(\lambda_n))|| \leq K .$$

If it were known that E_n is the spectral measure corresponding to one point of the spectrum of $T+S$, the proof of uniform boundedness would be complete. The next few lines shall be devoted to showing that, indeed, except for a finite number of indices, in every circle of radius $\frac{1}{3}d_n$ about $\lambda_n \in \sigma(T)$ there lies exactly one point $\lambda'_n \in \sigma(T+S)$ and the spectral measure $E'(\lambda'_n)$ corresponding to this point has a one dimensional range.

In (6) let the index set J have n for its only member. Then one sees on examining the estimates of the bound of the right member of (6),

$$||E'_n - E(\lambda_n)|| < \frac{K'}{d_n^{\tau/2}} . \tag{7}$$

For n sufficiently large

$$\frac{K'}{d_n^{\tau/2}} < \frac{1}{2} ,$$

which by Lemma 4 of [9] (also cf. [10, p. 320]) implies that E'_n and $E(\lambda_n)$ have the same dimension, which by hypothesis is 1. Therefore,

$(T+S)E_n'$ considered as an operator on the range of E_n is a scalar λ_n' and precisely one point $\lambda_n' \in \sigma(T+S)$ lies in the circle C_n.

Thus $T+S$ is a regular operator with uniformly bounded spectral measure and is therefore a spectral operator. (cf. [9, Lemma 2].

From the foregoing proof flow two consequences deserving of explicit mention.

COROLLARY 1. *The operator $T+S$ satisfies Condition B and for all n sufficiently large* $|\lambda_n' - \lambda_n| < \frac{1}{3}d_n$.

Proof. In virtue of the remark following inequality (3) of the proof, all but a finite number of the points of $\sigma(T+S)$ lie inside the circles C_n with center at $\lambda_n \in \sigma(T)$ and radius $\frac{1}{3}d_n$. Moreover, the discussion following (3) shows that except for a finite number of indices exactly one point λ_n' of $\sigma(T+S)$ lies in the circle C_n about λ_n. Now suppose $\lambda_k' \in \sigma(T+S)$ and its nearest neighbor is $\lambda_{k-1}' \in \sigma(T+S)$. Then

$$\begin{aligned} d_k' = |\lambda_k' - \lambda_{k-1}'| &\leq |\lambda_k' - \lambda_k| + |\lambda_k - \lambda_{k-1}| + |\lambda_{k-1} - \lambda_{k-1}'| \\ &\leq 1/3\, d_k + d_k + 1/3\, d_{k-1} \leq 5/3\, d_k\,, \end{aligned}$$

and

$$\sum_m^\infty d_k'^{-\tau} \leq \left(\frac{3}{5}\right)^\tau \sum_m^\infty d_k^{-\tau} < \infty\ .$$

It is of importance to know whether the perturbed operator $T+S$ still enjoys the "completeness" property

$$\sum_{k=1}^\infty E'(\lambda_k') \doteq I$$

with which the unperturbed operator T is endowed by hypothesis. The answer is given in the following.

THEOREM 2. *If T and S satisfy the conditions of Theorem 1, then*

$$\sum_{k=1}^\infty E'(\lambda_k') = I.$$

Proof. The proof rests on Lemma 16 of [9] which states:

The space $S_\infty(T) \equiv \{f \mid$ for each positive integer k, $E(\lambda_k)f = 0\}$ is the set of all $f \in H$ for which $f(\lambda) = R_\lambda(T)f$ is an entire function of λ.

Suppose C is a contour in the complex plane whose minimum distance from the spectrum $\sigma(T)$ is $d(C)$. Consider the function

$$f(\lambda, \mu)=\frac{\mu^{\nu}}{\lambda-\mu}$$

for $\lambda \in C$ and $\mu \in \sigma(T)$. Now let $\lambda' \in C$ be such that $\operatorname{dist}(\lambda', \sigma(T))=d(C)$, and let μ_n be the point in $\sigma(T)$ such that $\operatorname{dist}(\lambda', \mu_n)=d(C)$. Then

$$|f(\lambda, \mu)| \leq \frac{|\mu_n^{\nu}|}{|\lambda'-\mu_n|} < 1/d_n^{\tau/2}.$$

By choosing C properly one can achieve that $\|SR_\lambda(T)\| < 1/2$ for $\lambda \in C_n$ and, therefore, a fortiori, for $\lambda \in C$. Hence, by (3) $\|\{I-SR_\lambda(T)\}^{-1}\| < 2$ and by the above cited lemma, for $f \in S_\infty(T)$, one then has for $\lambda \in C$

$$\|R_\lambda(T+S)f\| = \|R_\lambda(T)\{I-SR_\lambda(T)\}^{-1}f\|$$

$$\leq \frac{k'}{d(C)}\|f\|.$$

Now, given $\varepsilon > 0$, choose C in such a way that $k'/d(C) < \varepsilon$. Then

$$\|R_\lambda(T+S)f\| < \varepsilon\|f\|.$$

The arbitrary nature of ε, the fact that $f(\lambda)=R_\lambda(T+S)f$ is an entire function of λ, and the permissible application of the maximum modulus principle allow one to assert that for all λ in the interior of C,

$$R_\lambda(T+S)f=0.$$

In particular at points $\lambda \in \rho(T)$, $R_\lambda(T+S)$ has an inverse. There are such points in the interior of C. Thus $f=0$ and the theorem is proved.

3. Application to differential operators of even order. $N=2\mu$. In appliying Theorems 1 and 2 to differential operators, the unperturbed operator T is identified with the operator $\tau \equiv d^N/dx^N$ with domain restricted by the two considerations:

(a) $f \in \mathscr{D}(T)$ only if $f \in C^{N-1}(0, 1)$ and $\dfrac{d^{N-1}f}{dx^{N-1}}$ is absolutely continuous, and

(b) $f \in \mathscr{D}(T)$ only if f satisfies $N=2\mu$ linearly independent boundary conditions of which μ bear on the point $x=0$ and μ on the point $x=1$. These boundary conditions can always, by linear combinations, be brought to the form

$$A_i(f)=f^{(k_i)}(0)+\sum_{j=0}^{k_i-1}\alpha_{ij}f^{(j)}(0) \qquad i=1, 2, \cdots, \mu$$

$$k_1 > k_2 > \cdots > k_\mu$$

$$(8) \qquad B_i(f)=f^{(l_i)}(1)+\sum_{j=0}^{l_i-1}\beta_{ij}f^{(j)}(1) \qquad i=1, 2, \cdots, \mu$$
$$l_1>l_2>\cdots>l_\mu$$

To show that T is a regular operator it is most convenient to refer to Lemma 10 of [9, p. 434] which states:

Let T be a differential operator and suppose that for some complex λ both $T-\lambda$ and $T^*-\lambda$ have an inverse. Then T and T^* are regular operators, T and T^* have spectra related by $\sigma(T)=\overline{\sigma}(T^*)$, and determine spectral measures E_1 and E_2 related by $E_1(\lambda)=E_2^*(\overline{\lambda})$.

Consider the differential equation $(\tau-\lambda)f=0$. By manipulating a tentative power series solution it can be shown in an elementary fashion that there exists a set of linearly independent solutions which are entire in the parameter λ. Let this set be $\{u_1, u_2, \cdots, u_N\}$. The general solution of the above equation can then be expressed in the form:

$$f(x)=\sum_{i=1}^{N} C_i u_i(x, \lambda).$$

On imposing the N linearly independent boundary conditions, one obtains a system of N homogeneous equations in the N unknowns C_i. This system has a nonvanishing solution vector if and only if the determinant of the matrix of the coefficients vanishes. This determinant, however, being a linear combination of entire functions in λ is itself entire. Hence its zeroes are isolated. Thus, for all but a countable set of points λ_k, one finds that $f(x)=0$, and thus $(T-\lambda)^{-1}$ exists. But, since the adjoint operator also has exactly N linearly independent boundary conditions associated with it, it follows by the same argument that there exists only a countable number of points μ_k where $(T^*-\mu_n)^{-1}$ fails to exist. Consequently, one can find a point λ such that both $(T-\lambda)^{-1}$ and $(T^*-\overline{\lambda})^{-1}$ exist and, therefore, by the cited lemma, T is regular.

It shall now be verified that T satisfies the spectral Condition B. This will be accomplished by showing that the above boundary conditions are what Birkhoff [2] has called regular. To clarify the meaning of this term the technique for obtaining an asymptotic development for the characteristic numbers and functions established in the general case by G. D. Birkhoff [2] and amplified and developed rigorously by J. Tamarkin shall here be briefly recapitulated.

Since there are N linearly independent solutions of the equation

$$\frac{d^N f}{dx^N}=\lambda f,$$

a solution of the boundary value problem must have the form

$$f(x)=\sum_{l=1}^{N} C_l u_l(x)\,.$$

The requirement that $A_i(f)=B_i(f)=0$ leads to a set of N linear equations in N unknowns $\{C_l\}$. A necessary and sufficient condition that a nontrivial solution $\{C_l\}$ of this system exist is the vanishing of the determinant of the coefficients:

$$\Delta(\lambda)=\begin{vmatrix} A_1(u_1) & \cdots & A_1(u_N) \\ \vdots & & \vdots \\ A_\mu(u_1) & \cdots & A_\mu(u_N) \\ B_1(u_1) & \cdots & B_1(u_N) \\ \vdots & & \vdots \\ B_\mu(u_1) & \cdots & B_\mu(u_N) \end{vmatrix}.$$

It should be noted that the solution is unique provided not all of the first minors of $\Delta(\lambda)$ vanish, that is, in this case, the characteristic value is simple.

A fundamental set of solutions of the differential equation

$$\frac{d^N f}{dx^N}=\lambda f$$

consists of

$$u_k(x;\ \lambda)=e^{\rho\omega_k x}$$

where

$$\lambda^{1/N}=\rho=|\lambda|^{1/N}e^{i(1/N)\arg\lambda}$$

and ω_k are the N distinct Nth roots of unity. The transformation $\rho^N=\lambda$ transports the entire λ-plane into a sector of angular width $2\pi/N$ of the ρ-plane. There is, then, a biunique correspondence between the zeroes of $\Delta(\lambda)$ in the λ-plane and the zeroes of $\delta(\rho)\equiv\Delta(\rho^N)$ in a sector of angular width $2\pi/N$ in the ρ-plane.

The elements of the determinant $\delta(\rho)$ can, by (8) be written as follows:

$$A_i(u_j)=\rho^{k_i}\left\{\omega_j^{k_i}+\sum_{t=0}^{k_i-1}\frac{\alpha_{it}\omega_j^t}{\rho^{k_i-t}}\right\}=\rho^{k_i}\{\omega_j^{k_i}+A_{ij}\}$$

$$B_i(u_j)=\rho^{l_i}e^{\rho\omega_j}\left\{\omega_j^{l_i}+\sum_{s=0}^{l_i-1}\frac{\beta_{is}\omega_j^s}{\rho^{l_i-s}}\right\}=\rho^{l_i}e^{\rho\omega_j}\{\omega_j^{l_i}+B_{ij}\}$$

where $\lim_{|\rho|\to\infty}A_{ij}=\lim_{|\rho|\to\infty}B_{ij}=0$. After removing the factors ρ^{k_i}, ρ^{l_i} from the rows with index i and $i+\mu$ one has

$$\delta(\rho)=\prod_{i=1}^{\mu}\rho^{k_i+l_i}\begin{vmatrix} \omega_1^{k_1}+A_{11} & \omega_2^{k_1}+A_{12} & \cdots\omega_N^{k_1}+A_{1N} \\ \vdots & \vdots & \vdots \\ \omega_1^{k_\mu}+A_{\mu 1} & \omega_2^{k_\mu}+A_{\mu 2} & \cdots\omega_N^{k_\mu}+A_{\mu N} \\ e^{\rho\omega_1}(\omega_1^{l_1}+B_{11}) & e^{\rho\omega_2}(\omega_2^{l_1}+B_{12}) & \cdots\ e^{\rho\omega_N}(\omega_N^{l_1}+B_{1N}) \\ \vdots & \vdots & \vdots \\ e^{\rho\omega_1}(\omega_1^{l_\mu}+B_{\mu 1}) & e^{\rho\omega_2}(\omega_2^{l_\mu}+B_{\mu 1}) & \cdots e^{\rho\omega_N}(\omega_N^{l_\mu}+B_{\mu N}) \end{vmatrix},$$

The sector S shall now be chosen in a convenient fashion. To this end, it is proper to distinguish between two cases:

(1) μ is even

$$S=\left\{\rho\,\middle|\,-\frac{\pi}{2\mu}\leq\arg\rho\leq\frac{\pi}{2\mu}\right\}$$

(2) μ is odd

$$S=\left\{\rho\,\middle|\,0\leq\arg\rho\leq\frac{\pi}{\mu}\right\}.$$

Let $\omega=e^{\pi i/\mu}$. Then in the first case, for $\arg\rho=0$

$$\Re(\rho\omega^{\mu/2})=\Re(\rho\omega^{-\mu/2})=0\,.$$

In the second case

$$\Re(\rho\omega^{(\mu-1)/2})=\Re(\rho\omega^{-(\mu+1)/2})=0$$

for $\arg\rho=\pi/2\mu$. Suppose the indexing is arranged so that in the first case

$$\omega_1=\omega^{-\mu/2},\ \omega_2=\omega^{-\mu/2+1},\ \cdots,\ \omega_\mu=\omega^{\mu/2-1},\ \omega_{\mu+1}=\omega^{\mu/2},\ \omega_{\mu+2}=\omega^{\mu/2+1},\ \cdots,\ \omega_{2\mu}=\omega^{3\mu/2-1}$$

and in the second,

$$\omega_1'=\omega^{-\mu/2-1/2},\ \omega_2'=\omega^{-\mu/2+1/2},\ \cdots,\ \omega_\mu'=\omega^{\mu/2-3/2},\ \omega_{\mu+1}'=\omega^{\mu/2-1/2},\ \cdots,\ \omega_{2\mu}'=\omega^{3\mu/2-3/2}\,.$$

Upon bringing $e^{\rho\omega_k}$ out of the determinant wherever $\Re(\rho\omega_k)>0$, one has

$$\delta(\rho)=\prod_{i=1}^{\mu}\rho^{k_i+l_i}\prod_{i=2}^{\mu+1}e^{\rho\omega_i}\delta'(\rho)$$

where $\delta'(\rho)$ has the appearance

$$\delta'(\rho)=\begin{vmatrix} \omega_1^{k_1}+A_{11} & \multirow{3}{*}{$\left\{\mathfrak{A}^I\right\}$} & \omega_{\mu+1}^{k_1}+A_{1\mu+1} & \multirow{3}{*}{$\left\{\mathfrak{A}^{II}\right\}$} \\ \vdots & & \vdots & \\ \omega_1^{k_\mu}+A_{\mu 1} & & \omega_{\mu+1}^{k_\mu}+A_{\mu\mu+1} & \\ e^{\rho\omega_1}(\omega_1^{l_1}+B_{11}) & \left\{\mathfrak{B}^I\right\} & e^{\rho\omega_{\mu-1}}(\omega_{\mu+1}^{l_1}+B_{1\mu+1}) & \left\{\mathfrak{B}^{II}\right\} \\ \vdots & & \vdots & \\ e^{\rho\omega_1}(\omega_1^{l_\mu}+B_{\mu 1}) & & e^{\rho\omega_{\mu+1}}(\omega_{\mu+1}^{l_\mu}+B_{\mu\mu+1}) & \end{vmatrix}.$$

Here, $\mathfrak{A}^I$ and $\mathfrak{B}^{II}$ are matrices consisting of $\mu-1$ columns and μ rows all of whose terms have for a factor an exponential term with negative real part. Asymptotically, these matrices are therefore negligible.

$$\mathfrak{A}^{II}=\begin{Bmatrix} \omega_{\mu+2}^{k_1}+A_{1\mu+2}\cdots\omega_{2\mu}^{k_1}+A_{12\mu} \\ \vdots \\ \omega_{\mu+2}^{k_\mu}+A_{\mu\mu+2}\cdots\omega_{2\mu}^{k_\mu}+A_{\mu 2\mu} \end{Bmatrix}$$

$$\mathfrak{B}^{I}=\begin{Bmatrix} e^{\rho\omega_2}(\omega_2^{l_1}+B_{12})\cdots e^{\rho\omega_\mu}(\omega_\mu^{l_1}+B_{1\mu}) \\ \vdots \\ e^{\rho\omega_2}(\omega_2^{l_\mu}+B_{\mu 2})\cdots e^{\rho\omega_\mu}(\omega_\mu^{l_\mu}+B_{\mu\mu}) \end{Bmatrix}.$$

Thus

$$\delta'(\rho)=e^{\rho\omega_1}\begin{vmatrix} \omega_{\mu+1}^{k_1} & \\ \vdots & \mathfrak{A}^{II} \\ \omega_{\mu+1}^{k_\mu} & \end{vmatrix}\begin{vmatrix} \omega_1^{l_1} & \\ \vdots & \mathfrak{B}^{I} \\ \omega_1^{l_\mu} & \end{vmatrix}+e^{\rho\omega_{\mu+1}}\begin{vmatrix} \omega_1^{k_1} & \\ \vdots & \mathfrak{A}^{II} \\ \omega_1^{k_\mu} & \end{vmatrix}\begin{vmatrix} & \omega_{\mu+1}^{l_1} \\ \mathfrak{B}^{I} & \vdots \\ & \omega_{\mu+1}^{l_\mu} \end{vmatrix}+O(1/\rho).$$

Only the case in which μ is even shall be considered explicitly since the treatment for μ odd is completely analagous. First note that

$$\omega_1=e^{-\pi i/2}=-i$$

and

$$\omega_{\mu+1}=e^{\pi i/2}=i.$$

Thus $\omega_1=-\omega_{\mu+1}$. Now let $x_i\equiv\omega^{k_i}$ and $y_i\equiv\omega^{l_i}$. The conditions that $k_1>k_2>\cdots>k_\mu$ and $l_1>l_2>\cdots l_\mu$ and the fact that $\omega=e^{i\pi/\mu}$ is a primitive root of unity imply that $x_i\neq x_j$, $y_i\neq y_j$ for $i\neq j$. Recall that by the arrangement of the indices, $\omega_s=\omega^{(-\mu/2)+s-1}=\omega^{-\mu/2}\omega^{s-1}$. Therefore,

$$\omega_s^{k_i}=\omega^{(-\mu/2)k_i}\omega^{(s-1)k_i}=x_i^{-\mu/2}x_i^{(s-1)}$$

and

$$\omega_s^{l_i}=\omega^{(-\mu/2)l_i(s-1)l_i}=y_i^{-\mu/2}y_i^{(s-1)}.$$

Using now the explicit representation of $\mathfrak{A}^{II}$ and $\mathfrak{B}^{I}$ given previously, but taking only zero order terms into account, one has

$$\delta'(\rho)=e^{\rho\omega_1}\begin{vmatrix} x_1^{-\mu/2}x_1^{\mu}, & x_1^{-\mu/2}x_1^{\mu+1}, & \cdots, & x_1^{-\mu/2}x_1^{2\mu-1} \\ x_2^{-\mu/2}x_2^{\mu}, & x_2^{-\mu/2}x_2^{\mu+1}, & \cdots, & x_2^{-\mu/2}x_2^{2\mu-1} \\ \cdots & & \cdots & \\ x_\mu^{-\mu/2}x_\mu^{\mu}, x_\mu^{-\mu/2}, & x_\mu^{-\mu/2}x_\mu^{\mu+1}, & \cdots, & x_\mu^{-\mu/2}x_\mu^{2\mu-1} \end{vmatrix}\begin{vmatrix} y_1^{-\mu/2}, & y_1^{-\mu/2}y_1, & \cdots, & y_1^{-\mu/2}y_1^{\mu-1} \\ y_2^{-\mu/2}, & y_2^{-\mu/2}y_2, & \cdots, & y_2^{-\mu/2}y_2^{\mu-1} \\ & & & \\ y_\mu^{-\mu/2}, & y_\mu^{-\mu/2}y_\mu, & \cdots, & y_\mu^{-\mu/2}y_\mu^{\mu-1} \end{vmatrix}$$

$$+e^{-\rho\omega_1}\begin{vmatrix} x_1^{-\mu/2}, & x_1^{-\mu/2}x_1^{\mu+1}, & \cdots, & x_1^{-\mu/2}x_1^{2\mu-1} \\ x_2^{-\mu/2}, & x_2^{-\mu/2}x_2^{\mu+1}, & \cdots, & x_2^{-\mu/2}x_2^{2\mu-1} \\ \vdots & \vdots & & \vdots \\ x_\mu^{-\mu/2}, & x_\mu^{-\mu/2}x_\mu^{\mu+1}, & \cdots, & x_\mu^{-\mu/2}x_\mu^{2\mu-1} \end{vmatrix}\begin{vmatrix} y_1^{-\mu/2}y_1, & \cdots, & y_1^{-\mu/2}y_1^{\mu-1}, & y_1^{-\mu/2}y_1^{\mu} \\ y_2^{-\mu/2}y_2, & \cdots, & y_2^{-\mu/2}y_2^{\mu-1}, & y_2^{-\mu/2}y_2^{\mu} \\ \vdots & & \vdots & \vdots \\ y_\mu^{-\mu/2}y_\mu, & \cdots, & y_\mu^{-\mu/2}y_\mu^{\mu-1}, & y_\mu^{-\mu/2}y_\mu^{\mu} \end{vmatrix}$$

$$+O(1/\rho)\,.$$

By bringing common factors outside the determinant, one can simplify the expression for $\delta'(\rho)$.

$$\delta'(\rho)=e^{\rho\omega_1}\prod_{i=1}^{\mu}x_i^{\mu/2}y_i^{-\mu/2}\begin{vmatrix} 1 & x_1 & \cdots & x_1^{\mu-1} \\ 1 & x_2 & \cdots & x_2^{\mu-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_\mu & \cdots & x_\mu^{\mu-1} \end{vmatrix}\begin{vmatrix} 1 & y_1 & \cdots & y_1^{\mu-1} \\ 1 & y_2 & \cdots & y_2^{\mu-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_\mu & \cdots & y_\mu^{\mu-1} \end{vmatrix}$$

$$+e^{-\rho\omega_1}\prod_{i=1}^{\mu}x_i^{-\mu/2}y_i^{-\mu/2}\begin{vmatrix} 1 & x_1^{\mu+1} & \cdots & x_1^{2\mu-1} \\ 1 & x_2^{\mu+1} & \cdots & x_2^{2\mu-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_\mu^{\mu+1} & \cdots & x_\mu^{2\mu-1} \end{vmatrix}\begin{vmatrix} 1 & y_1 & \cdots & y_1^{\mu-2}, & y_1^{\mu-1} \\ 1 & y_2 & \cdots & y_2^{\mu-2}, & y_2^{\mu-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & y_\mu & \cdots & y_\mu^{\mu-2}, & y_\mu^{\mu-1} \end{vmatrix}+O(1/\rho)\,.$$

The first determinantal factor of the second term above can be treated by noting that $1=x_i^{2\mu}$ and switching columns $\mu-1$ times and then bringing the factor x_i^{μ} outside. These manipulations yield

$$\delta'(\rho)=e^{\rho\omega_1}\prod_{i=1}^{\mu}x_i^{\mu/2}y_i^{-\mu/2}\begin{vmatrix} 1 & x_1 & \cdots & x_1^{\mu-1} \\ 1 & x_2 & \cdots & x_2^{\mu-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_\mu & \cdots & x_\mu^{\mu-1} \end{vmatrix}\begin{vmatrix} 1 & y_1 & \cdots & y_1^{\mu-1} \\ 1 & y_2 & \cdots & y_2^{\mu-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_\mu & \cdots & y_\mu^{\mu-1} \end{vmatrix}$$

$$-e^{-\rho\omega_1}\prod_{i=1}^{\mu}x_i^{\mu/2}y_i^{\mu/2}\begin{vmatrix} 1 & x_1 & \cdots & x_1^{\mu-1} \\ 1 & x_2 & \cdots & x_2^{\mu-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_\mu & \cdots & x_\mu^{\mu-1} \end{vmatrix}\begin{vmatrix} 1 & y_1 & \cdots & y_1^{\mu-1} \\ 1 & y_2 & \cdots & y_2^{\mu-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_\mu & \cdots & y_\mu^{\mu-1} \end{vmatrix}+O(1/\rho)\,.$$

Now note that the determinants involved in the above expression are Vandermonde determinants. But such determinants do not vanish provided only the entries $(x_1, x_2, \cdots, x_\mu)$ or $(y_1, y_2, \cdots, y_\mu)$ are distinct. That this is the case was demonstrated above. Therefore, the given boundary conditions are regular in the sense of Birkhoff since in the equation

$$\delta'(\rho)=\theta_0+\theta_1 e^{\rho\omega_1}+\theta_2 e^{-\rho\omega_1}$$

not both θ_1 and θ_2 vanish. Tamarkin [11] who examined "separated" boundary conditions failed to reach tbis general conclusion. By including common factors in the term $O(1/\rho)$, the equation $\delta'(\rho)=0$ can now be written in the form

$$e^{\rho\omega_1}-e^{-\rho\omega_1}\prod_{i=1}^{\mu} y_i^{\mu}+O(1/\rho)=0 .$$

But

$$y_k^{\mu}=\omega^{l_k}{}_{\mu}=e^{(\pi i/\mu)l_k\mu}=e^{\mu i l_k}=(-1)^{l_k} .$$

Hence, on multiplying by $e^{-\rho\omega_1}$, one obtains

$$e^{2\rho\omega_1}=(-1)^{\Sigma_{k=1}^{\mu} l_k}+O(1/\rho)$$

or

$$e^{2\rho i}=(-1)^{\Sigma_{k=1}^{\mu} l_k}+O(1/\rho) .$$

On taking square roots of both members, one finds that

$$e^{\rho i}=\pm i(1+O(1/\rho))$$

or

$$e^{\rho i}=\pm(1+O(1/\rho)) .$$

Taking logarithms of both members and noting that

$$\log(1+O(1/\rho))=O(1/\rho)$$

results in the expressions

$$\begin{aligned} &\rho_{Ik}=\pi/2+2\pi k+O(1/\rho) \\ (9)\qquad & \\ &\rho_{IIk}=\pi/2+2\pi k+O(1/\rho) \end{aligned}$$

or

$$\begin{aligned} &\rho_{Ik}=2\pi k+O(1/\rho) \\ (10)\qquad & \\ &\rho_{IIk}=\pi+2\pi k+O(1/\rho) . \end{aligned}$$

By neglecting the terms of order $1/\rho$ first estimates may be obtained which may then be inserted in (9) or (10) with the following results:

$$\rho_{Ik}=(2\pi k)\left\{1+\frac{1}{4k}+\frac{E_I(k)}{k^2}\right\}$$

(11)

$$\rho_{IIk}=(2\mu k)\left\{1-\frac{1}{4k}+\frac{E_{II}(k)}{k^2}\right\}$$

or

$$\rho_{Ik}=(2\pi k)\left\{1+\frac{E_I(k)}{k^2}\right\}$$

(12)

$$\rho_{IIk}=(2\pi k)\left\{1+\frac{1}{2k}+\frac{E_{II}(k)}{k^2}\right\}$$

where the $E_I(k)$ and $E_{II}(k)$ represent bounded functions of k.

It should be noted that (11) and (12) are valid not only in the case in which μ is even, but also when μ is odd.

Reverting to the λ-plane one finds that

$$\lambda_{Ik}=(2\pi k)^{2\mu}\left\{1+\frac{\mu}{2k}+\frac{E_I(k)}{k^2}\right\}$$

(13)

$$\lambda_{IIk}=(2\pi k)^{2\mu}\left\{1-\frac{\mu}{2k}+\frac{E_{II}(k)}{k^2}\right\}$$

or

$$\lambda_{Ik}=(2\pi k)^{2\mu}\left\{1+\frac{E_I(k)}{k^2}\right\}$$

(14)

$$\lambda_{IIk}=(2\pi k)^{2\mu}\left\{1+\frac{\mu}{k}+\frac{E_{II}(k)}{k^2}\right\}.$$

Since the zeroes of $\Delta(\lambda)$ furnish the poles of the Green's function, one sees that all except a finite set of characteristic values of T are simple poles of the resolvent. This does not, however, assert anything about the number of linearly independent characteristic functions associated with each characteristic value. This matter will be dealt with below.

From (13) and (14) one obtains expressions for the distance separating the points of the spectrum $\sigma(T)$.

$$|\lambda_{Ik}-\lambda_{IIk}|=(2\pi k)^{2\mu}\left\{\frac{\mu}{k}+\frac{E(k)}{k^2}\right\}$$

$$=(2\pi)^{2\mu}k^{2\mu-1}\left\{\mu+\frac{E(k)}{k}\right\}.$$

$$|\lambda_{Ik+1}-\lambda_{Ik}|=(2\pi)^{\mu}k^{2\mu-1}\left\{2\mu+\frac{E(k)}{k}\right\}$$

so that in any case, for $\tau>1/2\mu-1$

$$\sum_{k=1}^{\infty} d_k^{-\tau}<\infty .$$

This verifies that Condition B is satisfied.

In view of a prior remark it is merely necessary to exhibit a nonvanishing first minor of $\Delta(\lambda)$ in order to permit the conclusion that all but a finite number of characteristic values are simple. Moreover, since $\Delta(\lambda)=F(\rho)\delta'(\rho)$ where $F(\rho)\neq 0$ it is sufficient to find a nonvanishing first minor of $\delta'(\rho)$.

Reverting to the expression for $\delta'(\rho)$ and singling out the minor of the element in the first column and 2μth row results in the exhibit:

$$M_{2\mu,1}=(-1)^{2\mu+1}\begin{vmatrix} \mathfrak{A}^{I} & \begin{matrix} \omega_{\mu+1}^{k_1}+A_{1,\mu+1} \\ \vdots \\ \omega_{\mu+1}^{k_\mu}+A_{\mu,\mu+1} \end{matrix} & \mathfrak{A}^{II} \\ \mathfrak{B}'^{I} & \begin{matrix} e^{\rho\omega_{\mu+1}}(\omega_{\mu+1}^{l_1}+B_{1,\mu+1}) \\ \vdots \\ e^{\rho\omega_{\mu+1}}(\omega_{\mu+1}^{l_\mu}+B_{\mu,\mu+1}) \end{matrix} & \mathfrak{B}'^{II} \end{vmatrix}.$$

Here $\mathfrak{B}'^{I}$ and $\mathfrak{B}'^{II}$ are obtained from $\mathfrak{B}^{I}$ and $\mathfrak{B}^{II}$ by deleting the last row in each of these matrices. On expanding $M_{2\mu,1}$ in terms of the $\mu\times\mu$ minors occupying the first μ rows and their complements and noting that all the terms of $\mathfrak{A}^{I}$ and $\mathfrak{B}'^{II}$ are negligible in view of the fact that each has an exponential factor with negative real part, one has

$$M_{2\mu,1}=e^{\rho\omega_{\mu+1}}\begin{vmatrix} \omega_{\mu+1}^{k_1} & \omega_{\mu+2}^{k_1} & \cdots & \omega_{2\mu}^{k_1} \\ \vdots & \vdots & & \vdots \\ \omega_{\mu+1}^{k_\mu} & \omega_{\mu+2}^{k_\mu} & \cdots & \omega_{\mu}^{k_\mu} \end{vmatrix}\begin{vmatrix} \omega_2^{l_1} & \cdots & \omega_{\mu+1}^{l_1} \\ \vdots & & \vdots \\ \omega^{l_{\mu-1}} & \cdots & \omega_{\mu+1}^{l_{\mu-1}} \end{vmatrix}+O(1/\rho).$$

In the previously employed notation, $x_i=\omega^{k_i}$, $y_i=\omega^{l_i}$, one can write $M_{2\mu,1}$ in the form:

$$M_{2\mu,1}=e^{\rho\omega_{\mu+1}}\prod_{i=1}^{\mu}x_i^{\mu/2}\begin{vmatrix} 1 & x_1 & \cdots & x_1^{\mu-1} \\ 1 & x_2 & \cdots & x_2^{\mu-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_\mu & \cdots & x_\mu^{\mu-1} \end{vmatrix}\prod_{i=1}^{\mu-1}y_i^{\mu/2}\begin{vmatrix} 1 & y_1 & \cdots & y_1^{\mu-1} \\ 1 & y_2 & \cdots & y_2^{\mu-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_{\mu-1} & \cdots & y_{\mu-1}^{\mu-1} \end{vmatrix},$$

Again, the Vandermonde determinants appearing above do not vanish because the x_i and the y_i are distinct. Upon using the previously derived expressions for ρ_k, one sees that $|e^{\rho_k\omega_{\mu+1}}| \xrightarrow{k} 1$. Hence, it follows that $M_{2\mu,1} \neq 0$ except, possibly, for a finite number of indices k, and thus all except possibly a finite number of characteristic values are simple.

In order to show that T is spectral, it is necessary at this point merely to establish a uniform bound on the spectral resolution $E(e)$ of T. But because T is regular this is tantamount to giving a uniform bound for sums

$$\sum_{k \in J} E(\lambda_k)$$

whenever J is a finite set of indices. In establishing such a bound, the finite set of $\{\lambda_k\}$ which are multiple poles of the resolvent or multiple characteristic values cause no difficulty. Therefore, it shall be supposed that $E(\lambda_k)$ projects onto a one-dimensional range. One can construct $E(\lambda_k)$ explicitly by drawing on Lemma 12 of J. Schwartz which states:

"Let E be a projection of B-space X onto a finite dimensional range, and let E^*: $X^* \rightarrow X^*$ be its adjoint. Then if $\varphi_1, \cdots, \varphi_n$ is a basis of EX, we can find a unique basis of $\psi_1^*, \cdots, \psi_n^*$ of E^*X^* such that $\psi^*(\varphi_j) = \delta_{ij}$, and then

$$Ef = \sum_{i=1}^{n} \varphi_i \psi_i^*(f)$$

for any $f \in X$.

Now let $\varphi_m(x)$ be the mth characteristic function of T, and $\psi_m(x)$ the corresponding mth characteristic function of T^*. Then

$$E(\lambda_m)f = \frac{\int_0^1 \varphi_m(x)\overline{\psi_m(y)}f(y)dy}{\int_0^1 \varphi_m(x)\overline{\psi_m(x)}dx} .$$

Now suppose that

$$\varphi_m(x) = \theta_m(x) + \frac{1}{m} K_1(m, x) \tag{15}$$

$$\psi_m(x) = \theta_m(x) + \frac{1}{m} K_2(m, x) \tag{16}$$

where $K_1(m, x)$ and $K_2(m, x)$ are uniformly bounded in m. Then

$$\varphi_m(x)\overline{\psi_m(y)} = \theta_m(x)\overline{\theta_m(y)} + \frac{\theta_m(x)\overline{K_2(m, y)}}{m} + \frac{K_1(m, x)\overline{\theta_m(y)}}{m} + \frac{K_1(m, x)\overline{K_2(m, y)}}{m^2}$$

and

$$\int_0^1 \varphi_m(x)\overline{\psi_m(x)}dx=(\varphi_m, \psi_m)=\|\theta_m\|^2+\frac{1}{m}(\theta_m, K_2)+\frac{1}{m}(K_1, \theta_m)+\frac{1}{m^2}(K_1, K_2) .$$

Upon inserting these expansions in the expression for $E(\lambda_m)f$ above, we get

$$E(\lambda_m)f=\frac{\int_0^1 \theta_m(x)\overline{\theta_m(y)}f(y)dy}{\|\theta_m\|^2}+\frac{C_1(m)}{m}\frac{\int_0^1 \theta_m(x)\overline{K_2(m, y)}f(y)dy}{\|\theta_m\|^2}$$

$$+\frac{C_2(m)}{m}\frac{\int_0^1 K_1(m, x)\overline{\theta_m(y)}f(y)dy}{\|\theta_m\|^2}+\frac{C_3(m)}{m^2}\frac{\int_0^1 K_1(m, x)\overline{K_2(m, y)}f(y)dy}{\|\theta_n\|^2}$$

$$=\hat{E}_m f+\frac{1}{m}\hat{E}_m A_m f+\frac{1}{m}B_m\hat{E}_m f+\frac{K_m f}{m^2}$$

where $\hat{E}_m$ is a Hermitian projection since its norm is unity and it is an integral operator with symmetric kernel and A_m, B_m, K_m are multiplication operators that are uniformly bounded in m.

Now if J is a finite set of integers,

$$\|\sum_{n\in J} E(\lambda_n)\| \leq \|\sum_{n\in J}\hat{E}_n\|+\left\|\sum_{n\in J}\frac{1}{n}\hat{E}_n A_n\right\|$$

$$+\left\|\sum_{n\in J}\frac{1}{n}B_n\hat{E}_n\right\|+\left\|\sum_{n\in J}\frac{1}{n^2}K_n\right\| .$$

The first term is bounded by 1 because of the Hermitian character of the idempotents. Applying Lemma 3 to the second and third terms yields the bounds

$$\sup_n \|A_n\| \sum_{n=1}^{\infty}\frac{1}{n^2}$$

and

$$\sup_n \|B_n\| \sum_{n=1}^{\infty}\frac{1}{n^2} .$$

For the fourth term one has the bound

$$\sup_n \|K_n\| \sum_{n=1}^{\infty}\frac{1}{n^2} .$$

So that, granting the above representation for the characteristic functions of the operator T and the adjoint T^*, one may draw the conclusion that T is spectral.

In order to exhibit $\varphi_m(x)$ and $\psi_m(x)$ in the forms (15) and (16) it is necessary once more to resort to an asymptotic development. (cf. p.)

$$\varphi_m(x)=\sum_{i=1}^{2\mu} C_i u_i(x) \qquad u_i(x)=e^{\rho_m \omega_i x}$$

$$0=\sum_{i=1}^{2\mu} C_i A_j(u_i) \tag{17}$$

$$0=\sum_{i=1}^{2\mu} C_i B_j(u_i) \qquad j=1, 2, \cdots, \mu .$$

From the compatibility of equations (17) it follows that C_i is proportional to the minor $M_{\mu+1,i}$ of the element in the $\mu+1$ st column and the ith row of the matrix. This gives then the representation

$$\varphi_m(x) \sim \begin{vmatrix} \omega_1^{k_1}+A_{11} & \omega_2^{k_1}+A_{12} & \cdots & \omega_N^{k_1}+A_{1N} \\ \vdots & \vdots & & \vdots \\ \omega_1^{k_\mu}+A_{\mu 1} & \omega_2^{k_\mu}+A_{\mu 2} & \cdots & \omega_N^{k_\mu}+A_{\mu N} \\ e^{\rho_m \omega_1 x} & e^{\rho_m \omega_2 x} & \cdots & e^{\rho_m \omega_N x} \\ e^{\rho_m \omega_1}(\omega_1^{l_2}+B_{21}) & e^{\rho_m \omega_2}(\omega_2^{l_2}+B_{22}) & \cdots & e^{\rho_m {}_N}(\omega_N^{l_2}+B_{2N}) \\ \vdots & \vdots & & \vdots \\ e^{\rho_m \omega_1}(\omega_1^{l_\mu}+B_{\mu 1}) & e^{\rho_m \omega_2}(\omega_2^{l_\mu}+B_{\mu 1}) & \cdots & e^{\rho_m \omega_N}(\omega_N^{l_\mu}+B_{\mu N}) \end{vmatrix} .$$

Here and in similar expressions to follow, proportionality factors are freely discarded. Since the above determinant closely resembles $\delta'(\rho)$, essentially the same techniques that were successful before shall be applied again. For $k=2, 3, \cdots, \mu$ we have $\Re(\rho_m \omega_k) > 0$. Bring $e^{\rho_m \omega_k}$ outside the determinant.

$$\varphi_m(x) \sim \begin{vmatrix} \begin{matrix} \omega_1^{k_1}+A_{11} \\ \vdots \\ \omega_1^{k_1}+A_{\mu 1} \end{matrix} & \left\{\mathfrak{A}^I\right\} & \begin{matrix} \omega_{\mu+1}^{k_1}+A_{1,\mu+1} \\ \vdots \\ \omega_{\mu+1}^{k_\mu}+A_{\mu,\mu+1} \end{matrix} & \left\{\mathfrak{A}^{II}\right\} \\ e^{\rho_m \omega_1 x} & e^{\rho_m \omega_k (x-1)} & e^{\rho_m \omega_{\mu+1} x} & e^{\rho_m \omega_k x} \\ \begin{matrix} e^{\rho_m \omega_1}(\omega_1^{l_2}+B_{21}) \\ \vdots \\ e^{\rho_m \omega_1}(\omega_1^{l_n}+B_{\mu 1}) \end{matrix} & \left\{\overline{\mathfrak{B}}^I\right\} & \begin{matrix} e^{\rho_m \omega_{\mu+1}}(\omega_{\mu+1}^{l_2}+B_{2\,\mu+1}) \\ \vdots \\ e^{\rho_m \omega_{\mu+1}}(\omega_{\mu+1}^{l_\mu}+B_{\mu\,\mu+1}) \end{matrix} & \left\{\overline{\mathfrak{B}}^{II}\right\} \end{vmatrix} ,$$

The entries of the matrices $\mathfrak{A}^I$ and $\overline{\mathfrak{B}}^{II}$ are all negligible for large k. Expanding the above expression one obtains

$$\varphi_m(x) \sim \begin{vmatrix} \omega_1^{k_1} \\ \vdots & \mathfrak{A}^{II} \\ \omega_1^{k_\mu} \end{vmatrix} \begin{vmatrix} e^{\rho_m\omega_2(x-1)} & \cdots & e^{\rho_m\omega_\mu(x-1)} & e^{\rho_m\omega_{\mu+1}x} \\ \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} & e^{\rho_m\omega_{\mu+1}}\omega_{\mu+1}^{l_2} \\ \vdots & & \vdots & \vdots \\ \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} & e^{\rho_m\omega_{\mu+1}}\omega_{\mu+1}^{l_\mu} \end{vmatrix}$$

$$- \begin{vmatrix} \omega_{\mu+1}^{k_1} \\ \vdots & \mathfrak{A}^{II} \\ \omega_{\mu+1}^{k_\mu} \end{vmatrix} \begin{vmatrix} e^{\rho_m\omega_1 x} & e^{\rho_m\omega_2(x-1)} & \cdots & e^{\rho_m\omega_\mu(x-1)} \\ e^{\rho_m\omega_1}\omega_1^{l_2} & \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} \\ \vdots & \vdots & & \vdots \\ e^{\rho_m\omega_1}\omega_1^{l_\mu} & \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix} + O_x(1/\rho).$$

Recall that the two determinants involving $\mathfrak{A}^{II}$ are proportional to lowest order in $1/\rho$ and that the factor of proportionality is ± 1. Hence, on incorporating this factor in $O(1/\rho;\, x)$ and bringing $e^{\rho_m\omega_{\mu+1}}$ and $e^{\rho_m\omega_1}$ outside the determinants one has:

$$\varphi_m(x) \sim e^{-\rho_m\omega_1} \begin{vmatrix} e^{\rho_m\omega_2(x-1)} & \cdots & e^{\rho_m\omega_\mu(x-1)} & e^{\rho_m\omega_{\mu+1}(x-1)} \\ \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} & \omega_{\mu+1}^{l_2} \\ \vdots & & \vdots & \vdots \\ \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} & \omega_{\mu+1}^{l_2} \end{vmatrix}$$

$$\pm\, e^{\rho_m\omega_1} \begin{vmatrix} e^{\rho_m\omega_1(x-1)} & e^{\rho_m\omega_2(x-x)} & \cdots & e^{\rho_m\omega_\mu(x-1)} \\ \omega_1^{l_2} & \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} \\ \vdots & \vdots & & \vdots \\ \omega_1^{l_\mu} & \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix} + O_x(1/\rho)\,.$$

It should be noted that except at $x=1$, the terms $e^{\rho_m\omega_k(-1)x}$, $k \neq 1$ are negligible asymptotically since $\Re(\rho_m\omega_k(x-1)) < 0$. Now using the previously obtained asymptotic expressions for ρ_m, one finds that for $x \neq 1$

$$\varphi_{mI}(x) \sim e^{-2\pi i(m+1/4)x} \begin{vmatrix} \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} \\ \vdots & & \vdots \\ \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix}$$

$$- e^{2\pi i(m+1/4)x} \begin{vmatrix} \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} \\ \vdots & & \\ \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix} + O(1/m;\, x)$$

$$\varphi_{mII}(x) \sim e^{-2\pi i(m-1/4)x} \begin{vmatrix} \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} \\ \vdots & & \\ \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix}$$

$$- e^{2\pi i(m-1/4)x} \begin{vmatrix} \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} \\ \vdots & & \\ \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix} + O(1/m;\, x)$$

or else

$$\varphi_{mI}(x) \sim e^{-2\pi imx} \begin{vmatrix} \omega_2^{l_2} & \cdots & \omega_\mu^{l_\mu} \\ \vdots & & \\ \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix}$$

$$- e^{2\pi imx} \begin{vmatrix} \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} \\ \vdots & & \\ \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix} + O(1/m;\ x)$$

$$\varphi_{mII}(x) \sim e^{-2\pi i(m+1/2)x} \begin{vmatrix} \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} \\ \vdots & & \\ \omega_2^{l_k} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix}$$

$$- e^{2\pi i(m+1/2)x} \begin{vmatrix} \omega_2^{l_2} & \cdots & \omega_\mu^{l_2} \\ \vdots & & \vdots \\ \omega_2^{l_\mu} & \cdots & \omega_\mu^{l_\mu} \end{vmatrix} + O(1/m;\ x)\ .$$

On incorporating common factors that are uniformly bounded with respect to m into the terms $O(1/m;\ x)$ one has

$$\begin{aligned} \varphi_{mI}(x) &= \sin 2\pi\left(m + \frac{1}{4}\right)x + \frac{K_{1I}(m,\ x)}{m} \\ \varphi_{mII}(x) &= \sin 2\pi\left(m - \frac{1}{4}\right)x + \frac{K_{1II}(m,\ x)}{m} \end{aligned} \tag{18}$$

or

$$\begin{aligned} \varphi_{mI}(x) &= \sin 2\pi mx + \frac{K_{1I}(m,\ x)}{m} \\ \varphi_{mII}(x) &= \sin 2\pi\left(m + \frac{1}{2}\right)x + \frac{K_{1II}(m,\ x)}{m}\ . \end{aligned} \tag{19}$$

Thus the characteristic functions of T have been brought into the desired form. Note, however, that since $\tau = d^N/dx^N$ is formally self-adjoint, T and T^* differ only in the boundary conditions. But it is a simple matter to see that the boundary conditions of T^* will again be of the "separated" type (cf. [7, p. 186]) and that therefore all the developments leading to an asymptotic expression for ψ_{mI} and ψ_{mII} will be the same as those that served to find (18) and (19). Now note that the first terms in (18) and (19) in no way reflect the quantities occurring in the boundary conditions. Therefore it may be concluded that

(20)
$$\psi_{mI}(x)=\sin 2\pi\left(m+\frac{1}{4}\right)x+\frac{K_{2I}(m, x)}{m}$$
$$\psi_{mII}(x)=\sin 2\pi\left(m-\frac{1}{4}\right)x+\frac{K_{2II}(m, x)}{m}$$

or

(21)
$$\psi_{mI}(x)=\sin 2\pi mx+\frac{K_{2I}(m, x)}{m}$$
$$\psi_{mII}(x)=\sin 2\pi\left(m+\frac{1}{2}\right)x+\frac{K_{2II}(m, x)}{m} .$$

By what has preceded, then, it may be concluded that T is spectral.

To complete the verification of Condition A for T is still necessary to show that

$$\sum_{k=1}^{\infty} E(\lambda_k)=I.$$

To this end note that

$$\lim_{m\to\infty}\left\|\left(I-\sum_{k=m}^{\infty} E(\lambda_k)\right)-\left(I-\sum_{k=m}^{\infty}\hat{E}_k\right)\right\|=0$$

so that in virtue of the fact that $\hat{E}_k$ is Hermitian and the above cited lemma of J. Schwartz and F. Wolf, the range of

$$I-\sum_{k=m}^{\infty} E(\lambda_k)$$

for sufficiently large m is finite dimensional. But in his Lemma 15 J. Schwartz asserts that

$$S_\infty\equiv\{f|E(\lambda_k)f=0,\ 0<k<\infty\}$$

is either infinite dimensional or else the null space. But since

$$S_\infty\subseteqq \text{range}\left(I-\sum_{k=m}^{\infty} E(\lambda_k)\right),$$

the above implies that S_∞ is finite dimensional and hence consists of the null vector alone.

It remains to verify the special hypothesis placed on the spectrum of T in Theorem 1 and embodied in the requirement that for all sufficiently large N, for all

$$\lambda \in C_N \equiv \left\{\lambda \,\middle|\, |\lambda-\lambda_N| = \frac{1}{3} d_N\right\},$$

$$\max_{z_k \in \sigma(T)} \frac{|z_k|^\nu}{|\lambda - z_k|} < \frac{C}{d_N^{\tau/2}}$$

where

$$d_N = \operatorname{dist}[\lambda_N, \sigma(T) \sim \{\lambda_N\}].$$

First observe that if $N=k$,

$$\frac{|z_k|^\nu}{|\lambda - z_k|} = \frac{|\lambda_N|^\nu}{d_N}$$

and if $N \neq k$

$$\frac{|z_k|^\nu}{|\lambda_N - z_k|} = \frac{1}{|D_{Nk}|^{1-\nu}} \left|\frac{\lambda_N}{D_{Nk}} + e^{i\varphi}\right| \leq \frac{1}{|D_{Nk}|^{1-\nu}} \left\{\left|\frac{\lambda_N}{D_{Nk}}\right| + 1\right\}^\nu$$

$$\leq \frac{1}{d_N^{1-\nu}} \left\{\frac{|\lambda_N|}{d_N} + 1\right\}^\nu \leq \frac{\{|\lambda_N| + 1\}^\nu}{d_N}$$

$$D_{Nk} \equiv (z_k - \lambda_N) e^{-i\varphi}.$$

In any case, therefore, there exists a C such that

$$\max_{z_k \in \sigma(T)} \frac{|z_k|^\nu}{|\lambda - z_k|} < C \frac{|\lambda_N|^\nu}{d_N}.$$

Now recall that according to the previously obtained asymptotic formulas, $\lambda_N \sim N^{2\mu}$ and $d_N \sim N^{2\mu-1}$. Hence

$$\frac{|\lambda_N|^\nu}{d_N} \sim N^{2\mu(\nu-1)+1}.$$

Convergence of

$$\sum_{k=1}^{\infty} \frac{1}{d_k^\tau}$$

is assured for $\tau > 1/2\mu - 1$. It is thus required that $2\mu(\nu-1)+1 < -1/2$. This requirement is satisfied by taking $\nu < (\mu - 3/4)/\mu$ and, a fortiori, by the choice $\nu = (2\mu-2)/2\mu$.

Finally. it is necessary to determine the class of operators S for which $\mathscr{D}(S) \supset \mathscr{D}(T^\nu)$. To this end it shall be shown first that if $f(x) \in \mathscr{D}(T^\nu)$, then $f(x)$ is $\nu = 2\mu - 2$ times differentiable. Suppose $f \in \mathscr{D}(T^\nu)$. Then

$$f(x)=\sum_{k=1}^{\infty} E(\lambda_k)f.$$

If this series is differentiated termwise $2\mu-1$ times one has formally,

$$\sum_{k=1}^{\infty}(-1)^{\mu-1}(2\pi)^{2\mu-2}(k\pm 1/2)^{2\mu-2}$$
$$\times\left\{\int_0^1 \sin 2\pi(k\pm 1/4)x \sin 2\pi(k\pm 1/4)yf(y)dy+\frac{K\pm(x, f)}{k}\right\}.$$

But this expansion converges for $f\in\mathscr{D}(T^\nu)$ almost everywhere to $f^{(2\mu-2)}(x)$. Now let S be any closed operator whose domain consists of $2\mu-2$ times differentiable functions on the closed interval (0, 1) such that the $(2\mu-2)nd$, derivative is square-integrable. Theorem 1 applies.

Thus in conclusion one has the following.

THEOREM 3. *Let T be the operator $d^{2\mu}/dx^{2\mu}$ with boundary conditions*

$$A_i(f)=f^{(k_i)}(0)+\sum_{j=0}^{k_i-1}\alpha_{ij}f^{(j)}(0) \qquad i=1, 2, \cdots, \mu$$
$$k_1>k_2>\cdots>k_\mu$$

$$B_i(f)=f^{(l_i)}(1)+\sum_{j=0}^{l_i-1}\alpha_{ij}f^{(j)}(1) \qquad i=1, 2, \cdots, \mu$$
$$l_1>l_2>\cdots>l_\mu$$

then, if S is any closed operator whose domain consists of $2\mu-2$ times differentiable functions f with $f^{(2\mu-2)}(x)\in\mathscr{L}_2(0, 1)$, $T+S$ is a spectral operator and, if $E(\lambda_k)$ is its spectral measure, then

$$I=\sum_{k=1}^{\infty}E(\lambda_k).$$

In particular, one may make the following choice for S:

$$q_2(x)\frac{d^{2\mu-2}}{dx^{2\mu-2}}+q_3(x)\frac{d^{2\mu-3}}{dx^{2\mu-3}}+\cdots+q_{2\mu}(x)$$

where the coefficients $q_i(x)\in\mathscr{L}_2(0, 1)$[2]. More generally, S may be chosen in the form:

$$Q_2\frac{d^{2\mu-2}}{dx^{2\mu-2}}+Q_3\frac{d^{2\mu-3}}{dx^{2\mu-3}}+\cdots+Q_{2\mu}$$

[2] Note that the theorem actually holds for the wider class of boundary value problems in which the $2\mu-1$th derivative can be eliminated by a standard change of dependent variable [**7**, p. 72],

where Q_i is any bounded operator in $\mathscr{L}_2(0, 1)$.

Application of Theorem 2 shows that if $f \in \mathscr{L}_2(0, 1)$ and $E(\lambda_k)$ are the idempotents corresponding to $T+S$, then the series expansion

$$f = \sum_{k=1}^{\infty} E(\lambda_k) f$$

converges in $\mathscr{L}_2(0, 1)$ norm.

An additional consequence of Theorem 3 and Corollary 2 of [9, p. 448] is the following.

COROLLARY 1. *If $f \in C^{2\mu-1}$, $f^{(2\mu-1)}(x)$ is absolutely continuous, $f^{(2\mu-1)}(x) \in \mathscr{L}_2(0, 1)$, and $f(x)$ satisfies the boundary conditions above, then f can be expanded in the series*

$$f = \sum_{k=1}^{\infty} E(\lambda_k) f$$

where convergence is in the sense that, letting

$$S_n(x) = \sum_{k=1}^{n} E(\lambda_k) f ,$$

we have

$$\lim_{n\to\infty} \left\{ \int_0^1 |f(x)^{(2\mu)} - S_n^{(2\mu)}(x)|^2 \, dx \right\}^{1/2} + \max_{0 \leqq x \leqq 1} \max_{0 \leqq i \leqq 2\mu-1} |f^{(i)}(x) - S_n^{(i)}(x)| = 0 .$$

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DEVELOPMENT OF THE MAPPING FUNCTION AT AN ANALYTIC CORNER

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1. Introduction. In this paper we shall apply some theorems proved in [3] to study the following problem in conformal mapping. Let D be a domain of the complex plane, the boundary of which in the neighborhood of the origin consists of portions of two analytic curves Γ_1 and Γ_2. Suppose Γ_1 and Γ_2 meet at the origin and form a corner with opening $\pi\alpha > 0$, and suppose the origin is a regular point of both curves. Let $F(z)$ be a function which maps conformally the upper half plane $\Im z > 0$ onto the domain D, and suppose that $F(0)=0$. How does the mapping function $F(z)$ behave in the neighborhood of the origin?

A partial answer to this question is given by a theorem stated by Lichtenstein [5]. Let $F^{-1}(z)$ be the inverse function which maps D onto the upper half plane. Then Lichtenstein stated that for z in the neighborhood of the origin

$$\frac{dF^{-1}(z)}{dz} = z^{1/\alpha - 1}\varphi(z)$$

where $\varphi(z)$ is a continuous function with $\varphi(0) \neq 0$.[1] This same result can, however, be obtained with much weaker requirements on the boundary curve as has been shown by the work of Kellogg [2] and Warschawski [6].

In the case $\alpha = 1$ where the curves Γ_1 and Γ_2 meet at a straight angle Lewy [4] has proved a much stronger result—that $F(z)$ has an asymptotic expansion in powers of z and $\log z$. The method used in this paper is a generalization of that used by Lewy. We find that for all $\alpha > 0$ the function $F(z)$ has an asymptotic expansion in the neighborhood of the origin. If α is irrational then the expansion is in integral powers of z, and z^α. If α is rational then the expansion is in integral powers of z, z^α, and $\log z$.

2. Notation. First let us make clear what type of asymptotic expansions we will be considering. Let $\chi_n(z)$, $(n=0, 1, 2, \cdots)$ be a sequence of functions such that $\chi_{n+1}(z)/\chi_n(z) \to 0$ as $z \to 0$ in the sector

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[1] Lichtenstein proved this result only in the case of irrational α. The complete theorem has been proved recently by Warschawski [7].

$\theta_1 \leq \arg z \leq \theta_2$. A series $\sum_{n=0}^{\infty} A_n \chi_n(z)$ is called an asymptotic expansion for $f(z)$ valid in the sector $\theta_1 \leq \arg z \leq \theta_2$, and we write

$$f(z) \sim \sum_{n=0}^{\infty} A_n \chi_n(z) ,$$

if for every integer $N \geq 0$

$$f(z) = \sum_{n=0}^{N} A_n \chi_n(z) + o(\chi_N(z))$$

as $z \to 0$, $\theta_1 \leq \arg z \leq \theta_2$.

Clearly, in a sector $\theta_1 \leq \arg z \leq \theta_2$ a function $f(z)$ cannot have more than one asymptotic expansion in terms of such a sequence of functions $\chi_n(z)$.

We shall sometimes be concerned with asymptotic expansions which are valid in every finite sector on the logarithmic Riemann surface with the origin as branch point. By this we mean that the limits hold for $z \to 0$ in any finite sector $\theta_1 \leq \arg z \leq \theta_2$ where θ_1 and θ_2 are arbitrary constants. Otherwise expressed, we consider any sequence $z_1, z_2, z_3, \cdots$ such that there exist constants θ_1 and θ_2 for which

$$\theta_1 \leq \arg z_n \leq \theta_2 \qquad (n=1, 2, 3, \cdots)$$

and

$$\lim_{n \to \infty} |z_n| = 0 .$$

Thus we exclude any sequence for which $\limsup_{n \to \infty} |\arg z_n| = \infty$.

Throughout this paper we will use the letter c to denote a typical coefficient in a series when the exact value of the coefficient is not important in the discussion. For example, instead of writing $\sum_{n=0}^{\infty} c_{mn} z^n$, we may write simply $\sum_{n=0}^{\infty} c z^n$. Thus we avoid a multiplicity of subscripts.

3. **Principal results.** Let $F(z)$ be the mapping function which maps the upper half plane onto the domain D, and let Γ_1 be the image of a portion of the negative real axis and Γ_2 the image of a portion of the positive real axis. We shall prove the following theorem.

THEOREM 1. *If $\alpha > 0$ is irrational, then for $z \to 0$ in any finite sector*

$$F(z) \sim \sum A_{kl} z^{k+l\alpha}$$

where k and l run over integers, $k \geq 0$, $l \geq 1$; and the coefficient $A_{01} \neq 0$. If $\alpha = p/q > 0$, a fraction reduced to lowest terms, then for $z \to 0$ in any finite sector

$$F(z) \sim \sum A_{klm} z^{k+l\alpha} (\log z)^m$$

where k, l, and m run over integers for which

$$k \geqq 0\,, \qquad 1 \leqq l \leqq q\,, \qquad 0 \leqq m \leqq k/p\,;$$

and the coefficient $A_{010} \neq 0$.

In this theorem the terms in the series are supposed to be arranged in an order such that a term of the form $z^{k+l\alpha}(\log z)^m$ precedes one of the form $z^{k'+l'\alpha}(\log z)^{m'}$ if either $k+l\alpha < k'+l'\alpha$ or $k+l\alpha = k'+l'\alpha$ and $m > m'$. Arranged in this order, these products of powers of z and $\log z$ form a sequence of functions χ_n. The coefficients in these expansions are complex constants, some of which may be zero.

From Theorem 1 an asymptotic expansion for the inverse function $F^{-1}(z)$, which maps the domain D onto a portion of the upper half plane, can be obtained easily by replacing the asymptotic expansions by finite developments with error terms and proceeding as usual in the inversion of functions. The result obtained is stated in the following theorem.

THEOREM 2. *If α is irrational, then for $z \to 0$ in any finite sector the inverse of $F(z)$,*

$$F^{-1}(z) \sim \sum B_{kl} z^{k+l/\alpha}$$

where k and l run over integers, $k \geqq 0$, $l \geqq 1$; and $B_{01} \neq 0$. If $\alpha = p/q$, a fraction reduced to lowest terms, then for $z \to 0$ in any finite sector

$$F^{-1}(z) \sim \sum B_{klm} z^{k+l/\alpha} (\log z)^m$$

where k, l and m run over integers for which $k \geqq 0$, $1 \leqq l \leqq p$, $0 \leqq m \leqq k/q$; and $B_{010} \neq 0$.

There is another way to state Theorems 1 and 2 in the case of rational α. We can write

$$F(z) \sim z^{1/\alpha} M_1(z, z^{1/\alpha}, z^q \log z)$$

and

$$F^{-1}(z) \sim z^{\alpha} M_2(z, z^{\alpha}, z^p \log z)$$

where M_1 and M_2 are triple power series in their three arguments. In the case $\alpha = 1$ the triple power series reduces to a double series in z and $z \log z$ as found by Lewy [4].

Observe that the function $F(z)$, defined originally for $0 \leqq \arg z \leqq \pi$, can be extended by the reflection principle across both the positive x-axis and the negative x-axis since the curves Γ_1 and Γ_2 are analytic curves. The images of Γ_1 and Γ_2 in such reflections are again analytic curves. Hence $F(z)$ can again be extended by reflection, and in fact can be continued near the origin onto the entire logarithmic Riemann surface with branch point at the origin. The function $F(z)$ is regular for $|z|$ sufficiently small, say, $0<|z|<\rho$, on any sheet of this Riemann surface; but, generally speaking, ρ depends on the sheet of the surface.

4. Extension of developments to larger sectors. If the asymptotic expansions of Theorems 1 and 2 hold for $z\to 0$ in $0\leqq \arg z\leqq \pi$, they hold for $z\to 0$ in any finite sector $\theta_1\leqq \arg z\leqq \theta_2$. Suppose, indeed, that for given $r>0$, $F(z)$ has a finite development of the form

$$(4.1)\qquad F(z)=\sum A_{klm}\, z^{k+l\alpha}(\log z)^m+o(z^r)$$

as $z\to 0$, $0\leqq \arg z\leqq \pi$, where the sum is extended over integers k, l, and m such that $k+l\alpha\leqq r$, $k\geqq 0$, $l\geqq 1$; and $0\leqq m\leqq k/p$ when $\alpha=p/q$, $m=0$ when α is irrational. Then the same development is valid for $z\to 0$ with $-\pi\leqq \arg z\leqq 0$. To see this let ζ^* be the image of ζ in an analytic reflection on the curve Γ_1. Then $\bar{\zeta}^*$, the complex conjugate of ζ^*, is an analytic function of ζ, say $\Phi(\zeta)$, which is regular for $|\zeta|$ sufficiently small. By the reflection principle, since $F(z)$ takes the positive real axis, $\arg z=0$, into the analytic curve Γ_1, we have

$$F(\bar{z})=(F(z))^*=\overline{\Phi(F(z))}$$

for $0\leqq \arg z\leqq \pi$. Observe that this formula continues $F(z)$ for $|z|$ sufficiently small into the sector $-\pi\leqq \arg z\leqq \pi$. Since $\Phi(\zeta)$ is regular for $|\zeta|$ sufficiently small and $\Phi(0)=0$, we have

$$\Phi(\zeta)=\sum_{n=1}^{[r/\alpha]} c\zeta^n+o(\zeta^{r/\alpha})$$

for $z\to 0$. Then with

$$\zeta=F(z)=z^\alpha\left[\sum cz^{k+l\alpha}(\log z)^m+o(z^{r-\alpha})\right], \qquad (k\geqq 0,\ l\geqq 0,\ k+l\alpha\leqq r-\alpha)$$

we have by (4.1) for $z\to 0$, $0\leqq \arg z\leqq \pi$,

$$\zeta^n=z^{n\alpha}\left[\sum cz^{k+l\alpha}(\log z)^m+o(z^{r-n\alpha})\right]$$

where $k\geqq 0$, $l\geqq 0$, $k+l\alpha\leqq r-n\alpha$; m is limited as before. Also

$$o(\zeta^{r/\alpha})=o((O(z^\alpha))^{r/\alpha})=o(z^r)$$

as $z\to 0$. Consequently for $z\to 0$, $0\leqq \arg z\leqq \pi$,

$$F(\bar{z})=\overline{\Phi(F(z))}=\sum c\bar{z}^{k+l\alpha}(\log \bar{z})^m+o(\bar{z}^r)$$

where k, l, and m are restricted in the same way as in (4.1). But this means that $F(z)$ has a development of the same type as (4.1) for $-\pi\leqq \arg z\leqq 0$. This new development must coincide with that given by (4.1) since both hold for $z\to 0$ with $\arg z=0$.

In the same way we can reflect across the line $\arg z=\pi$ and establish that (4.1) holds in the larger sector thus obtained. By induction we can prove that (4.1) holds in any finite sector $\theta_1 \leqq \arg z \leqq \theta_2$. Thus we see that if Theorem 1 holds for $z \to 0$ in the sector $0 \leqq \arg z \leqq \pi$, it holds for $z \to 0$ in any finite sector.

5. **Some lemmas.** We now state some lemmas which will be used in the proof of Theorem 1. Lemmas 1 and 2 are special cases of Theorems 4.1 and 4.2 of [3]. The integrals are Lebesgue integrals extended over positive values of t. The range of z considered is $0<|z|<A$, $-2\pi \leqq \arg z \leqq 0$. We take the branch of the analytic function of z, $\log(1-z/t)$ which is real for $0<z<t$, $\arg z=0$.

LEMMA 1. *Let A be a positive real number, μ a real number >-1, and n a nonnegative integer ; and let*

$$\varphi(z)=\int_0^A t^\mu (\log t)^n \log(1-z/t)dt \ .$$

Then there is a power series $q(z)$, which converges for $|z|<A$, and a polynomial in $\log z$, $P(\log z)$, *such that*

$$\varphi(z)+z^{\mu+1}P(\log z)+q(z) \ .$$

If μ is an integer then, the polynomial P is of degree $n+1$; and if μ is not an integer, it is of degree n.

LEMMA 2. *Let $\beta(t)$ be a measurable function, bounded absolutely for $0<t<A$ and such that $\beta(t)\to 0$ as $t \to 0$ through positive real values. Let μ be a real number >-1 which is not an integer, and let*

$$\beta_1(z)=\int_0^A \beta(t)t^\mu \log(1-z/t)\,dt \ .$$

Then there is a power series $q(z)$ such that for $z \to 0$

$$\beta_1(z)=q(z)+o(z^{\mu+1}) \ .$$

LEMMA 3. *Let μ be a real number. Let $\eta(z)$ be an analytic function, regular for $0<|z|<R$, $\theta_1 \leqq$* $\arg z$ *$\leqq \theta_2$ and such that $\eta(z)=o(z^\mu)$ for $z\to 0$ in the sector $\theta_1 \leqq$* $\arg z$ *$\leqq \theta_2$. Then the derivative*

$$\eta'(z)=o(z^{\mu-1})$$

for $z \to 0$ in any sector in the interior of the sector $\theta_1 \leqq$ $\arg z$ *$\leqq \theta_2$.*

A proof of Lemma 3 is obtained by estimating a Cauchy integral with path a circle about z with radius $\delta|z|$, δ small.

LEMMA 4. *Let λ be a real number. Then for $z \to 0$ with $|\arg z|$ bounded, $|z^{-\lambda}F(z)|$ tends to zero if $\lambda < \alpha$ and tends to infinity if $\lambda > \alpha$.*

A proof of Lemma 4 can be obtained by a study of the Poisson integral (see Gross [**1**, pp. 57–61]; the requirement that $z \to 0$ in an angle in the interior of $0 \leqq \arg z \leqq \pi$ can be eliminated by using the fact that Γ_1 and Γ_2 are analytic curves).

This lemma also follows from the theorem of Lichtenstein mentioned in the introduction.

6. Preliminary transformations. First we establish that the general case can be reduced to the special case in which the curve Γ_2 is an analytic curve tangent to the positive real axis and Γ_1 is a portion of the ray $\arg \zeta = -\pi\alpha$ in the ζ plane. Consider a function $\psi(\zeta)$, regular for $|\zeta|$ sufficiently small, for which $\psi(0)=0$, $\psi'(0)=b\neq 0$, and which takes the analytic curve Γ_1 into the line $\arg\zeta = -\pi\alpha$. The function ψ maps Γ_2 into an analytic curve tangent to the positive real axis. For the sake of simplicity of notation we carry through the proof in detail only for irrational α.

Suppose that we know Theorem 1 in the special case in which Γ_1 is the line $\arg\zeta = -\pi\alpha$, then for $z \to 0$ we have

$$\psi(F(z)) = z^{\alpha}\{\sum C_{kl}\, z^{k+l\alpha} + o(z^r)\}$$

where the sum is extended over integers k and l for which $k \geqq 0$, $l \geqq 0$, $k + l\alpha \leqq r$. In addition, we can suppose that $C_{00} \neq 0$. Then since the inverse

$$\psi^{-1}(\zeta) = \frac{\zeta}{b} + \sum_{n=2}^{N} c\zeta^n + o(\zeta^N)$$

as $\zeta \to 0$, we have

$$F(z) = \psi^{-1}(\psi(F(z))) = \frac{1}{b} z^{\alpha}\{\sum C_{lk}\, z^{k+l\alpha} + o(z^r)\} + z^{2\alpha}\{\sum c z^{k+l\alpha} + o(z^r)\}$$
$$+ \cdots + z^{N\alpha}\{\sum c z^{k+l\alpha} + o(z^r)\} + o(z^{N\alpha})$$

for $z \to 0$. Hence by taking N large enough, we obtain

$$F(z) = z^{\alpha}\{\sum C'_{kl}\, z^{k+l\alpha} + o(z^r)\}$$

where $C'_{00} = \frac{1}{b} C_{00} \neq 0$. All of the sums considered are extended over integers $k \geqq 0$, $l \leqq 0$, $k + l\alpha \leqq r$. Thus we need consider only the special case in which Γ_1 is a portion of the line $\arg\zeta = -\pi\alpha$.

Now we make another preliminary transformation. Let $w = \zeta^{1/\alpha}$, so that the line $\arg\zeta = -\pi\alpha$ goes into the negative real axis. The analytic curve Γ_2 goes into a curve Γ' tangent to the positive real axis. This new curve Γ' is not analytic at the origin; we will find it useful to have the equation of Γ'.

Let $\zeta=\xi+i\eta$. The analytic curve Γ_2 is given by an equation with real a_j

$$\eta=a_2\xi^2+a_3\xi^3+a_4\xi^4+\cdots$$

for $\xi>0$, where the series is convergent for ξ sufficiently small. Then on Γ' we have

$$\begin{aligned} w=\zeta^{1/\alpha}=(\xi+i\eta)^{1/\alpha}&=\xi^{1/\alpha}(1+ia_2\xi+ia_3\xi^2+\cdots)^{1/\alpha} \\ &=\xi^{1/\alpha}(1+c\xi+c\xi^2+\cdots)\ . \end{aligned}$$

Separating real and imaginary parts, we have with $w=u+iv$

$$u=\xi^{1/\alpha}(1+c\xi^2+c\xi^3+\cdots)$$

$$v=\xi^{1/\alpha}(c\xi+c\xi^2+c\xi^3+\cdots)\ .$$

Consequently,

$$u^\alpha=\xi+c\xi^2+c\xi^3+\cdots$$

and thus

$$\xi=u^\alpha+cu^{2\alpha}+cu^{3\alpha}+\cdots\ .$$

Hence we obtain finally that the curve Γ' is given by an equation of the form

$$v=u\sum_{k=1}^{\infty}b_k u^{k\alpha} \tag{6.1}$$

for $u\leqq 0$, where the series converges for u sufficiently small.

7. Obtaining the asymptotic expansion. Let D' be the image of D under the transformation $w=\zeta^{1/\alpha}$; we can now assume that near the origin D' is bounded by the negative real axis and the curve Γ' given by the equation (6.1). We consider the function $w=G(z)=(F(-z))^{1/\alpha}$ which is a univalent conformal mapping of a semi-neighborhood $y<0$ of the $z=x+iy$ plane into the domain D' of the $w=u+iv$ plane. Observe that $G(0)=0$, a portion $-A\leqq x\leqq 0$ of the negative x-axis is mapped into a portion of the negative u-axis, and a portion $0\leqq x\leqq A$ of the positive x-axis goes into Γ'.

We will need an estimate for $G(z)$ and its derivative $G'(z)$. By Lemma 4 we have for $z\to 0$, $|\arg z|$ bounded

$$G(z)=[F(-z)]^{1/\alpha}=[o(z^\lambda)]^{1/\alpha}$$

for any $\lambda<\alpha$. Hence for any $\varepsilon>0$

$$G(z)=o(z^{1-\varepsilon}) \tag{7.1}$$

as $z\to 0$ with $|\arg z|$ bounded. Using Lemma 3 we conclude further that for $z\to 0$

$$G'(z)=o(z^{-\varepsilon}) . \tag{7.2}$$

Now we construct a certain function $H(z)$ which differs from $G(z)$ by a single-valued function. Observe that the function

$$G(z)=u(x, y)+iv(x, y)$$

can be continued across the negative real axis, $\arg z=-\pi$, by the reflection principle. In particular, we have for $\arg z=0$

$$\begin{aligned}G(z)-(Gze^{-2\pi i})&=u(z, 0)+iv(z, 0)-[u(z, 0)-iv(z, 0)]\\&=2iv(z, 0) .\end{aligned}$$

Consider for $-2\pi\leqq\arg z\leqq 0$ the analytic function

$$H(z)=\frac{1}{\pi}\int_0^A\frac{\partial v(t, 0)}{\partial t}\log(1-z/t)\,dt \tag{7.3}$$

where the integral is extended over positive real values and the branch of $\log(1-z/t)$ considered is the one which is real for $0<z<t$, $\arg z=0$. That the integral converges follows from the estimate (7.2).

For $\arg z=0$ we have

$$H(z)-H(ze^{-2\pi i})=\frac{2\pi i}{\pi}\int_0^z\frac{\partial v(t, 0)}{\partial t}dt=2iv(z, 0)$$

since

$$\log(1-z/t)-\log(1-ze^{-2\pi i}/t)=\begin{cases}2\pi i & \text{for } t<z ,\\ 0 & \text{for } t>z .\end{cases}$$

Thus the difference $p(z)=G(z)-H(z)$ satisfies the condition $p(z)=p(ze^{-2\pi i})$ for $\arg z=0$. Furthermore $p(z)$ is regular for $0<|z|<A$, $-2\pi<\arg z<0$; it is continuous as z approaches a point of the positive real axis for $\arg z=0$ or $\arg z=-2\pi$, and it is bounded for $z\to 0$. Hence by Riemann's theorem on removable singularities $p(z)$ is equal to a power series convergent for $|z|<A$.

From (6.1) and (7.3) we conclude that for $-2\pi\leqq\arg z\leqq 0$

$$G(z)=\frac{1}{\pi}\int_0^A\left\{\frac{\partial u(t, 0)}{\partial t}\sum_{n=1}^{\infty}b_n(1+n\alpha)u^{n\alpha}\right\}\log(1-z/t)\,dt+p(z) \tag{7.4}$$

where $p(z)$ is a power series with constant term equal to zero.

By (7.1) and (7.2) we have $\frac{\partial u(t,0)}{\partial t}=o(t^{-\varepsilon})$ and $u^{\alpha}=o(t^{(1-\varepsilon)\alpha})$ for $t\to 0$, ε an arbitrary positive number. Hence for $t\to 0$

$$\frac{\partial u(t,0)}{\partial t}\sum_{n=1}^{\infty}b_n(1+n\alpha)u^{n\alpha}=o(t^{(1+\alpha)(1-\varepsilon)-1})\ .$$

Inserting this estimate in (7.4) and applying Lemma 2, we obtain for $z\to 0$, $-2\pi\leqq\arg z\leqq 0$

$$G(z)=az+z^2q(z)+o(z^{(1+\alpha)(1-\varepsilon)}) \tag{7.5}$$

where $q(z)$ is a power series in z which converges for $|z|$ sufficiently small. We conclude that $a\neq 0$ by applying Lemma 4 with λ slightly larger than α. Knowing this, we can conclude further that a is positive from the fact that $G(z)$ maps the positive real axis into Γ', a curve which at the origin makes an angle of π with the negative real axis. Since $G(z)=[F(-z)]^{1/\alpha}$ the result of § 4 shows that the estimate (7.5) holds for $z\to 0$ in any finite sector.

Now we prove Theorem 1 by induction. We consider first the case in which α is irrational. We shall prove that there are constants a_{kl} such that for every integer N,

$$G(z)=\sum_{k+l\alpha\leqq N\alpha}a_{kl}z^{k+l\alpha}+o(z^{N\alpha})\ ,\qquad k\geqq 1,\ l\geqq 0 \tag{7.6}$$

as $z\to 0$ with $|\arg z|$ bounded. We begin by noting that $G(z)$ has such a development for $N=N_0$ where N_0 is the integer for which $\frac{1}{\alpha}\leqq N_0<1+\frac{1}{\alpha}$. This follows directly from (7.5) since for ε sufficiently small $(1+\alpha)(1-\varepsilon)\geqq N_0\alpha$ and hence $o(z^{(1+\alpha)(1-\varepsilon)}=o(z^{N_0\alpha})$. Consequently, to prove (7.6) by induction it will be sufficient to show that if $G(z)$ has a development of the type (7.6) with an error term $0(z^{N\alpha})$, then $G(z)$ has such a development with an error term $o(z^{(N+1)\alpha})$. In proving (7.6) by induction we will simultaneously obtatin a proof of Theorem 1 by using the fact that $F(z)=[G(-z)]^{\alpha}$.

By the induction hypothesis we have

$$u(t,0)=\sum_{k+l\alpha\leqq N\alpha}\Re\{a_{kl}\}t^{k+l\alpha}+o(t^{N\alpha})\ ,\qquad (k\geqq 1,\ l\geqq 0)\ ,$$

and thus since $a_{10}=a>0$

$$u(t,0)=at\{1+\sum ct^{k+l\alpha}+o(t^{N\alpha-1})\}$$

where the sum is over $k\geqq 0$, $l\geqq 0$, for which $(k,l)\neq(0,0)$ and $k+l\alpha\leqq N\alpha-1$. Using the binomial theorem, we find

$$u^{n\alpha}=a^{n\alpha}t^{n\alpha}\{\sum_{k+l\alpha\leqq N\alpha-1} ct^{k+l\alpha}+o(t^{N\alpha-1})\}\ , \qquad (k\geqq 0,\ l\geqq 0)\ .$$

Moreover, by Lemma 3 and the induction hypothesis we have for $t\to 0$

$$\frac{\partial u(t,0)}{\partial t}=\Re G'(t)=\sum_{k+l\alpha\leqq N\alpha-1}(k+1+l\alpha)\Re\{a_{k+1,l}\}t^{k+l\alpha}+o(t^{N\alpha-1})$$

where $k\geqq 0$, $l\geqq 0$. Inserting these estimates in (7.4), we obtain

$$G(z)=p(z)+\int_0^A\{\sum ct^{k+l\alpha}+o(t^{(N+1)\alpha-1})\}\log(1-z/t)\,dt$$

where the sum is over integers $k\geqq 0$, $l\geqq 1$, for which $k+l\alpha\leqq(N+1)\alpha-1$. Now we apply Lemmas 1 and 2, observing that since $l\geqq 1$ and α is irrational, $k+l\alpha$ cannot be an integer. We find for $z\to 0$, $-2\pi\leqq\arg z\leqq 0$,

$$G(z)=\sum_{k+l\alpha\leqq(N+1)\alpha}a_{kl}z^{k+l\alpha}+o(z^{(N+1)\alpha}), \qquad (k\geqq 1\ ,\quad l\geqq 0)\ . \tag{7.7}$$

When k and l are integers for which $k+l\alpha\leqq N\alpha$, the coefficient a_{kl} must, of ceurse, be the same as that appearing in the development with error term $o(z^{N\alpha})$.

We wish to prove that (7.7) holds for $z\to 0$ in any finite sector. We note that for $z\to 0$, $0\leqq\arg z\leqq 2\pi$

$$F(z)=[G(-z)]^\alpha=a^\alpha(-z)^\alpha\{1+\sum c(-z)^{k+l\alpha}+o(z^{(N+1)\alpha-1})\}^\alpha$$

where the sum is over $k\geqq 0$, $l\geqq 0$, which for $(k,l)\neq(0,0)$ and $k+l\alpha\leqq(N+1)\alpha-1$. Hence by the binomial theorem

$$F(z)=\sum A_{kl}z^{k+l\alpha}+o(z^{(N+2)\alpha-1}) \tag{7.8}$$

where the sum is extended over $k\geqq 0$, $l\geqq 1$, for which $k+l\alpha\leqq(N+2)\alpha-1$. Note further that $A_{01}\neq 0$. We have proved (7.8) for $z\to 0$ with $0\leqq\arg z\leqq 2\pi$, but by the result of § 4 this formula must hold for $z\to 0$ in any finite sector. Consequently, from (7.8) by using the binomial theorem we can obtain (7.7) for $z\to 0$ in any finite sector. Thus $G(z)$ has a development with error term $o(z^{(N+1)\alpha})$. Hence by induction (7.6) and also (7.8) hold for all N. This proves Theorem 1 for irrational α.

Now we prove Theorem 1 for $\alpha=p/q$, a fraction reduced to lowest terms. Let γ be a positive irrational number less than α. We shall prove that there are constants a_{klm} such that for every integer N, as $z\to 0$, in a finite sector

$$G(z)=\sum_{k+l\alpha\leqq N\gamma}a_{klm}z^{k+l\alpha}(\log z)^m+o(z^{N\gamma}) \tag{7.9}$$

where $k\geqq 1$, $0\leqq l\leqq q-1$, and $0\leqq m\leqq\dfrac{k-1}{p}$. We begin by noting that $G(z)$ has such a development for $N=N_0$ where N_0 is the integer for which $\frac{1}{\gamma} \leq N_0 < 1 + \frac{1}{\gamma}$, as can be seen directly from (7.5). Consequently, to prove (7.9) by induction it will be sufficient to show that if $G(z)$ has a development of the above type with error term $o(z^{N\gamma})$, then it has such a development with error term $o(z^{(N+1)\gamma})$.

By the induction hypothesis we have for positive $t \to 0$

$$u(t, 0) = \sum_{k+l\alpha \leq N\gamma} \Re\{a_{klm}\} t^{k+l\alpha} (\log t)^m + o(t^{N\gamma})$$

where $k \geq 1$, $0 \leq l \leq q-1$, $0 \leq m \leq \frac{k-1}{p}$. Since $a_{100} = a \neq 0$, we have

$$u(t, 0) = at\{1 + \sum ct^{k+l\alpha}(\log t)^m + o(t^{N\gamma-1})\}$$

where the sum is over integers for which

$$(7.10) \qquad k \geq 0\ ; \quad 0 \leq l \leq q-1\ ; \quad 0 \leq m \leq k/p\ ; \quad k + l\alpha \leq N\gamma - 1\ .$$

Using the binomial theorem, we obtain

$$u^{n\alpha} = a^{n\alpha} t^{n\alpha} \{\sum ct^{k+l\alpha}(\log t)^m + o(t^{N\gamma-1})\}$$

where k, l, and m are restricted by the conditions (7.10). Moreover, by Lemma 3 and the induction hypothesis we have

$$\frac{\partial u(t, 0)}{\partial t} = \sum ct^{k+l\alpha}(\log t)^m + o(t^{N\gamma-1})$$

where again k, l, and m are restricted by the conditions (7.10).

Inserting these estimates in (7.4) we have, since

$$o(t^{N\gamma+\alpha-1}) = o(t^{(N+1)\gamma-1})\ ,$$

the formula

$$G(z) = p(z) + \int_0^A \{\sum ct^{k+l\alpha} + o(t^{(N+1)\gamma-1})\} \log(1 - z/t)\, dt\ .$$

The sum in the integrand is extended over integers k, l, and m for which

$$k \geq 0\ ; \quad 1 \leq l \leq q\ ; \quad 0 \leq m \leq k/p\ ; \quad k + l\alpha \leq (N+1)\gamma - 1\ .$$

Now we apply Lemmas 1 and 2 to obtain a better development for $G(z)$. Note that $k + l\alpha = k + lp/q$ cannot be an integer unless $l = q$. Consequently terms of the form

$$ct^{k+l\alpha}(\log t)^m$$

in the integrand, with $l \neq q$, produce besides a power series only terms of the form

$$cz^{k+1+l\alpha}(\log z)^{m'}$$

with

$$0 \leqq m' \leqq m \leqq \frac{(k+1)-1}{p}$$

in the development for $G(z)$. On the other hand, when $l=q$ they produce besides a power series only terms of the form

$$cz^{k+1+l\alpha}(\log z)^{m'} = cz^{k+p+1}(\log z)^{m'}$$

with

$$0 \leqq m' \leqq m+1 \leqq \frac{k}{p}+1 = \frac{(k+p+1)-1}{p} .$$

In applying Lemma 2 we observe that $(N+1)\gamma - 1$ is not an integer because γ is irrational. Hence we conclude that for $z \to 0$, $-2\pi \leqq \arg z \leqq 0$,

$$G(z) = \sum_{k+l\alpha \leqq (N+1)\gamma} a_{klm} z^{k+l\alpha}(\log z)^m + o(z^{(N+1)\gamma}) \tag{7.11}$$

where $k \geqq 1$, $0 \leqq l \leqq q-1$, and $0 \leqq m \leqq \dfrac{k-1}{p}$.

As in the case of irrational α we obtain from this the result

$$F(z) = \sum A_{klm} z^{k+l\alpha}(\log z)^m + o(z^{(N+1)\gamma+\alpha-1}) , \qquad (A_{010} \neq 0) \tag{7.12}$$

where the sum is over integers k, l, and m for which

$$k \geqq 0 , \quad 1 \leqq l \leqq q , \quad 0 \leqq m \leqq k/p ; \quad k + l\alpha \leqq (N+1)\gamma + \alpha - 1 .$$

By (7.11) this result holds for $z \to 0$ with $0 \leqq \arg z \leqq 2\pi$, but by the result of § 4, it must hold for $z \to 0$ in any finite sector. From this we then obtain (7.11) for $z \to 0$ in any finite sector. Hence $G(z)$ has a development with error term $o(z^{(N+1)\gamma})$. Thus by induction (7.9) and also (7.12) hold for all N. This completes the proof of Theorem 1.

We note finally that by Lemma 3 derivatives of $F(z)$ and $F^{-1}(z)$ of arbitrary order have asymptotic expansions which can be obtained by differentiating the expansion for $F(z)$ and $F^{-1}(z)$ termwise and then rearranging the terms in the new series in an appropriate order.

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CONVEXITY OF ORLICZ SPACES

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In a paper [1] which appeared in 1936, J. A. Clarkson defined a property of Banach spaces known as uniform convexity. Let $\|f\|$ denote the norm of an element f of such a space and let $\{f_n', f_n''\}$ be any sequence of pairs of elements such that $\|f_n'\|=\|f_n''\|=1$ and $\lim_{n\to\infty} \frac{1}{2}\|f_n'+f_n''\|=1$. The space is said to be *uniformly convex* if these conditions imply that $\lim_{n\to\infty}\|f_n'-f_n''\|=0$. It has been shown [2] that an equivalent definition is one in which the condition $\|f_n'\|=\|f_n''\|=1$ may be replaced with the weaker $\|f_n'\|\leq 1$ and $\|f_n''\|\leq 1$. Clarkson has been successful in showing that the Lebesgue spaces L_p are uniformly convex if $p\neq 1$ and that L_1 is not uniformly convex. The convexity properties of more general classes of Banach spaces have been investigated by M. M. Day [3], I. Halperin [4] and E. J. McShane [7].

A concept of convexity related to uniform convexity has been described and is termed *strict convexity*. It is defined in the following manner. Let f', f'' be any pair of elements in a Banach space such that $\|f'\|=\|f''\|=1$ and $\frac{1}{2}\|f_n'+f_n''\|=1$. The space is said to be strictly convex if these conditions imply that $\|f'-f''\|=0$. In a Euclidean space, strict convexity corresponds geometrically to the property that the unit sphere $\|f\|=1$ does not contain a segment. We remark that, if a space has the property of uniform convexity, then it possesses that of strict convexity as well; however, the converse implication is generally untrue.

The principal objective of this paper is to investigate the conditions which an Orlicz space [9] must satisfy to be uniformly convex. Also the related problem of determining the conditions for strict convexity is considered. A solution to both of these questions has been presented which may be regarded as complete in the sense that both the necessary and sufficient criteria are developed.

We begin by formulating the definitions of Orlicz spaces in accordance with the notations to be used subsequently. Except in minor details we shall adopt the standard conventions. Let $v=\varphi(u)$ be a monotonically nondecreasing function not identically zero, defined for all $0\leq u$ such that $\varphi(u)=\varphi(u-)$ and $\varphi(0)=0$; also, let $\bar{\varphi}(u)$ denote the associated function $\bar{\varphi}(u)=\varphi(u+)$. Let $u=\psi(v)$ be the function inverse to $\varphi(u)$ which is defined by the relations:

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(i) $\psi(0)=0$,
(ii) $\psi(v)=u$ if $\varphi(u)=v$ and u is a point of continuity for $\varphi(u)$,
(iii) $\psi(v)=\psi(v-)$,
(iv) if $\varphi(u)\neq\overline{\varphi}(u)$, then $\psi(v)=u$ for all $\varphi(u)\leqq v\leqq\overline{\varphi}(u)$,
(v) if $\lim_{u\to\infty}\varphi(u)=l<\infty$, then $\psi(v)=+\infty$ for all $l\leqq v$.

Also, let $\overline{\psi}(v)=\psi(v+)$. Since $\varphi(u)$ and $\psi(v)$ are monotonic functions they are Lebesgue measurable and their indefinite Lebesgue integrals define the functions:

$$\Phi(u)=\int_0^u\varphi(\bar{u})d\bar{u} \quad \text{and} \quad \Psi(v)=\int_0^v\psi(\bar{v})d\bar{v}\ .$$

Let Δ be a measure space with a σ-finite nonatomic measure μ and a σ-ring of measurable subsets. Let $f(x)$ be a μ-measurable function defined on Δ; then, the functions $\varphi(|f(x)|)$, $\overline{\varphi}(|f(x)|)$, $\Psi(|f(x)|)$, etc., are also μ-measurable on Δ. For each function $f(x)$, we define:

$$\|f\|_\Phi=\sup\int_\Delta|f(x)|g(x)\,d\mu$$

where the supremum is taken for all $g(x)\geqq 0$ satisfying $\int_\Delta\Psi(g)\,d\mu\leqq 1$. The Orlicz space $L_\Phi=L_\Phi(\Delta,\mu)$ is defined to be the collection of all functions $f(x)$, μ-measurable on Δ, for which $\|f\|_\Phi<\infty$. It may be shown (Zaanen, [10]) that the space L_Φ is a Banach space with the norm $\|f\|_\Phi$. If $\Phi(u)=u^p$, $1\leqq p<\infty$ then L_Φ is the classical Lebesgue space L_p.

Necessary and sufficient conditions for both types of convexity will be expressed directly in terms of the functions φ, ψ, etc. For strict convexity of L_Φ these conditions are simply that $\psi(v)$ and $\Psi(v)$ should be continuous in the extended sense. By this we mean that if V_0 is defined by $V_0=\sup_{\psi(v)<\infty} v$ then $\psi(v)$ and $\Psi(v)$ are continuous for $v<V_0$ and $\lim_{v\to V_0-}\psi(v)=\infty$ and $\lim_{v\to V_0-}\Psi(v)=\infty$. Of course, requirements additional to those for strict convexity must be satisfied to imply uniform convexity. It is found that these conditions are alternative according as Δ is assumed to have finite or infinite measure. If Δ is of infinite measure it is necessary and sufficient that the space satisfy the following requirements: not only must the functions $\psi(v)$ and $\Psi(v)$ be continuous in the extended sense but the function $\varphi(u)$ may neither increase too rapidly nor too slowly. Precisely stated, there must be a constant $0<N<\infty$ such that $\Phi(2u)/\Phi(u)\leqq N$ for all $0<u$ (or what is readily shown to be an equivalent statement, that there exist a constant $0<N<\infty$ such that $\varphi(2u)/\varphi(u)\leqq N$ for all $0<u$), and also that for each constant $0<\varepsilon<1/4$ there is a corresponding constant $1<R_\varepsilon<\infty$ such that if $0<u$ then $R_\varepsilon<\varphi(u)/\varphi((1-\varepsilon)u)$. When Δ is of finite measure, the functions $\psi(v)$ and $\Psi(v)$, as before, must be continuous in the extended sense; however, slightly less stringent conditions apply to the functions $\varphi(u)$ and $\Phi(u)$. It is required merely that the conditions stated for Δ of infinite measure apply only in the limiting sense; namely, that there exist a constant $0<N<\infty$ such that $\limsup_{u\to\infty} \Phi(2u)/\Phi(u)\leqq N$ and that for each constant $0<\varepsilon<1/4$ there is a corresponding constant $1<R_\varepsilon<\infty$ such that $R_\varepsilon<\liminf_{u\to\infty} \varphi(u)/\varphi((1-\varepsilon)u)$.

We begin the demonstration by establishing first the statements relative to strict convexity.

LEMMA 1. *If $f(x)\in L_\Phi$ is a step function, then*

$$\|f\|_\Phi=\sup_{g(x)\geqq 0}\int_\Delta |f|g\,d\mu$$

where $\int_\Delta \Psi(g)\,d\mu\leqq 1$ and where $g(x)$ is also a step function with the same regions of constancy as $f(x)$ and $g(x)=0$ whenever $f(x)=0$.

Proof. Let $|f(x)|=f_i$ on sets e_i of measure $\mu(e_i)=\lambda_i>0$ $i=1, 2, \cdots, n$. Let $h(x)\geqq 0$ be any function such that $\int_\Delta \Psi(h)\,d\mu\leqq 1$. Define:

$$g(x)=\lambda_i^{-1}\int_{e_i} h(x)\,d\mu=g_i \quad \text{on} \quad e_i\ .$$

Since $\Psi(v)$ is the integral of a monotone nondecreasing function, it is a convex function so that by Jensen's inequality [10]:

$$\lambda_i^{-1}\int_{e_i} \Psi(h)\,d\mu\geqq \Psi\Big(\lambda_i^{-1}\int_{e_i} h(x)\,d\mu\Big)=\Psi(g_i)$$

and therefore:

$$\int_\Delta \Psi(g)\,d\mu=\sum_{i=1}^n \Psi(g_i)\lambda_i\leqq\sum_{i=1}^n\int_{e_i}\Psi(h)\,d\mu$$
$$=\int_\Delta \Psi(h)\,d\mu\leqq 1\ .$$

On the other hand:

$$\int_\Delta |f|g\,d\mu=\sum_{i=1}^n f_i g_i\lambda_i=\sum_{i=1}^n f_i\Big(\lambda_i^{-1}\int_{e_i} h\,d\mu\Big)\lambda_i$$
$$=\int_\Delta |f|h\,d\mu\ .$$

It is clear that we may take $g(x)=0$ where $f(x)=0$ since the integral $\int_\Delta |f|g\,d\mu$ will remain unaltered in value while $\int_\Delta \Psi(g)\,d\mu$ can only become smaller.

THEOREM 1. *If $\Psi(v)$ is discontinuous, then L_Φ is not strictly convex.*

Proof. Since $\Psi(v)$ is defined as the integral of a positive function, the only type of discontinuity which can arise is of the form $\Psi(V_0)<\infty$ while $\Psi(V_0+)=\infty$ where $0<V_0<\infty$. (It is to be remarked that the definition of the space L_Φ excludes the case $V_0=0$ as trivial). Let $\lambda\leqq\min[\mu(\Delta)/2, 1/2\Psi(V_0)]$ be a finite number, $A\subseteqq\Delta$, $B\subseteqq\Delta$ be two sets such that $A\cap B=0$ and $\mu(A)=\mu(B)=\lambda$. Define $f'(x)=1/\lambda V_0$ on A and 0 elsewhere, $f''(x)=1/\lambda V_0$ on B and 0 elsewhere. By Lemma 1, if c', c'' represent positive numbers:

$$\|f'\|_\Phi=\sup_{c'\geqq 0}\frac{1}{\lambda V_0}\cdot\lambda c' \quad \text{where} \quad \Psi(c')\leqq\frac{1}{\lambda},$$

$$\|f''\|_\Phi=\sup_{c'\geqq 0}\frac{1}{\lambda V_0}\cdot\lambda c'' \quad \text{where} \quad \Psi(c'')\leqq\frac{1}{\lambda}.$$

Since $\Psi(V_0)\leqq 1/2\lambda$ and $\Psi(v)=\infty$ for $V_0<v$ the largest value of c' or c'' that may be chosen is $c'=c''=V_0$. Thus $\|f'\|_\Phi=\|f''\|_\Phi=1$.

But $\frac{1}{2}|f'(x)+f''(x)|=\frac{1}{2}|f'(x)-f''(x)|=1/2\lambda V_0$ on $A\cup B$ and 0 elsewhere, so that by Lemma 1:

$$\left\|\frac{f'+f''}{2}\right\|_\Phi=\left\|\frac{f'-f''}{2}\right\|_\Phi=\sup_{c\geqq 0}\frac{1}{2\lambda V_0}\cdot 2\lambda c$$

where c represents any positive number with $\Psi(c)\leqq 1/2\lambda$. As before, it follows the largest value of c it is possible to choose is: $c=V_0$ so that $\frac{1}{2}\|f'+f''\|_\Phi=\frac{1}{2}\|f'-f''\|_\Phi=1$.

THEOREM 2. *If $\psi(v)$ is discontinuous, then L_Φ is not strictly convex.*

Proof. Let $v_0=\sup_{\psi(v)=0} v$, $V_0=\sup_{\psi(v)<\infty} v$. By Theorem 1 it may be assumed that $\Psi(v)$ is continuous in the extended sense, so that $\lim_{v\to V_0-}\psi(v)=\infty$. Two cases are distinguished according as $\psi(v)$ is or is not continuous at $v=v_0$.

(A) *v_0 is a point of discontinuity for $\psi(v)$.* Let $v_0<\beta<V_0$ be a point of continuity for $\psi(v)$ and choose β large enough so that the relation $\Psi(\beta)=1/\lambda$ defines $\lambda\leqq\mu(\Delta)/2$ and $\lambda<\infty$. It is then possible to determine sets $A\subseteqq\Delta$, $B\subseteqq\Delta$ such that: $\mu(A)=\mu(B)=\lambda$ and $A\cap B=0$. For each value of a parameter $0<p<1$ define $a_p=[1-p\overline{\psi}(v_0)v_0]/\beta$, $b_p=p\overline{\psi}(v_0)$ and

$$f_p(x)=\begin{cases} \dfrac{a_p}{\lambda} & \text{on} \quad A\,, \\ \dfrac{b_p}{\lambda} & \text{on} \quad B\,, \\ 0 & \text{on} \quad (A\cup B)'\,. \end{cases}$$

By Lemma 1, $\|f_p\|_\Phi=\sup_{\substack{\xi\geqq 0\\ \eta\geqq 0}}(a_p\xi+b_p\eta)$ where $\Psi(\xi)+\Psi(\eta)\leqq 1/\lambda$. Since $\Psi(v)$ is the integral of a positive non-decreasing function not identically zero, $\Psi(v)$ increases continuously to infinity; hence, it is possible to replace the condition $\Psi(\xi)+\Psi(\eta)\leqq 1/\lambda$ by $\Psi(\xi)+\Psi(\eta)=1/\lambda$ with $\xi\geqq v_0$, $\eta\geqq v_0$ for it ξ_1, η_1 satisfy $\Psi(\xi_1)+\Psi(\eta_1)<1/\lambda$ it is always possible to find $\xi_2\geqq\xi_1$, $\eta_2\geqq\eta_1$ so that $\Psi(\xi_2)+\Psi(\eta_2)=1/\lambda$ and $\xi_2\geqq v_0$, $\eta_2\geqq v_0$ while $a_p\xi_2+b_p\eta_2\geqq a_p\xi_1+b_p\eta_1$. Thus $\eta=\eta(\xi)$ is determined as a single-valued function with $v_0\leqq\xi\leqq\beta$. If $I_p(\xi)=a_p\xi+b_p\eta(\xi)$ then $\|f_p\|_\Phi=\sup_{v_0\leqq\xi\leqq\beta} I_p(\xi)$. But $I_p(\xi)$ assumes its maximum subject to $v_0\leqq\xi\leqq\beta$ either for $\xi=v_0$, $\xi=\beta$ or at points for which $d^+I_p(\xi)/d\xi$ and $d^-I_p(\xi)/d\xi$ simultaneously change their signs, where $d^+/d\xi$, $d^-/d\xi$ denote respectively upper and lower derivatives. That is, the maximum must be assumed either at a boundary of the interval $v_0\leqq\xi\leqq\beta$ or at a turning value or a cusp. But

$$\frac{d^-\eta}{d\xi}=\frac{-\bar\psi(\xi)}{\psi(\eta(\xi))}$$

so that

$$\frac{d^-I_p(\xi)}{d\xi}=a_p-b_p\frac{\bar\psi(\xi)}{\psi(\eta(\xi))}=\frac{1}{\beta}\left[1-p\bar\psi(v_0)\left(v_0+\frac{\bar\psi(\xi)\beta}{\psi(\eta(\xi))}\right)\right].$$

If p is any value $0<p<P$ where

$$0<P\leqq\min\left[1,\ (\bar\psi(v_0)v_0+\psi(\beta)\beta)^{-1}\right]$$

then $d^-I_p(\xi)/d\xi>0$, $v_0<\xi<\beta$; and since $I_p(v_0)<I_p(\beta)$ it follows: $\|f_p\|_\Phi=I_p(\beta)=1$ for all such p. Choose $0<p'<p''<P/2$ and define

$$f'(x)=f_{p'}(x)\,,\qquad f''(x)=f_{p''}(x)\,.$$

Then $\frac{1}{2}(f'(x)+f''(x))=\frac{1}{2}f_{p'+p''}(x)$ so that $\|f'\|_\Phi=\|f''\|_\Phi=\frac{1}{2}\|f'+f''\|_\Phi=1$. On the other hand $|f'(x)-f''(x)|$ is different from zero on B and therefore its norm is not zero. Thus L_Φ fails to be strictly convex in this case.

(B) *v_0 is a point of continuity for $\psi(v)$.* Let α be a point of discontinuity for $\psi(v)$ so that $\psi(\alpha)<\bar\psi(\alpha)\neq 0$. Also, let $\alpha<\beta<V_0$ be a point of continuity for $\psi(v)$ and choose β large enough so that the relation

$\Psi(\alpha)+\Psi(\beta)=1/\lambda$ defines $\lambda\leq\mu(\Delta)/2$ and $0<\lambda<\infty$. It is then possible to determine sets $A\subseteqq\Delta$, $B\subseteqq\Delta$ such that $\mu(A)=\mu(B)=\lambda$ and $A\cap B=0$.

Consider the equations

$$\begin{cases} a'\alpha+b'\beta=1 \\ a'\phi(\beta)-b'\bar{\phi}(\alpha)=0 \end{cases} \qquad \begin{cases} a''\alpha+b''\beta=1 \\ a''\phi(\beta)-b''\left(\dfrac{\phi(\alpha)+\bar{\phi}(\alpha)}{2}\right)=0\,. \end{cases}$$

Since $\phi(\beta)\geqq\phi(\alpha)>0$ it follows that the determinants of these equations do not vanish; therefore, the equations may be solved and it may be observed that the values of a', b', a'', b'' are all greater than zero. Define

$$f'(x)=\begin{cases} \dfrac{a'}{\lambda} & \text{on } A \\ \dfrac{b'}{\lambda} & \text{on } B \\ 0 & \text{on } (A\cup B)' \end{cases} \qquad f''(x)=\begin{cases} \dfrac{a''}{\lambda} & \text{on } A \\ \dfrac{b''}{\lambda} & \text{on } B \\ 0 & \text{on } (A\cup B)' \end{cases}$$

By Lemma 1, $\|f'\|_{\Phi}=\sup_{\substack{\xi\geqq 0\\ \eta\geqq 0}}(a'\xi+b'\eta)$ where $\Psi(\xi)+\Psi(\eta)\leqq 1$. As in (A) above the condition $\Psi(\xi)+\Psi(\eta)\leqq 1/\lambda$ may be replaced with $\Psi(\xi)+\Psi(\eta)=1/\lambda$ and $\xi\geqq v_0$, $\eta\geqq v_0$ and these relationships determine $\xi=\xi(\eta)$ and $\eta=\eta(\xi)$ as single-valued functions with $v_0\leqq\xi\leqq\delta$ and $v_0\leqq\eta\leqq\delta$ respectively where $\Psi(\delta)=1/\lambda$. If $d^+/d\xi$, $d^-/d\xi$ denote respectively upper and lower derivatives, it may be seen:

$$\frac{d^+\eta}{d\xi}=-\frac{\phi(\xi)}{\bar{\phi}(\eta(\xi))}\,, \qquad \frac{d^-\eta}{d\xi}=-\frac{\bar{\phi}(\xi)}{\phi(\eta(\xi))}\,,$$

$$\frac{d^+\xi}{d\eta}=-\frac{\phi(\eta)}{\bar{\phi}(\xi(\eta))}\,, \qquad \frac{d^-\xi}{d\eta}=-\frac{\bar{\phi}(\eta)}{\phi(\xi(\eta))}\,,$$

where $\Psi(\xi)+\Psi(\eta)=1$.

If $I'(\xi)=a'\xi+b'\eta(\xi)$ then $\|f'\|_{\Phi}=\sup_{v_0\leqq\xi\leqq\delta} I'(\xi)$. As in (A), $I'(\xi)$ assumes its maximum subject to $v_0\leqq\xi\leqq\delta$ either for $\xi=v_0$, $\xi=\delta$ or at points for which $d^+I(\xi)/d\xi$ and $d^-I(\xi)/d\xi$ simultaneously change their signs. Now

$$\lim_{\xi\to v_0^+}\left[\frac{d^-I'(\xi)}{d\xi}\right]=a'+b'\left[\frac{d^-\eta(\xi)}{d\xi}\right]_{\xi\to v_0^+}=a'-\frac{\phi(v_0)}{\phi(\delta)}b'=a'>0$$

since $\eta=\delta$ when $\xi=v_0$ and $\phi(\delta)>0$. Thus, $I'(\xi)$ increases in the immediate neighborhood of $\xi=v_0$ and $\xi=v_0$ cannot give a maximum. Also

$$\lim_{\xi \to \delta^-}\left[\frac{d^+I'(\xi)}{d\xi}\right] = a' + b'\left[\frac{d^+\eta(\xi)}{d\xi}\right]_{\xi \to \delta^-} = a' - b' \lim_{\substack{\xi \to \delta^- \\ \eta \to v_0^+}} \frac{\psi(\xi)}{\bar{\psi}(\eta)} = -\infty$$

since $b' > 0$. Thus $I'(\xi)$ decreases in the immediate neighborhood of $\xi = \delta$ to the value $I'(\delta)$ and $\xi = \delta$ cannot give a maximum.

Now

$$\frac{d^+I'(\xi)}{d\xi} = a' - b'\frac{\psi(\xi)}{\bar{\psi}(\eta(\xi))} \quad \text{and} \quad \frac{d^-I'(\xi)}{d\xi} = a' - b'\frac{\bar{\psi}(\xi)}{\psi(\eta(\xi))} .$$

Since $a'\psi(\beta) - b'\bar{\psi}(\alpha) = 0$ and the condition $\Psi(\xi) + \Psi(\eta(\xi)) = 1/\lambda$ implies that as ξ increases, η cannot increase and conversely, a critical examination of the expressions above establishes the following relations:

$$\text{if} \quad \xi > \alpha \quad \text{then} \quad \frac{d^+I'(\xi)}{d\xi} \leqq 0 , \quad \frac{d^-I'(\xi)}{d\xi} \leqq 0 ,$$

$$\text{if} \quad \xi = \alpha \quad \text{then} \quad \frac{d^+I'(\xi)}{d\xi} > 0 , \quad \frac{d^-I'(\xi)}{d\xi} = 0 ,$$

$$\text{if} \quad \xi < \alpha \quad \text{then} \quad \frac{d^+I'(\xi)}{d\xi} > 0 , \quad \frac{d^-I'(\xi)}{d\xi} > 0 .$$

Since that $d^+I'(\xi)/d\xi$, $d^-I'(\xi)/d\xi$ can change sign only once, it follows that the value $\xi = \alpha$, $\eta = \beta$ gives unique maximum to $I'(\xi)$. Thus $\|f'\|_\Phi = a'\alpha + b'\beta = 1$.

If $I''(\xi) = a''\xi + b''\eta(\xi)$ then an analogous argument leads to the relations

$$\text{if} \quad \xi > \alpha \quad \text{then} \quad \frac{d^+I''(\xi)}{d\xi} < 0 , \quad \frac{d^-I''(\xi)}{d\xi} < 0 ,$$

$$\text{if} \quad \xi = \alpha \quad \text{then} \quad \frac{d^+I''(\xi)}{d\xi} > 0 , \quad \frac{d^-I''(\xi)}{d\xi} < 0 ,$$

$$\text{if} \quad \xi < \alpha \quad \text{then} \quad \frac{d^+I''(\xi)}{d\xi} > 0 , \quad \frac{d^-I''(\xi)}{d\xi} > 0 ,$$

so that $\|f''\|_\Phi = a''\alpha + b''\beta = 1$.

Consider

$$\frac{f'(x) + f''(x)}{2} = \begin{cases} \dfrac{a' + a''}{2\lambda} & \text{on} \quad A \\[2ex] \dfrac{b' + b''}{2\lambda} & \text{on} \quad B \\[2ex] 0 \quad \text{on} & (A \cup B)' \end{cases}$$

Let

$$g(x)=\begin{cases}\alpha & \text{on } A\\ \beta & \text{on } B\\ 0 & \text{on } (A\cup B)'\end{cases}$$

then

$$\int_\Delta \Psi(g)\,d\mu=\lambda[\Psi(\alpha)+\Psi(\beta)]=1$$

and

$$\left\|\frac{f'+f''}{2}\right\|_\Phi \geq \frac{a'+a''}{2}\alpha+\frac{b'+b''}{2}\beta=1\ .$$

Thus by the triangle inequality $\frac{1}{2}\|f'+f''\|_\Phi=1$

A consideration of the defining equation shows that $b'\neq b''$. Therefore $|f'(x)-f''(x)|$ is not zero on B and it may be concluded immediately that $\|f'-f''\|_\Phi\neq 0$. Thus L_Φ fails to be strictly convex in this case also.

LEMMA 2. *If $\psi(v)$ is continuous in the extended sense and $0\leq v\leq V_0$, $0\leq v'\leq V_0$ then $\Psi(v')\geq\Psi(v)+\psi(v)(v'-v)$.*

Proof. If $v=v'$ then the relation certainly holds. If $v<v'$ then

$$\Psi(v')=\int_0^{v'}\psi(\bar v)d\bar v=\int_0^v\psi(\bar v)d\bar v+\int_v^{v'}\psi(\bar v)d\bar v\geq\Psi(v)+\psi(v)(v'-v)$$

If $v>v'$ then

$$\Psi(v')=\int_0^{v'}\psi(\bar v)\,d\bar v=\int_0^{v}\psi(\bar v)\,d\bar v-\int_{v'}^{v}\psi(\bar v)d\bar v\geq\Psi(v)+\psi(v)(v'-v)\ .$$

THEOREM 3. *If*

(i) *$\Psi(v)$ is continuous in the extended sense,*

(ii) *$\psi(v)$ is continuous in the extended sense,*

(iii) *$f(x)\in L_\Phi$ and $|f(x)|=\text{ess sup}\,|f(x)|=M<\infty$ on some set of positive measure and also, when Δ is of infinite measure, $f(x)$ vanishes outside a set of finite measure; then there is a constant $0<C_f<\infty$ and a function $g_f(x)\geq 0$ such that $\|f\|_\Phi=\int_\Delta f(x)g_f(x)d\mu$, where $\psi(g_f(x))=C_f|f(x)|$ and $\int_\Delta\Psi(g_f)d\mu=1$.*

Proof. We first establish the existence of a constant C_f and a function $g_f(x)$ which satisfy the last two relations of the theorem. Let

$E \equiv \underset{x}{E}[|f(x)|>0]$ and let $S \equiv \underset{x}{S}[|f(x)|=M]$ with $\mu(S)=\delta \neq 0$. Since $\Psi(v)$ is continuous in the extended sense and increases from zero to infinity, there is a value $v' < V_0$ such that $\Psi(v')=1/\delta$. With $C'=\psi(v')M^{-1}$ define $g'(x)=v'$ for $x \in S$ and $g'(x)=\varphi(C'|f(x)|)$ for $x \in S'$. Then $\psi(g'(x)) = C'|f(x)|$ and

$$1 \leqq \int_\Delta \Psi(g'(x))d\mu \leqq \mu(E)\cdot\delta^{-1} < \infty \ .$$

Two cases will be distinguished according as

$$\text{(A)} \qquad \int_\Delta \Psi(\varphi(C'|f|))d\mu \leqq 1 \ ,$$

or

$$\text{(B)} \qquad \int_\Delta \Psi(\varphi(C'|f|))\, d\mu > 1 \ .$$

(A) For each value of the parameter $1 \leqq k < \infty$ define

$$g_k(x)=\begin{cases} \varphi(C'|f(x)|) & \text{for} \quad x \in S' \ , \\ \min\,[k\varphi(C'|f(x)|),\ v'] & \text{for} \quad x \in S \ . \end{cases}$$

The family of functions $g_k(x)$ is then a continuous one satisfying $\psi(g_k(x))=C'|f(x)|$ and increasing with k from $g_1(x)=\varphi(C'|f(x)|)$ to $g'(x)$. The integral $I(k)=\int_\Delta \Psi(g_k)d\mu$ increases continuously from value $\leqq 1$ to values $\geqq 1$. There is then a value k_0 such that $\int_\Delta \Psi(g_{k_0})d\mu=1$. The function $g_{k_0}(x)=g_f(x)$ and the constant $C'=C_f$ are those of the theorem.

(B) Let $C_0=\inf C$ where $\int_\Delta \Psi(\varphi(C|f(x)|))\, d\mu \geqq 1$. Since

$$\lim_{C\to C_0^+} \varphi(C|f(x)|)=\overline{\varphi}(C_0|f(x)|)$$

it follows by Lebesgue's theorem that

$$\int_\Delta \Psi(\overline{\varphi}(C_0|f(x)|)d\mu \geqq 1.$$

Again, let $C^0=\sup C$ where

$$\int_\Delta \Psi(\overline{\varphi}(C|f(x)|))\, d\mu \leqq 1 \ .$$

By the continuity of $\psi(v)$ it follows that $C^0=C_0$ and since

$$\lim_{C\to C_0^-} \overline{\varphi}(C|f(x)|)=\varphi(C_0|f(x)|)$$

then by Lebesgue's theorem $\int_\Delta \Psi(\varphi(C_0|f(x)|))d\mu \leqq 1$. For each value of the parameter $0 \leqq k \leqq 1$ define

$$g_k(x) = (1-k)\varphi(C_0|f(x)|) + k\bar{\varphi}(C_0|f(x)|)$$

then the family of functions $g_k(x)$ satisfies $\psi(g_k(x)) = C_0|f(x)|$ and increases continuously with k from $\varphi(C_0|f(x)|)$ to $\bar{\varphi}(C_0|f(x)|)$. The integral $I(k) = \int_\Delta \Psi(g_k)d\mu$ increases continuously from values $\leqq 1$ so that there is a value k_0 such that $\int_\Delta \Psi(g_{k_0})d\mu = 1$. The constant $C_0 = C_f$ and function $g_{k_0}(x) = g_f(x)$ are those of the theorem.

It is easily seen in either case (A) or (B) that $0 < C_f$ for if $C_f = 0$ then the corresponding function $g_f(x) \leqq v_0$ and hence $\int_\Delta \Psi(g_f)d\mu = 0$ which is a contradiction of the proof already made that $\int_\Delta \Psi(g_f)d\mu = 1$.

Finally, it follows from Lemma 2 that

$$\|f\|_\Phi = \int_\Delta |f(x)|g_f(x)d\mu .$$

Let $h(x) \geqq 0$ be any function such that $\int_\Delta \Psi(h)d\mu \leqq 1$. In Lemma 2 let $v = g_f(x)$, $v' = h(x)$; then, integrating over Δ gives

$$\int_\Delta \Psi(h)\,d\mu \geqq \int_\Delta \Psi(g_f)\,d\mu + C_f \int_\Delta |f(x)|(h(x) - g_f(x))\,d\mu$$

or

$$\int_\Delta |f(x)|g_f(x)\,d\mu \geqq \int_\Delta |f(x)|h(x)\,d\mu + \frac{1 - \int_\Delta \Psi(h)d\mu}{C_f} .$$

Since $C_f > 0$ we obtain $\|f\|_\Phi = \int_\Delta |f(x)|g_f(x)\,d\mu$.

THEOREM 4. *If*

(i) *the hypotheses of Theorem* 3 *are satisfied,*

(ii) $\|f\|_\Phi > 0$,

then

$$\|f\|_\Phi = \frac{\int_\Delta \Phi(C_f|f|)\,d\mu + 1}{C_f}$$

where C_f *is the associated constant of Theorem* 3.

Proof. By Young's inequality, for arbitrary $0 \leqq u$, $0 \leqq v$,

$$uv \leqq \Phi(u) + \Psi(v)$$

with equality if and only if at least one of the relations $v = \varphi(u)$ or $u = \psi(v)$ is satisfied. Let $u = C_f |f(x)|$, $v = g_f(x)$ then since $\psi(g_f(x)) = C_f |f(x)|$ the inequality becomes an equality and

$$C_f |f(x)| g_f(x) = \Phi(C_f |f(x)|) + \Psi(g_f(x)) .$$

Since $0 < C_f < \infty$, integration over Δ gives the stated result.

THEOREM 5. *If*
(i) $\Psi(v)$ *is continuous in the extended sense,*
(ii) $\psi(v)$ *is continuous in the extended sense,*
then L_Φ *is strictly convex.*

Proof. Let $f'(x)$, $f''(x)$ be any pair of elements of L_Φ such that $\frac{1}{2}\|f' + f''\|_\Phi = 1$, $\|f'\|_\Phi = 1$, $\|f''\|_\Phi = 1$. Let $f(x) = f'(x) + f''(x)$ and

$$S \equiv \underset{x}{S}[|f(x)| = \text{ess sup } |f(x)|] .$$

If $\mu(S) = 0$ let

$$\underset{n}{E} \equiv \underset{x}{E}\left[|f(x)| \leqq \min\left(n, \left(1 - \frac{1}{n}\right) \text{ess sup } |f(x)|\right)\right] \qquad (n = 1, 2, \cdots) ;$$

if $\mu(S) > 0$ let $\underset{n}{E} = S'$ $(n = 1, 2, \cdots)$. Let Δ_n be a sequence of sets snch that $\Delta_n \subseteqq \Delta_{n+1} \subseteqq \Delta$, $\mu(\Delta_n) < \infty$, $\mu(\Delta_n - E_n) > 0$ and $\lim_n \Delta_n = \Delta$. If $\mu(S) = 0$ define

$$F_n(x) = \begin{cases} \min\left[n, \left(1 - \frac{1}{n}\right) \text{ess sup } |f(x)|\right] & \text{on } (\Delta_n - E_n) , \\ |f(x)| & \text{on } (\Delta_n \cap E_n) , \\ 0 & \text{on } \Delta_n' \end{cases}$$

while if $\mu(S) > 0$ define

$$F_n(x) = \begin{cases} \text{ess sup } |f(x)| & \text{on } (\Delta_n - E_n) \\ |f(x)| & \text{on } (\Delta_n \cap E_n) \\ 0 & \text{on } \Delta_n' \end{cases}$$

observing that since $\|f\|_\Phi < \infty$ then ess sup $|f(x)| < \infty$ in this case. It follows easily from the definition of the norm in L_Φ that $\|F_n\|_\Phi \to \|f' + f''\|_\Phi$. The functions $F_n(x)$ have been constructed in such a way that they satisfy postulate (iii) of Theorem 3 so that by this theorem and also Theorem

4, there are constants $\frac{1}{2} \leqq C_n < \infty$ and functions $g_n(x)$ satisfying the relations: $\psi(g_n(x)) = C_n F_n(x)$, $\int_\Delta \Psi(g_n)\,d\mu = 1$ and $\|F_n\|_\Phi = \int_\Delta F_n(x) g_n(x)\,d\mu$. Since $F_n(x) \leqq F_{n+1}(x)$ and $\int_\Delta \Psi(g_n)\,d\mu = 1$, ($n = 1, 2$, etc.) it follows that the sequence C_n decreases to a limit $\frac{1}{2} \leqq C < \infty$. Since $\psi(g_n(x)) = C_n F_n(x)$, $F_n(x) \leqq F_{n+1}(x)$ and $\int_\Delta \Psi(g_n)\,d\mu = 1$ it follows by the monotone properties of $\psi(v)$ that for each arbitrarily chosen but fixed m_0 the sequence $g_n(x)$ ultimately decreases on $(\Delta_m \cap E_m)$. When $\mu(S) > 0$ we see $(\Delta_n \cap E_n) \to (\Delta - S)$ and $(\Delta_n - E_n) \to S$ so that in this event the sequence $g_n(x)$ decreases on S also. When $\mu(S) = 0$ we see as before $(\Delta_n \cap E_n) \to (\Delta - S)$ and $(\Delta_n - E_n) \to S$. Thus the sequence $g_n(x)$ in both cases converges in measure to its limit inferior which we denote by $g(x)$.

By Theorem 3

$$\|F_n\|_\Phi = \int_\Delta |F_n| g_n\,d\mu = \int_\Delta |f' + f''| g_n\,d\mu$$

$$\leqq \int_\Delta |f'| g_n\,d\mu + \int_\Delta |f''| g_n\,d\mu \leqq \|f'\|_\Phi + \|f''\|_\Phi\ .$$

Since

$$\|F_n\|_\Phi \to \|f' + f''\|_\Phi = \|f'\|_\Phi + \|f''\|_\Phi\ ,$$

it follows that

$$\lim_n \int_\Delta |f'| g_n\,d\mu = \|f'\|_\Phi \quad \text{and} \quad \lim_n \int_\Delta |f''| g_n\,d\mu = \|f''\|_\Phi\ .$$

We show that there is a constant $0 < D' < \infty$ such that $\psi(g(x)) = D'|f'(x)|$ almost everywhere. If this were not the case there is a constant $0 < B < \infty$ and sets T_1', T_2' of finite positive measure such that

$$\psi(g(x)) > B|f(x)| \quad \text{on} \quad T_1'$$

$$0 < \psi(g(x)) < B|f(x)| \quad \text{on} \quad T_2'\ .$$

By Egoroff's theorem we may extract subsets $T_1'' \subseteqq T_1'$, $T_2'' \subseteqq T_2'$ such that the sequence $g_n(x)$ ultimately tends uniformly to $g(x)$ on T_1'' and T_2''. From T_1'', T_2'' we may extract subsets T_1''', T_2''' of positive measure such that the sequence is not only bounded on T_1''' and T_2''' but, since $\int_\Delta \Psi(g_n)\,d\mu \leqq 1$ and $g(x) = \liminf g_n(x)$, it is also bounded away from V_0. We may again find subsets $T_1 \subseteqq T_2'''$ and $T_2 \subseteqq T_2'''$ such that for suitably small constants $0 < t < \infty$, $0 < \alpha < \infty$:

$$\psi(g_n(x) - t') > B(|f(x)| + \alpha)\ , \qquad x \in T_1,\ 0 \leqq t' \leqq t\ , \tag{1}$$

$$0 < \psi(g_n(x)+t'') < B(|f(x)|)\ , \qquad x \in T_2\ ,\ 0 \leqq t'' \leqq t\ ,$$

for all n sufficiently large.

Since $\int_\Delta |f'| g_n \, d\mu \to \|f'\|_\Phi$, for each $0 < \varepsilon$ there is an n_ε such such that if $n_\varepsilon \leqq n$ then: $\int_\Delta |f'| h \, d\mu - \int_\Delta |f'| g_n \, d\mu < \varepsilon$ where $h(x) \geqq 0$ is any function with $\int_\Delta \Psi(h) \, d\mu \leqq 1$. Also, since $g(x)$ is bounded away from V_0 and the sequence $g_n(x)$ converges uniformly on T_1 to $g(x)$, there is a constant $0 < \beta < \infty$ such that for sufficiently large n

$$\int_{T_1} \psi(g_n(x)) \, d\mu \leqq \int_{T_1} \psi(g(x)+\beta) \, d\mu < \infty\ . \tag{2}$$

Let

$$0 < \varepsilon < \alpha\mu(T_1) \min \left[t,\ \frac{t\int_{T_2} \psi(g(x)) \, d\mu}{\int_{T_1} \psi(g(x)+\beta) \, d\mu} \right]$$

and choose $n \geqq n_\varepsilon$ so that (2) holds. Then, if

$$\int_{T_1} \Psi(g_n) \, d\mu + \int_{T_2} \Psi(g_n) \, d\mu = b \tag{3}$$

and if $0 < t_1 < \infty$, $0 < t_2 < \infty$ also satisfy

$$\int_{T_1} \Psi(g_n(x)-t_1) \, d\mu + \int_{T_2} \Psi(g_n(x)+t_2) \, d\mu = b \tag{4}$$

we have by the mean value theorem, for some $0 \leqq \theta_1 \leqq 1$, $0 \leqq \theta_2 \leqq 1$,

$$-t_1 \int_{T_1} \psi(g_n(x)-\theta_1 t_1) \, d\mu + t_2 \int_{T_2} \psi(g_n(x)+\theta_2 t_2) \, d\mu = 0\ . \tag{5}$$

Thus, if

$$t_1 = \min \left[t,\ \frac{t\int_{T_2} \psi(g(x)) \, d\mu}{\int_{T_1} \psi(g(x)+\beta) \, d\mu} \right]$$

then $t_2 \leqq t$. Now by (1)

$$-t_1 \int_{T_1} \psi(g_n(x)-\theta_1 t_1) \, d\mu \leqq -Bt_1 \int_{T_1} |f| \, d\mu - Bt_1 \alpha\mu(T_1)$$

$$t_1 \int_{T_2} \psi(g_n(x) + \theta_2 t_2) \, d\mu \leqq Bt_2 \int_{T_2} |f| d\mu\ ,$$

so that by (5)

$$-t_1\int_{T_1}|f|\,d\mu+t_2\int_{T_2}|f|\,d\mu\geqq t_1\alpha\mu(T_1)>\varepsilon\ .$$

But if

$$h(x)=\begin{cases} g_n(x)-t_1 & \text{on}\quad T_1\\ g_n(x)+t_2 & \text{on}\quad T_2\\ g_n(x) & \text{on}\quad (T_1\cup T_2)'\end{cases}$$

then by (3) and (4) $\int_\Delta \Psi(h)\,d\mu=\int_\Delta \Psi(g_n)\,d\mu=1$ while

$$\int_\Delta |f'|h\,d\mu=\int_\Delta |f'|g_n\,d\mu-t_1\int_{T1}|f'|\,d\mu+t_1\int_{T_2}|f'|\,d\mu$$
$$\geqq\int_\Delta |f'|g_n\,d\mu+\varepsilon\ ,$$

which contradicts the demonstration already made that $\int_\Delta |f'|h\,d\mu-\int_\Delta |f'|g_n\,d\mu<\varepsilon$. Thus, there is a constant $0<D'<\infty$ such that $\psi(g(x))=D'|f'(x)|$. Similarly, there is a constant $0<D''<\infty$ such that $\psi(g(x))=D''|f''(x)|$.

Since $|f'(x)|=\psi(g(x))/D'$ and $|f''(x)|=\psi(g(x))/D''$ we see $|f'(x)|$ and $|f''(x)|$ differ at most by a constant factor. But $\|f'\|_\Phi=\|f''\|_\Phi=1$ so that this factor is unity. Thus, $f'(x)$ and $f''(x)$ differ at most in sign. But $\psi(g(x))=C|f'(x)+f''(x)|$ so that if $f'(x)=-f''(x)$ at any point, then $\psi(g(x))=D'|f'(x)|=D''|f''(x)|=0$ at this same point. Hence $f'(x)=f''(x)$ almost everywhere and $\|f'-f''\|_\Phi=0$.

Theorems 1, 2 and 5 together have established the necessary and sufficient conditions for the strict convexity of the spaces L_Φ. In order to proceed with the more difficult demonstrations for uniform convexity we shall require the following important proposition relating to the norm of an element in L_Φ.

THEOREM 6. *If*

(i) *$\psi(v)$ is continuous in the extended sense,*

(ii) *$\Psi(v)$ is continuous in the extended sense,*

(iii) (a) *there is a constant $0<N<\infty$ such that $\Phi(2u)/\Phi(u)\leqq N$, $(0<u)$, when Δ is of infinite measure,*

(b) $\limsup_{u\to\infty}\Phi(2u)/\Phi(u)<+\infty$ *when Δ is of finite measure, then for each $f\in L_\Phi$ different from zero there is a constant C_f and a function $g_f(x)\geqq 0$ such that*

$$\|f\|_\Phi=\int_\Delta |f|\,d\mu\ ,$$

$$\psi(g_f(x))=C_f|f(x)| \quad and \quad \int_\Delta \Psi(g_f)\,d\mu=1.$$

Proof. Let

$$S=\underset{x}{S}[|f(x)|=\text{ess sup}\,|f(x)|]\ .$$

If $\mu(S)=0$, let

$$\underset{n}{E}=\underset{x}{E}\left[|f(x)|\leqq\min\left(n,\left(1-\frac{1}{n}\right)\text{ess sup}\,|f(x)|\right)\right],\qquad (n=1,2,\cdots);$$

if $\mu(S)>0$ let $\underset{n}{E}=S'$, $(n=1,2,\cdots)$. Let $\varDelta_n$ be a sequence of sets such that $\varDelta_n\leqq\varDelta_{n+1}\leqq\varDelta$, $\mu(\varDelta_n)<\infty$, $\mu(\varDelta_n-E_n)>0$ and $\lim_n\varDelta_n=\varDelta$. If $\mu(S)=0$, define

$$F_n(x)=\begin{cases}\min\left[n,\left(1-\frac{1}{n}\right)\text{ess sup}\,|f(x)|\right] & \text{on}\quad(\varDelta_n-E_n)\ ,\\ |f(x)| & \text{on}\quad(\varDelta_n\cap E_n)\ ,\\ 0 & \text{on}\quad\varDelta_n'\end{cases}$$

while if $\mu(S)>0$, define

$$F_n(x)=\begin{cases}\text{ess sup}\,|f(x)| & \text{on}\quad(\varDelta_n-E_n)\ ,\\ |f(x)| & \text{on}\quad(\varDelta_n\cap E_n)\ ,\\ 0 & \text{on}\quad\varDelta_n'\end{cases}$$

observing that in this case ess sup $|f(x)|<\infty$ since $\|f\|_\Phi<\infty$. The functions $F_n(x)$ satisfy the postulates of Theorems 3 and 4 so that there are constants $\|F_n\|_\Phi^{-1}\leqq C_n<\infty$ and functions $g_n(x)\geqq0$ such that $\|F_n\|_\Phi^{-1}\leqq C_n<\infty$ and functions $g_n(x)\geqq0$ such that $\|F_n\|_\Phi=\int_\Delta F_ng_n\,d\mu$ where $\psi(g_n(x))=C_n\cdot F_n(x)$ and $\int_\Delta\Psi(g_n)\,d\mu=1$. Since $F_n(x)\leqq F_{n+1}(x)\leqq|f(x)|$ it follows from the condition $\int_\Delta\Psi(g_n)\,d\mu=1$ that the sequence C_n cannot increase and since $\|f\|_\Phi\geqq\|F_n\|_\Phi$ it has a limit $\|f\|_\Phi^{-1}\leqq C<\infty$. Since $\psi(g_n(x))=C_nF_n(x)$, $F_n(x)\leqq F_{n+1}(x)$ and $\int_\Delta\Psi(g_n)\,d\mu=1$ it follows by the monotone properties of $\psi(v)$ that for each arbitrarily chosen but fixed m the sequence $g_n(x)$ ultimately decreases on $(\varDelta_m\cap E_m)$. When $\mu(S)>0$ we see $(\varDelta_n\cap E_n)\rightarrow(\varDelta-S)$ and $(\varDelta_n-E_n)\rightarrow S$, so that in this event the sequence $g_n(x)$ decreases on S also. When $\mu(S)=0$ we see $(\varDelta_n\cap E_n)\rightarrow(\varDelta-S)$ and $(\varDelta_n-E_n)\rightarrow S$. Thus, the sequence $g_n(x)$ in both cases converges in measure to its limit inferior, which we denote by $g(x)$.

(a) Assume that postulate (iii) (a) holds. In this case there is a constant $0 < M < \infty$ such that $\bar{\varphi}(2u) \leqq M\varphi(u)$ for $0 < u$. Thus, if $\Phi(2u) \leqq N\Phi(u)$ for $0 < u$ then $\Phi(4u) \leqq N^2\Phi(u)$. Suppose there were a sequence $0 < u_n$ such that for each natural number $\bar{\varphi}(2u_n) > n\varphi(u_n)$, then

$$\Phi(4u_n) \geqq \int_{2u_n}^{4u_n} \varphi(\bar{u})\, d\bar{u} \geqq 2u_n\bar{\varphi}(2u_n) \geqq 2nu_n\varphi(u_n) \geqq 2n\Phi(u_n)$$

since $u_n\varphi(u_n) \geqq \Phi(u_n)$. Now

$$\int_\Delta |f|\bar{\varphi}(2C|f|)\, d\mu \leqq M\int_\Delta |f|\varphi(C|f|)\, d\mu \leqq M\|f\|_\Phi$$

since

$$\int_\Delta \Psi(\varphi(C|f|))\, d\mu \leqq \int_\Delta \liminf \Psi(g_n)\, d\mu \leqq \liminf \int_\Delta \Psi(g_n)\, d\mu = 1$$

by Fatou's lemma. By Young's inequality

$$\int_\Delta \Psi(\bar{\varphi}(2C|f|)\, d\mu = 2C\int_\Delta |f|\bar{\varphi}(2C|f|) d\mu - \int_\Delta \Phi(2C|f|) d\mu$$

$$\leqq 2CM\|f\|_\Phi < \infty .$$

But for all n sufficiently large $\bar{\varphi}(2C|f(x)|) \geqq g_n(x) \geqq g(x)$ therefore by the monotone property of $\Psi(v)$ and Lebesgue's theorem

$$1 = \liminf \int_\Delta \Psi(g_n)\, d\mu = \int_\Delta \liminf \Psi(g_n)\, d\mu = \int_\Delta \Psi(g)\, d\mu .$$

Let $h(x) \geqq 0$ be any function such that $\int_\Delta \Psi(h)\, d\mu \leqq 1$. In Lemma 2 let $v = g(x)$, $v' = h(x)$; then integrating over Δ gives

$$\int_\Delta \Psi(h)\, d\mu \geqq \int_\Delta \Psi(g)\, d\mu + C\int_\Delta |f|(h-g)\, d\mu$$

or

$$\int_\Delta |f| g\, d\mu \geqq \int_\Delta |f| h\, d\mu + \frac{1 - \int_\Delta \Psi(h)\, d\mu}{C} .$$

Since $C > 0$ we have $\|f\|_\Phi = \int_\Delta |f(x)| g(x)\, d\mu$. The function $g(x)$ and the constant C are those of the theorem.

(b) Assume that postulate (iii) (b) holds. Since $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$, there is a u' such that for $u' \leqq u$, $\Phi(2u)/\Phi(u) \leqq 2N$. Then for

$u' < u$, $\Phi(4u) \leq (2N)^2\Phi(u)$. With appropriate modifications of the corresponding demonstration in (a) above we easily show that there is a constant $0 < M < \infty$ and a value u_1 such that for $u_1 \leq u$, $\bar{\varphi}(u)/\varphi(u) \leq M$. Recalling that Δ is of finite measure, let $v_1 > 0$ be a value such that $\Psi(v_1) \leq 1/\mu(\Delta)$, then

$$\int_\Delta |f|\bar{\varphi}(2C|f|)\,d\mu \leq \int_\Delta |f|\bar{\varphi}(u_1)\,d\mu + M\int_\Delta |f|\varphi(C|f|)\,d\mu$$
$$\leq \left[\frac{\bar{\varphi}(u_1)}{v_1} + M\right]\|f\|_\Phi$$

since $\varphi(C|f(x)|) \leq g(x)$ and by Fatou's lemma

$$\int_\Delta \Psi(g)\,d\mu = \int_\Delta \liminf \Psi(g_n)\,d\mu \leq \liminf \int_\Delta \Psi(g_n)\,d\mu = 1\ .$$

By Young's inequality

$$\int_\Delta \Psi(\bar{\varphi}(2C|f|))\,d\mu = 2C\int_\Delta |f|\bar{\varphi}(2C|f|)\,d\mu - \int_\Delta \Phi(2C|f|)\,d\mu$$
$$\leq 2C\int_\Delta |f|\bar{\varphi}(2C|f|)\,d\mu \leq 2C\left[\frac{\bar{\varphi}(u_1)}{v_1} + M\right]\|f\|_\Phi < \infty\ .$$

But $\bar{\varphi}(2C|f(x)|) \geq g_n(x) \geq g(x)$ for all n sufficiently large, so that by Lebesgue's theorem and the monotone property of $\Psi(v)$

$$\int_\Delta \Psi(g)\,d\mu = \int_\Delta \liminf \Psi(g_n)\,d\mu = \liminf \int_\Delta \Psi(g_n)\,d\mu = 1\ .$$

The remainder of the proof is as in (a) above. The constant C and the function $g(x)$ are those of the theorem.

The above theorem may be generalized in several ways. The author has secceeded in obtaining a number of analogous conclusions [8] when the function $\psi(v)$ is discontinuous and when the hypotheses relative to the function $\Phi(u)$ do not hold. It is interesting to observe that for spaces in which conditions (iii) (a) or (iii) (b) do not apply, there is always an element f of the space for which the norm is not attained; that is to say, there is no function $h(x) \geq 0$ such that $\|f\|_\Phi = \int_\Delta |f| h\,d\mu$ with $\int_\Delta \Psi(h)\,d\mu = 1$. In this case, however, there is a constant $0 < C$ such that

$$\int_\Delta \Psi(\bar{\varphi}(C|f|))\,d\mu = 1 - a\ ,$$

where $0 < a < 1$ is a constant; for any larger constant $D > C$ the integral $\int_{\Delta} \Psi(\varphi(D|f|)d\mu$ is infinite. It is further remarkable that in this case

$$\|f\|_{\Phi}=\int_{\Delta} |f|\overline{\varphi}(C|f|)d\mu + aC^{-1} .$$

The proofs and complete statements of these propositions will not be presented since they are not essential to the discussions relating to convexity. Theorem 4 admits an obvious generalization not only to spaces which satisfy the postulates of Theorem 6, but to the more general case when only the first of these conditions holds. The problem of determining the constant C which appears in all of these theorems in terms of elementary properties of $f(x)$ has hot met with a suitable and satisfying solution despite the author's attempts to find one.

We proceed now to a consideration of the necessary and sufficient conditions for uniform convexity of Orlicz spaces. It was remarked in the introduction that every uniformly convex space is strictly convex but that the converse statement need not be true; therefore, it is clear that any necessary condition for strict convexity must be also a necessary condition for uniform convexity. Thus by Theorems 1 and 2 we must assume at least that $\Psi(v)$ is continuous in the extended sense and $\psi(v)$ is continuous in the extended sense. For a similar reason, the following theorems furnish us with further necessary conditions.

THEOREM 7. [5] *Every uniformly convex space is reflexive.*

THEOREM 8. [6] *Necessary and sufficient conditions that an Orlicz space be reflexive are that there exist a constant $0<N<\infty$, such that*

(a) $\Phi(2u)/\Phi(u)\leqq N$ *and* $\Psi(2v)/\Psi(v)\leqq N$, $(0<u,\ 0<v)$ *when* Δ *is of infinite measure;*

(b) $\limsup_{u\to\infty} \Phi(2u)/\Phi(u)\leqq N$ *and* $\limsup_{u\to\infty} \Psi(2v)/\Psi(v)\leqq N$ *when* Δ *is of finite measure.*

The conditions implied in Theorems 1, 2 and 8 must be supplemented with an additional necessary condition in order to insure uniform convexity. This is expressed in the next theorem.

THEOREM 9. *A necessary condition that* L_{Φ} *should be uniformly convex is that for every constant* $0<a<\infty$ *there is a constant* $1<K_a<\infty$ *such that* (a) *when* Δ *is of infinite measure then* $\varphi(u+au)/\varphi(u)>K_a$, $(0<u)$; *and* (b) *when* Δ *is of finite measure then* $\liminf_{u\to\infty} \varphi(u+au)/\varphi(u) > K_a$.

Proof. By Theorems 1, 2 and 8 and our above remarks we may and shall assume that $\Psi(v)$ and $\psi(v)$ are continuous in the extended sense and that $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$ and $\limsup_{v\to\infty} \Psi(2v)/\Psi(v) \leqq N$ for some constant $0<N<\infty$. We see then that $\psi(v)\to\infty$ for if $\sup \psi(v) \leqq A$, $0<A<\infty$ then $\varphi(u)=\infty$, $A<u$ so that $\Phi(2u)=\int_0^{2u} \varphi(\bar{u})d\bar{u}=\infty$ for $A/2<u$ which contradicts $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$. Similarly the condition $\limsup_{v\to\infty} \Psi(2v)/\Psi(v) \leqq N$ implies that $\varphi(v)\to\infty$.

Suppose there were a value $0<a<\infty$ and a sequence u_n such that alternatively according to the respective hypotheses

(a) $$\frac{\varphi(u_n+au_n)}{\varphi(u_n)} \to 1$$

(b) $$\liminf_{u_n\to\infty} \frac{\varphi(u_n+au_n)}{\varphi(u_n)} = 1 \, .$$

There is then a sequence of pairs: $\{v_n=\varphi(u_n),\ \bar{v}_n=\varphi(u_n+au_n)\}$ such that $\bar{v}_n/v_n \to 1$. Let $\lambda_n = 1/(\Psi(v_n)+\Psi(\bar{v}_n))$ and define w_n by $2\Psi(w_n)=1/\lambda_n$; then $\bar{v}_n \geqq w_n \geqq v_n$ and $\bar{v}_n/w_n \to 1$, $v_n/w_n \to 1$. We remark that in the second case $u_n\to\infty$ so that $v_n\to\infty$ and ultimately $\lambda_n<\mu(\Delta)/2$. Determine sets: A_n, B_n of positive measure such that $A_n \cap B_n = 0$, $\mu(A_n)=\mu(B_n)=\mu_n = \min[\mu(\Delta)/2, \lambda_n]$; and define functions $f_n'(x)$, $f_n''(x)$ respectively as

$$f'(x)=\begin{cases} \dfrac{(1+a)}{[(1+a)\bar{v}_n+v_n]\mu_n} & \text{on } A_n\,, \\ \dfrac{1}{[(1+a)\bar{v}_n+v_n]\mu_n} & \text{on } B_n\,, \\ 0 & \text{on } (A_n\cup B_n)' \end{cases}$$

$$f''(x)=\begin{cases} \dfrac{1}{[(1+a)\bar{v}_n+v_n]\mu_n} & \text{on } A_n\,, \\ \dfrac{(1+a)}{[(1+a)\bar{v}_n+v_n]\mu_n} & \text{on } B_n\,, \\ 0 & \text{on } (A_n\cup B_n)' \,. \end{cases}$$

With $C_n'=C_n''=[(1+a)\bar{v}_n+v_n]\mu_n u_n$ we see: $\psi(g_n')=C_n'f_n'$, $\psi(g_n'')=C_n f_n''$ where

$$g_n'(x)=\begin{cases} \bar{v}_n & \text{on } A_n \\ v_n & \text{on } B_n \\ 0 & \text{on } (A_n\cup B_n)' \end{cases} \qquad g_n''(x)=\begin{cases} v_n & \text{on } A_n \\ \bar{v}_n & \text{on } B_n \\ 0 & \text{on } (A_n\cup B_n)' \end{cases}$$

and for all n sufficiently large so that $\lambda_n=\mu_n$ we have

$$\int_\Delta \Psi(g_n')\,d\mu = \lambda_n(\Psi(v_n)+\Psi(\bar{v}_n)) = 1 ,$$

$$\int_\Delta \Psi(g_n'')\,d\mu = \lambda_n(\Psi(v_n)+\Psi(\bar{v}_n)) = 1 .$$

Thus, by Theorem 6

$$\|f_n'\|_\Phi = \int_\Delta f_n' g_n'\,d\mu = \frac{(1+a)\bar{v}_n + v_n}{(1+a)\bar{v}_n + v_n} = \frac{\mu_n}{\mu_n} = 1$$

$$\|f_n''\|_\Phi = \int_\Delta f_n'' g_n''\,d\mu = \frac{(1+a)\bar{v}_n + v_n}{(1+a)\bar{v}_n + v_n} = \frac{\mu_n}{\mu_n} = 1 .$$

Now

$$\frac{f_n'(x)+f_n''(x)}{2} = \begin{cases} \dfrac{(1+a/2)}{[(1+a)\bar{v}_n + v_n]\mu_n} & \text{on } A_n \cup B_n \\ 0 & \text{on } (A_n \cup B_n)' \end{cases}$$

so that by Lemma 1 and Theorem 6

$$\left\|\frac{f_n'+f_n''}{2}\right\|_\Phi = \frac{(1+a/2)w_n}{[(1+a)\bar{v}_n+v_n]} \cdot \frac{2\lambda_n}{\lambda_n} \to 1$$

since $\bar{v}_n/w_n \to 1$ and $v_n/w_n \to 1$. Again

$$|f_n'(x)-f_n''(x)| = \begin{cases} \dfrac{a}{[(1+a)\bar{v}_n+v_n]} \cdot \dfrac{1}{\lambda_n} & \text{on } A_n \cup B_n \\ 0 & \text{on } (A_n \cup B_n)' \end{cases}$$

so that

$$\|f_n'-f_n''\|_\Phi = \frac{a}{(1+a/2)}\left\|\frac{f_n'+f_n''}{2}\right\|_\Phi \to \frac{a}{1+a/2} > 0$$

and L_Φ is not uniformly convex.

LEMMA 3. *Let* $0<\varepsilon<1/4$ *and* $1<K_\varepsilon<T<\infty$, $0<b$ *be constants such that alternatively*

(a) $$K_\varepsilon < \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} < T , \qquad (0<u) ,$$

(b) $$K_\varepsilon < \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} < T , \qquad (b<u) ,$$

then there is a constant $0<L_\varepsilon$ *such that*

$$\Psi(v') \geq \Psi(v) + \psi(v)(v'-v) + L_\varepsilon \Phi(|u'-u|)$$

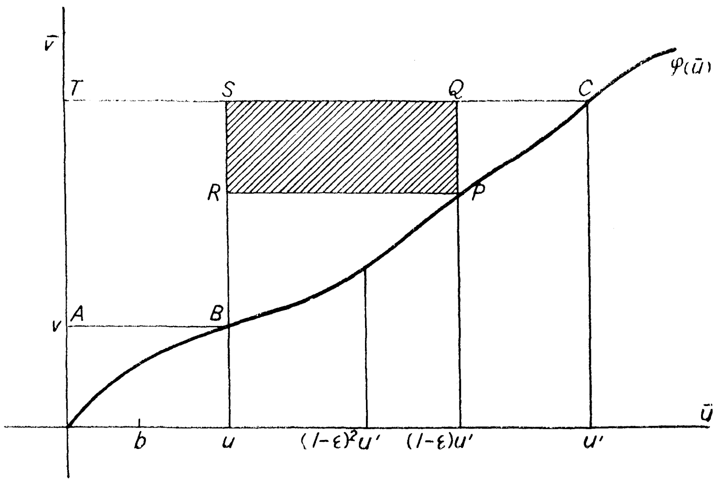
when respectively

(a) $$|u'-u| \geq \frac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2} u > 0$$

(b) $$|u'-u| \geq \max\left[b, \frac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2} u \right]$$

where (u, v), (u', v') *are related by either* $v=\varphi(u)$ *or* $u=\psi(v)$ *and* $v'=\varphi(u')$ *or* $u'=\psi(v')$.

Proof. Assume $u' \geq u$ and consider the first diagram



We note first

$$\text{(a)} \quad (1-\varepsilon)^2 u' \geq u > 0\,; \qquad \text{(b)} \quad (1-\varepsilon)^2 u' \geq u \geq b$$

according to the respective hypotheses. Since $\varphi(u)$ is a monotone non-decreasing function we find from the definition of $\Psi(v)$

$$\Psi(v') - \Psi(v) = \text{Area } (OCT) - \text{Area } (OBA)$$

$$\geq \text{Area } (ABST) + \text{Area } (PQRS)$$

so that:

$$\Psi(v') \geq \Psi(v) + \psi(v)(v'-v) + \overline{QP} \cdot \overline{PR}\,.$$

Observing that respectively

(a) $$\frac{u'}{(1-\varepsilon)} > u' \geq u > 0\,,$$

(b)
$$\frac{u'}{(1-\varepsilon)} > u' \geqq u > b ,$$

by corresponding hypotheses with $u'/(1-\varepsilon)$ instead of u, we see that $\varphi((1-\varepsilon)u') \leqq (1/K_\varepsilon)\varphi(u')$. Thus

$$\overline{QP} = \varphi(u') - \varphi((1-\varepsilon)u') \geqq \left(1 - \frac{1}{K_\varepsilon}\right)\varphi(u') .$$

Also

$$\overline{PR} = (1-\varepsilon)u' - u \geqq [(1-\varepsilon) - (1-\varepsilon)^2]u' = \varepsilon(1-\varepsilon)u' .$$

Hence

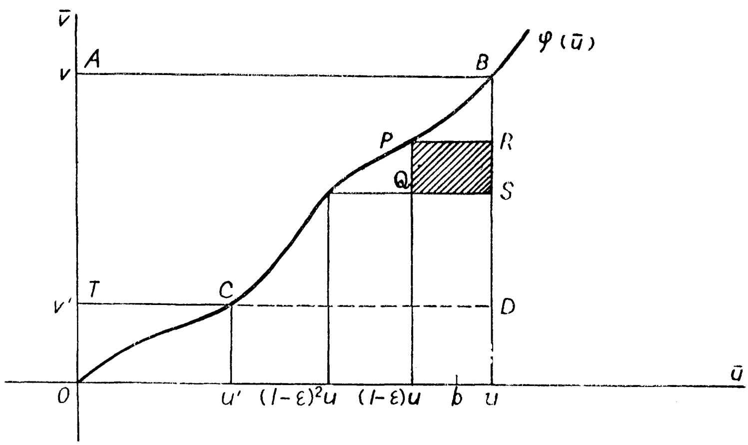
$$\overline{QP}\cdot\overline{PR} \geqq \left(1 - \frac{1}{K_\varepsilon}\right)\varepsilon(1-\varepsilon)u'\varphi(u')$$

$$\geqq \left(1 - \frac{1}{K_\varepsilon}\right)\varepsilon(1-\varepsilon)\Phi(u') \geqq \left(1 - \frac{1}{K_\varepsilon}\right)\varepsilon(1-\varepsilon)\Phi(|u'-u|) .$$

Thus with $P_\varepsilon = (1 - 1/K_\varepsilon)\varepsilon(1-\varepsilon) > 0$ we have

$$\Psi(v') \geqq \Psi(v) + \psi(v)(v'-v) + P_\varepsilon\Phi(|u'-u|) .$$

Assume $u' < u$ and consider the second diagram.



We note first that

$$u - u' \geqq \frac{(2\varepsilon - \varepsilon^2)u}{(1-\varepsilon)^2} \geqq (2\varepsilon - \varepsilon^2)u$$

so that $(1-\varepsilon)^2 u \geqq u'$. Since $\varphi(u)$ is monotone nondecreasing, from the definition of $\Psi(v)$ we find:

$$\begin{aligned}\Psi(v)-\Psi(v') &= \text{Area } (OBA)-\text{Area } (OCT)\\ &\leq \text{Area } (ABDT)-\text{Area } (PQRS)\end{aligned}$$

so that

$$\begin{aligned}\Psi(v') &\geq \Psi(v)-\text{Area } (ABDT)+\text{Area } (PQRS)\\ &= \Psi(v)+\psi(v')(v'-v)+\overline{PQ}\cdot\overline{RP}\,.\end{aligned}$$

But

$$\begin{aligned}\overline{PQ} &= \varphi((1-\varepsilon)u)-\varphi((1-\varepsilon)^2u)\\ &\geq \left(1-\frac{1}{K_\varepsilon}\right)\varphi((1-\varepsilon)u) \geq \left(1-\frac{1}{K_\varepsilon}\right)\frac{1}{T}\,\varphi(u)\end{aligned}$$

where, if we are considering the second set of hypotheses, we make use of the fact that $b \leq u$. Also $\overline{RP}=\varepsilon u$; therefore

$$\begin{aligned}\Psi(v') &\geq \Psi(v)+\psi(v)(v'-v)+\left(1-\frac{1}{K_\varepsilon}\right)\frac{\varepsilon}{T}\,u\varphi(u)\\ &\geq \Psi(v)+\psi(v)(v'-v)+\left(1-\frac{1}{K_\varepsilon}\right)\frac{\varepsilon}{T}\,\Phi(u)\\ &\geq \Psi(v)+\psi(v)(v'-v)+Q_\varepsilon\Phi(|u'-u|)\end{aligned}$$

where $Q_\varepsilon=(1-1/K_\varepsilon)(\varepsilon/T)>0$.

Taking $L_\varepsilon=\min(P_\varepsilon, Q_\varepsilon)$ we have the stated result.

THEOREM 10. *Let $\psi(v)$ be continuous, $u=\psi(v)$ and $u'=\psi(v')$ and let $0<\varepsilon<1/4$, $1<R_\varepsilon\leq N<\infty$ be constants such that alternatively*

(a) (i) $\dfrac{\Phi(2u)}{\Phi(u)}\leq N\,,\qquad (0<u)$

(ii) $R_\varepsilon<\dfrac{\varphi(u)}{\varphi((1-\varepsilon)u)}\,,\qquad (0<u)$

(iii) $|u'-u|\geq\dfrac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2}\,u>0$

or

(b) (i) $\limsup\limits_{u\to\infty}\dfrac{\Phi(2u)}{\Phi(u)}\leq N\,,$

$$\text{(ii)}\quad \limsup_{u\to\infty} \frac{\varphi(u)}{\varphi((1-\varepsilon)u)} > R_\varepsilon ,$$

$$\text{(iii)}\quad |u'-u| \geqq \max\left[\frac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2},\ \frac{(2\varepsilon-\varepsilon^2)}{(1-\varepsilon)^2}u\right]$$

then there is a constant $L_\varepsilon > 0$ such that

$$\Psi(v') \geqq \Psi(v) + \psi(v)(v'-v) + L_\varepsilon \Phi(|u'-u|) .$$

Proof. (*a*) By the same reasoning employed in Theorem 6, we may use hypothesis (a) (i) to show that there is a constant $0 < M < \infty$ such that $\varphi(2u)/\varphi(u) \leqq M$, $0 < u$. Writing $(1-\varepsilon)^2 u$ for u and noting that $\varphi(2(1-\varepsilon)^2 u) \geqq \varphi((1-\varepsilon)u)$, $0 < \varepsilon < 1/4$ we have

$$M \geqq \frac{\varphi(2(1-\varepsilon)^2 u)}{\varphi((1-\varepsilon)^2 u)} \geqq \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2 u)} , \qquad (0 < u) .$$

Again writing $(1-\varepsilon)u$ for u in (ii) we have

$$R_\varepsilon < \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2 u)} , \qquad (0 < u) .$$

With $M = T$, $R_\varepsilon = K_\varepsilon$ we may apply Lemma 3 to obtain the stated result.

(b) As in the proofs of Theorem 6 we may use (b) (i) to show that there is a constant $0 < M < \infty$ such that $\limsup_{u\to\infty} \varphi(2u)/\varphi(u) \leqq M$; this implies that for each $0 < \varepsilon < 1/4$ there is a value $u_1 < \infty$ such that if $(1-\varepsilon)^2 u_1 < u$ then $\varphi(2u)/\varphi(u) \leqq 2M$.

Writing $(1-\varepsilon)^2 u$ for u we see

$$\frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2 u)} \leqq \frac{\varphi(2(1-\varepsilon)u)}{\varphi((1-\varepsilon)^2 u)} \leqq 2M , \qquad (u_1 \leqq u) .$$

Since $\psi(v)$ is continuous, it follows that if $0 < b$ is any constant then $\varphi((1-\varepsilon)^2 b) > 0$; since $\varphi(2u)/\varphi(u) \leqq 2M$ when $(1-\varepsilon)^2 u_1 \leqq u$ it follows that $\varphi(2u_1) < \infty$; therefore

$$\frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2 u)} \leqq \frac{\varphi(2u_1)}{\varphi((1-\varepsilon)^2 b)} < \infty \qquad (b \leqq u \leqq u_1) .$$

Thus with

$$T = 1 + \max\left[2M,\ \frac{\varphi(2u_1)}{\varphi((1-\varepsilon)^2 b)}\right]$$

we have

$$\frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} < T \qquad (b \leqq u) .$$

The second hypothesis implies that there is a $u_2 < \infty$ such that

$$\frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} > \frac{R_\varepsilon+1}{2} > 1 \qquad (u_2 \leqq u) .$$

Let $\eta=(2\varepsilon-\varepsilon^2)/(1-\varepsilon)^2$, then $\eta>0$. Let

$$S = \inf_{\eta<u<u_2} \frac{\varphi((1-\varepsilon)u)}{\varphi((1-\varepsilon)^2u)} .$$

Suppose $S=1$, then there would exist a sequence $\eta \leqq u_n \leqq u_2$ such that $\varphi((1-\varepsilon)u_n)/\varphi((1-\varepsilon)^2u_n) \to 1$. From this sequence a subsequence u_{n1} could be extracted which either increases or decreases to a limit $\eta \leqq u' \leqq u_2$. If u_{n1} increases to u', then the left continuity of $\varphi(u)$ implies that $\varphi((1-\varepsilon)u')/\varphi((1-\varepsilon)^2u')=1$; while if u_{n1} decreases to u' then the right continuity of $\overline{\varphi}(u)$ implies that $\overline{\varphi}((1-\varepsilon)u')/\overline{\varphi}((1-\varepsilon)^2u')=1$. In either event this would imply that $\psi(v)$ had a discontinuity at alternatively $v=\varphi((1-\varepsilon)u')$ or $v=\overline{\varphi}((1-\varepsilon)u')$, since $\psi(v) \leqq (1-\varepsilon)^2u' < (1-\varepsilon)u' \leqq \overline{\psi}(v)$. Since $\psi(v)$ is continuous by hypothesis, we conclude: $S>1$. If we let $K_\varepsilon=\min[S, (R_\varepsilon+1)/2]$, $b=(2\varepsilon-\varepsilon^2)/(1-\varepsilon)^2$ and T as above, we see that the hypotheses of Lemma 3 are satisfied and we have established the proposition.

We shall suppress the proofs of the two following lemmas since they may be found readily in the reference cited.

LEMMA 4. [10] *If $f(x) \in L_\Phi$ and $f(x) \neq 0$ on a set of positive measure, then*

$$\int_\Delta \Phi\left(\frac{|f(x)|}{\|f\|_\Phi}\right) d\mu \leqq 1 .$$

LEMMA 5. [10] *If $f(x) \in L_\Phi$ and if there is a constant $0<N<\infty$ such that* (a) *$\Phi(2u)/\Phi(u) \leqq N$, $0<u$ when Δ is of infinite measure; or* (b) *$\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$ when Δ is of finite measure, then*

$$\int_\Delta \Phi(|f(x)|)\, d\mu < \infty .$$

LEMMA 6. *If $\{f_n(x)\}$ is a sequence of elements of L_Φ such that $\int_\Delta \Phi(|f_n(x)|)\, d\mu \to 0$ and if there is a constant $0<N<\infty$ such that either* (a) *$\Phi(2u)/\Phi(u) \leqq N$, $0<u$ when Δ is of infinite measure; or*

(b) $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$ *and also* $\psi(v)$ *is continuous, when* Δ *is of finite measure, then* $\|f_n\|_\Phi \to 0$.

Proof. (a) let $p \geqq 1$ be any positive integer and choose n_p sufficiently small so that $\int_\Delta \Phi(|f_n(x)|)d\mu \leqq 1/N^p$ for all $n_p \leqq n$. Then

$$\int_\Delta \Phi(2^p|f_n|)d\mu \leqq N^p \int_\Delta \Phi(|f_n|)d\mu \leqq 1$$

so that if $g_n(x) \geqq 0$ and $\int_\Delta \Psi(g_n)d\mu \leqq 1$ by Young's inequality

$$\int_\Delta 2^p|f_n|g_n d\mu \leqq \int_\Delta \Phi(2^p|f_n|)d\mu + \int_\Delta \Psi(g_n)d\mu \leqq 2$$

so that $\|f_n\|_\Phi \leqq 2/2^p$, $(n_p \leqq n)$. Since p may be chosen arbitrarily large, the proposition is demonstrated.

(b) Since $\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N$ it follows that there is a $u' < \infty$ such that: $\Phi(2u)/\Phi(u) \leqq 2N$, $u' \leqq u$ and $\Phi(2u') < \infty$. Since $\psi(v)$ is continuous and $\psi(0)=0$, it follows that $\varphi(u) > 0$, $0 < u$ and hence $\Phi(u) > 0$, $0 < u$; therefore, if $0 < u'' \leqq u'$ be any number we see $\Phi(2u)/\Phi(u) \leqq \Phi(2u')/\Phi(u'') < \infty$ when $u'' \leqq u \leqq u'$ so that if $N_{u''} = \max[2N, \Phi(2u')/\Phi(u'')]$ we have $\Phi(2u)/\Phi(u) \leqq N_{u''} < \infty$, $u'' < u$. Let $p \geqq 1$ be any number and choose $0 < u'' \leqq 1/2^p$; let $S_n = E_x[|f_n(x)| \geqq u'']$. If $g_n(x) \geqq 0$ and $\int_\Delta \Psi(g_n)d\mu \leqq 1$ then by Young's inequality

$$\begin{aligned}\int_\Delta 2^p|f_n|g_n d\mu &\leqq \int_\Delta \Phi(2^p|f_n|)d\mu + \int_\Delta \Psi(g_n)d\mu \\ &\leqq \int_{S_n} \Phi(2^p|f_n|)d\mu + \int_{S_n'} \Phi(2^p|f_n|)d\mu + 1 \\ &\leqq (N_{u''})^p \int_{S_n} \Phi(|f_n|)d\mu + \Phi(1)\mu(\Delta) + 1 .\end{aligned}$$

By choosing n sufficiently large, we have $\int_{S_n} \Phi(|f_n|)d\mu \leqq (N_{u''})^{-p}$ so that

$$\|f_n\|_\Phi \leqq \frac{2+\Phi(1)\mu(\Delta)}{2^p} .$$

Taking p sufficiently large we see that $\|f_n\|_\Phi \to 0$ since $\mu(\Delta) < \infty$.

LEMMA 7. *If* α, β *are real, then*

$$|\alpha-\beta|=|\alpha|+|\beta|-|\alpha+\beta|+||\alpha|-|\beta|| .$$

Proof. If $\alpha \geqq \beta \geqq 0$ then $|\alpha-\beta|=||\alpha|-|\beta||$ and $|\alpha|+|\beta|=|\alpha+\beta|$. If $\alpha \geqq 0 \geqq \beta$ then $|\alpha|+|\beta|=|\alpha-\beta|$ and $|\alpha+\beta|=||\alpha|-|\beta||$. If $0>\alpha \geqq \beta$ then $|\alpha|+|\beta|=|\alpha+\beta|$ and $|\alpha-\beta|=||\alpha|-|\beta||$. The remaining cases in which $\beta \geqq \alpha$ hold by symmetry.

LEMMA 8. *If $\eta \leqq 1$ then $\Phi(\eta u) \leqq \eta\Phi(u)$ and if $\eta \geqq 1$ then $\Phi(\mu u) \geqq \eta\Phi(u)$.*

Proof. Since $\varphi(u)$ is monotone nondecreasing if $\eta \leqq 1$ we have

$$\Phi(\eta u)=\int_0^{\eta u} \varphi(\bar{u})\, d\bar{u} \leqq \eta \int_0^{u} \varphi(\bar{u})\, d\bar{u}=\eta\Phi(u) .$$

If $\eta \geqq 1$ and $\xi=1/\eta \leqq 1$ then $\Phi(\xi u') \leqq \xi\Phi(u')$; so that, if $\xi u'=u$, we have $\eta\Phi(u) \leqq \Phi(\eta u)$.

LEMMA 9. (a) *If there is a constant $0<N<\infty$ such that $\Phi(2u)/\Phi(u) \leqq N$ then*

$$\Phi(u_1+u_2) \leqq N[\Phi(u_1)+\Phi(u_2)] ,$$

for arbitrary $0 \leqq u_1$, $0 \leqq u_2$;

(b) *if for each $0<u''$ there is a constant $0<N_{u''}<\infty$ such that $\Phi(2u)/\Phi(u) \leqq N_{u''}$, $u'' \leqq u$, then*

$$\Phi(u_1+u_2) \leqq \Phi(2u'')+N_{u''}[\Phi(u_1)+\Phi(u_2)]$$

for arbitrary $0 \leqq u_1$, $0 \leqq u_2$.

Proof. (a) Let $u_3=\max[u_1, u_2]$ so that $u_1+u_2 \leqq 2u_3$. Then

$$\Phi(u_1+u_2) \leqq \Phi(2u_3) \leqq N\Phi(u_3) \leqq N[\Phi(u_1)+\Phi(u_2)] .$$

(b) Let $u_3=\max[u_1, u_2]$. If $u_3 \leqq u''$, then

$$\Phi(u_1+u_2) \leqq \Phi(2u_3) \leqq \Phi(2u'') .$$

If $u_3>u''$ then

$$\Phi(u_1+u_2) \leqq \Phi(2u_3) \leqq N_{u''}\Phi(u_3) \leqq N_{u''}[\Phi(u_1)+\Phi(u_2)] .$$

THEOREM 10. *Let $\Psi(v)$ and $\psi(v)$ continuous and let $0<\varepsilon<1/4$, $1<R_\varepsilon \leqq N<\infty$ be constants such that alternatively*

(a) *when Δ is of infinite measure*

$$\text{(i)} \quad \Phi(2u)/\Phi(u) \leq N\,, \qquad (0<u)\,,$$

$$\text{(ii)} \quad R_\varepsilon < \frac{\varphi(u)}{\varphi((1-\varepsilon)u)}\,, \qquad (0<u)\,;$$

or

(b) *when Δ is of finite measure*

$$\text{(i)} \quad \limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leq N\,,$$

$$\text{(ii)} \quad \limsup_{u\to\infty} \frac{\varphi(u)}{\varphi((1-\varepsilon)u)} > R_\varepsilon$$

then L_Φ is uniformly convex.

Proof. We first assume that $f'_n(x) \geq 0$, $f''_n(x) \geq 0$, that $\|f'_n\|_\Phi = \|f''_n\|_\Phi = 1$ and that $\frac{1}{2}\|f'_n + f''_n\|_\Phi \to 1$; and we shall prove that $\|f'_n - f''_n\|_\Phi \to 0$.

Let $\eta = (2\varepsilon - \varepsilon^2)/(1-\varepsilon)^2$; we observe that $\lim_{\varepsilon\to 0} \eta = 0$. By Theorem 9 there is a constant $0 < L_\eta < \infty$ such that, corresponding to the alternative hypotheses, when

(a) $$|u'-u| \geq \eta u > 0\,,$$

(b) $$|u'-u| \geq \max(\eta, \eta u)\,,$$

then

$$(*) \qquad \Psi(v') \geq \Psi(v) + \psi(v)(v'-v) + L_\eta \Phi(|u'-u|)$$

where $u = \psi(v)$ and $u' = \psi(v')$. By Theorem 6, let $1 < D_n < \infty$, $1 < C'_n < \infty$ be constants, $g_n(x) \geq 0$, $h'(x) \geq 0$ be functions such that

$$\psi(g_n(x)) = D_n\left(\frac{f'_n(x) + f''_n(x)}{2}\right), \qquad \int_\Delta \Psi(g_n)\,d\mu = 1$$

$$\psi(h'_n(x)) = C'_n f'_n(x)\,, \qquad \int_\Delta \Psi(h'_n)\,d\mu = 1\,.$$

Let alternatively

(a) $$E_n(\eta) \equiv \underset{x}{E}\left[\left|C'_n f'_n(x) - \frac{D_n}{2}(f'_n(x) + f''_n(x))\right| \geq \eta C'_n f'_n(x)\right]$$

(b) $$E_n(\eta) \equiv \underset{x}{E}\left[\left|C'_n f'_n(x) - \frac{D_n}{2}(f'_n(x) + f''_n(x))\right| \geq \max[\eta, \eta C'_n f'_n(x)]\right].$$

Write $v' = g_n(x)$, $v = h'_n(x)$ in $(*)$ above and integrate over $E_n(\eta)$ to obtain

$$\int_{E_n(\eta)} \Psi(g_n)\,d\mu \geq \int_{E_n(\eta)} \Psi(h_n')\,d\mu + \int_{E_n(\eta)} C_n' f_n'(x)(g_n(x) - h_n'(x))\,d\mu$$

$$+ L_\eta \int_{E_n(\eta)} \Phi\left(\left|C_n' f_n'(x) - \frac{D_n}{2}(f_n'(x) + f_n''(x))\right|\right) d\mu \,.$$

But, by Lemma 2, $\Psi(v') \geq \Psi(v) + \psi(v)(v' - v)$; so that making the same substitutions as before and integrating over $E_n'(\eta)$ we may assert

$$\int_{E_n'(\eta)} \Psi(g_n)\,d\mu \geq \int_{E_n'(\eta)} \Psi(h_n')\,d\mu + \int_{E_n'(\eta)} C_n' f_n'(x)(g_n(x) - h_n'(x))\,d\mu \,.$$

Hence

$$\int_\Delta \Psi(g_n)\,d\mu \geq \int_\Delta \Psi(h_n')\,d\mu + C_n' \int_\Delta C_n' f_n'(x)(g_n(x) - h_n'(x))\,d\mu$$

$$+ L_\eta \int_{E_n(\eta)} \Phi\left(\left|C_n' f_n'(x) - \frac{D_n}{2}(f_n'(x) + f_n''(x))\right|\right) d\mu \,,$$

so that

$$C_n' \int_\Delta f_n'(x)(h_n'(x) - g_n(x))\,d\mu \geq L_\eta \int_{E_n(\eta)} \Phi\left(\left|C_n' f_n' - \frac{D_n}{2}(f_n' + f_n'')\right|\right) d\mu \,.$$

By Lemma 8, since $1 \leq C_n' < \infty$ we have

$$(+) \qquad \int_\Delta f_n'(x)(h_n'(x) - g_n(x))\,d\mu \geq L_\eta \int_{E_n(\eta)} \Phi\left(\left|f_n' - \frac{D_n}{2C_n'}(f_n' + f_n'')\right|\right) d\mu \,.$$

Now

$$\|f_n' + f_n''\|_\Phi = \int_\Delta (f_n'(x) + f_n''(x)) g_n(x)\,d\mu$$

$$= \int_\Delta f_n'(x) g_n(x)\,d\mu + \int_\Delta f_n''(x) g_n(x)\,d\mu \leq \|f_n'\|_\Phi + \|f_n''\|_\Phi \,.$$

But

$$\int_\Delta f_n'(x) g_n(x)\,d\mu \leq \int_\Delta f_n'(x) h_n'(x)\,d\mu \leq \|f_n'\|_\Phi$$

and

$$\int_\Delta f_n''(x) g_n(x)\,d\mu \leq \|f_n''\|_\Phi$$

and also

$$\|f_n' + f_n''\|_\Phi \to \|f_n'\|_\Phi + \|f_n''\|_\Phi$$

so that:

$$\int_{\Delta} f_n'(x)(h_n'(x) - g_n(x))\, d\mu \to 0 .$$

Thus by (+) since $0 < L_\eta < \infty$,

$$(**) \qquad \int_{E_n(\eta)} \Phi\left(\left| f_n' - \frac{D_n}{2C_n} (f_n' + f_n'')\right|\right) d\mu \to 0 .$$

on $E_n'(\eta)$ alternatively

$$\text{(a)} \qquad \left| f_n'(x) - \frac{D_n}{2C_n'} (f_n'(x) + f_n''(x))\right| \leqq \eta f_n'(x) ;$$

$$\text{(b)} \qquad \left| f_n'(x) - \frac{D_n}{2C_n'} (f_n'(x) + f_n''(x))\right| \leqq \max\left[\frac{\eta}{C_n'}, \eta f_n'(x)\right] \leqq \max\, [\eta, \eta f_n'(x)]$$

so that, since $\eta \leqq 1$, by Lemma 4 and 8 we have alternatively as $\varepsilon \to 0$

$$\text{(a)} \qquad \int_{E_n'(\eta)} \Phi\left(\left| f_n' - \frac{D_n}{2C_n'} (f_n' + f_n'')\right|\right) d\mu \leqq \int_{E_n'(\eta)} \Phi(\eta f_n')\, d\mu$$

$$\leqq \eta \int_{E_n'(\eta)} \Phi(f_n')\, d\mu \leqq \eta \int_{\Delta} \Phi(f_n')\, d\mu \leqq \eta \to 0$$

$$\text{(b)} \qquad \int_{E_n'(\eta)} \Phi\left(\left| f_n' - \frac{D_n}{2C_n'} (f_n' + f_n'')\right|\right) d\mu \leqq \int_{E_n'(\eta)} \Phi(\eta)\, d\mu$$

$$+ \int_{E_n'(\eta)} \Phi(\eta f_n')\, d\mu \leqq \Phi(\eta)\mu(\Delta) + \int_{\Delta} \Phi(\eta f_n')\, d\mu$$

$$\leqq \Phi(\eta)\mu(\Delta) + \eta \int_{\Delta} \Phi(f_n')\, d\mu \leqq \Phi(\eta)\mu(\Delta) + \eta \to 0$$

since $\mu(\Delta)$ is finite. Combining these results with (**) we see that

$$\int_{\Delta} \Phi\left(\left| f_n' - \frac{D_n}{2C_n'} (f_n' + f_n'')\right|\right) d\mu \to 0 .$$

Thus by Lemma 6

$$\left\| f_n' - \frac{D_n}{2C_n'} (f_n' + f_n'')\right\|_{\Phi} \to 0$$

and this implies in turn that

$$\frac{D_n}{C_n'} \left\| \frac{f_n' + f_n''}{2} \right\|_{\Phi} \to \|f_n'\|_{\Phi} .$$

But $\frac{1}{2}\|f_n'+f_n''\|_\Phi \to 1=\|f_n'\|_\Phi$ so that $D_n/C_n' \to 1$. It then follows that

$$\left\|f_n'-\left(\frac{f_n'+f_n''}{2}\right)\right\|_\Phi \to 0$$

from which we have immediately that

$$\|f_n'-f_n''\|_\Phi \to 0 \ .$$

We now prove the theorem for the general case when the functions $f_n'(x), f_n''(x)$ are not necessarily positive. We use the equivalent definition of uniform convexity which has been noted in the introduction. Let $\|f_n'\|_\Phi=\|f_n''\|_\Phi=1$ and suppose $\|f_n'+f_n''\|_\Phi \to 2$. We define

$$F_n'(x)=\begin{cases} |f_n'(x)| & \text{if } \ (f_n'(x)+f_n''(x)) \text{ has the sign of } f'(x) \\ 0 & \text{otherwise ;} \end{cases}$$

$$F_n''(x)=\begin{cases} |f_n''(x)| & \text{if } \ (f_n'(x)+f_n''(x)) \text{ has the sign of } f_n''(x) \ , \\ 0 & \text{otherwise .} \end{cases}$$

Clearly

$$0 \leqq F_n'(x) \leqq |f_n'(x)| \ , \qquad 0 \leqq F_n''(x) \leqq |f_n''(x)| \ ,$$

$$F_n'(x)+F_n''(x) \geqq |f_n'(x)+f_n''(x)|$$

and

$$2|F_n'(x)-F_n''(x)| \geqq |f_n'(x)-f_n''(x)| \ ,$$

so that $\|F_n'\|_\Phi \leqq 1$, $\|F_n''\|_\Phi \leqq 1$, $\liminf \|F_n'+F_n''\|_\Phi \geqq 2$ and $\|f_n''-f_n'\| \leqq 2\|F_n'-F_n''\|_\Phi$. Our result for positive functions applied to $F_n'(x)$ and $F_n''(x)$ now gives $\|f_n'-f_n''\|_\Phi \to 0$ and L_Φ is uniformly convex.

THEOREM 11. *Necessary and sufficient conditions that L_Φ be uniformly convex are*

(a) *in case Δ is of infinite measure*

(i) *$\Psi(v)$ is continuous,*

(ii) *$\psi(v)$ is continuons,*

(iii) *there is a constant $0<N<\infty$ such that $\Phi(2u)/\Phi(u) \leqq N$, $\Psi(2v)/\Psi(v) \leqq N$, $(0<u, \ 0<v)$.*

(iv) *for each constant $0<\varepsilon<1/4$ there is a constant $1<R_\varepsilon<\infty$ such that $\varphi(u)/\varphi((1-\varepsilon)u)>R_\varepsilon$, $(0<u)$;*

r:

(b) *in case Δ is of finite measure*

(i) *$\Psi(v)$ is continuons,*

(ii) *$\psi(v)$ is continuous,*

(iii) *there is a constant $0<N<\infty$ such that*

$$\limsup_{u\to\infty} \Phi(2u)/\Phi(u) \leqq N\,, \quad \textit{and} \quad \limsup_{v\to\infty} \Psi(2v)/\Psi(v) \leqq N$$

(iv) *for each constant* $0<\varepsilon<1/4$ *there is a constant* $1<R_\varepsilon<\infty$ *such that* $\liminf_{u\to\infty} \varphi(u)/\varphi((1-\varepsilon)u)>R_\varepsilon$.

Proof. The theorem is simply a summary of the results of Theorems 1, 2, 7, 8 and 9 and of Theorem 10.

It is interesting to remark (a) that the condition $\varphi((1+\varepsilon)u)/\varphi(u)> R_\varepsilon>1$ implies that $\Psi(2v)/\Psi(v)\leqq N$ for some constant $0<N<\infty$, and (b) that the condition $\liminf_{u\to\infty} \varphi((1+\varepsilon)u)/\varphi(u)>R_\varepsilon>1$ implies that $\limsup_{v\to\infty} \Psi(2v)/\Psi(v)\leqq N$ for some constant $0<N<\infty$; but the implications converse to (a) and (b) are untrue. To prove the direct statement we choose an integer $0<p$ such that $((R_\varepsilon+1)/2)^p>2$. Now, respectively (a) for all $0<u$,

$$\frac{\varphi((1+\varepsilon)u)}{\varphi(u)}>\left(\frac{R_\varepsilon+1}{2}\right);$$

and (b) there is a value $0<u_\varepsilon$ such that if $u_\varepsilon\leqq u$, then

$$\frac{\varphi((1+\varepsilon)u)}{\varphi(u)}>\left(\frac{R_\varepsilon+1}{2}\right).$$

Then if (a) $0<u$, (b) $u_\varepsilon<u$, we see that

$$\varphi((1+\varepsilon)^p u)\geqq\left(\frac{R_\varepsilon+1}{2}\right)^p\varphi(u)\geqq 2\varphi(u)\,.$$

Letting $v=\varphi(u)$ we have $(1+\varepsilon)^p\psi(v)\geqq\psi(2v)$ when alternatively:
(a) $0<v$,
(b) $v_\varepsilon<v$ where $v_\varepsilon=\varphi(u_\varepsilon)$. But then

$$\text{(a)} \qquad (1+\varepsilon)^p\Psi(v)=(1+\varepsilon)^p\int_0^v\psi(\bar v)\,d\bar v\geqq\int_0^v\psi(2\bar v)\,dv\geqq\tfrac12\Psi(2v)$$

where $0<v$,

$$\text{(b)} \qquad (1+\varepsilon)^p\Psi(v)=(1+\varepsilon)^p\int_{v_\varepsilon}^v\psi(\bar v)\,d\bar v+(1+\varepsilon)^p\Psi(v_\varepsilon)$$

$$\geqq\int_{v_\varepsilon}^v\psi(2\bar v)\,d\bar v+(1+\varepsilon)^p\Psi(v_\varepsilon)>\tfrac12\Psi(2v)-\tfrac12\Psi(2v_\varepsilon)$$

where $v_\varepsilon\leqq v$, and since $\Psi(2v_\varepsilon)<\infty$ and $\Psi(v)\to\infty$ we see that $\limsup_{v\to\infty}\Psi(2v)/\Psi(v)\leqq 2(1+\varepsilon)^p<\infty$. To prove the converse construct the following function. Let $u_1=0$, $v_1=0$, $u_n=2^n$, $v_n=2^n$, $u_n'=(1+\varepsilon)u_n$, $v_n'=(2^n+\frac{1}{2})$; $(n=2, 3, \cdots)$. Join the points (u_1v_1) to (u_2v_2); (u_nv_n) to $(u_n'v_n')$; (u_n', v_n') to (u_{n+1}, u_{n+1}) each by straight line segments and let this function be $\varphi(u)$. Then

$$\frac{\varphi((1+\varepsilon)u_n)}{\varphi(u_n)}=\frac{\varphi(u_n')}{\varphi(u_n)}=\frac{v_n'}{v_n}=\frac{2^n+\frac{1}{2}}{2^n}\to 1$$

while $\psi(2v)\leqq 4\psi(v)$, $(0<v)$ and therefore $\Psi(2v)/\Psi(v)\leqq 8$, $(0<v)$. It is also clear that condition (i) is implied by the condition $\limsup_{v\to\infty}\Psi(2v)/\Psi(v)\leqq N$ and consequently by (iv). Thus, if we wished to do so, we might delete any statement relative to the function $\Psi(v)$ from Theorem 11. It is true, however, that the remaining conditions are independent for none of them is implied by any combination of the other hypotheses.

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WAYNE UNIVERSITY

INTERIOR VARIATIONS AND SOME EXTREMAL PROBLEMS FOR CERTAIN CLASSES OF UNIVALENT FUNCTIONS

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1. Introduction. The theory of regular univalent functions in the unit circle U_z has been developed for various subclasses, for example, the class of real univalent functions which leads to symmetric domains, the class of bounded univalent functions whose image domain lies with in the unit circle and the functions for which the image domains are convex or star-like. The approach through the calculus of variations has been used very successfully towards the solution of extremal problems belonging to the various classes and also towards the determination of the extremal domains. The purpose of the present paper is to show how the method of interior variations due to Schiffer [1] can be adapted for the following subclasses:

(i) The class V of symmetric regular univalent-functions $f(z)$ in U_z which have the form $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ with real a_n. In particular we show that if $\varphi(a_2, a_3, \cdots, a_n; \bar{a}_2, \bar{a}_3, \cdots, \bar{a}_n)$ is a real valued function which is symmetric and analytic in a_ν and $\bar{a}_\nu$ $(\nu=2, 3, \cdots, n)$ and where $\{a_n\}$ are the coefficients in the power series expansion of the more general class V_1 of regular univalent functions then, under the assumption that the function $f(z)$ whose coefficients $\{a_\nu\}$ maximize $\varphi(a_2, \cdots, a_n, \bar{a}_2, \cdots, \bar{a}_n)$ is symmetric, the functional differential equation satisfied by $f(z)$ in the general class V_1 is the same as the functional differential equation satisfied by $f(z)$ in the class V.

(ii) The class S of bounded univalent functions $f(z)$ in U_z which are normalized so that $f(0)=0$, $|f(z)|\leq 1$ and at a fixed point $\zeta \in U_z$, $f(\zeta)=\omega$. In particular we find the functions which maximize or minimize $|f'(\zeta)|$.

(iii) The class Σ of bounded univalent functions $f(z)$ in U_z which are real on the real axis and are normalized so that $f(0)=0$, $|f(z)|\leq 1$ and at a fixed point ζ on the real axis $f(\zeta)=\omega$. In particular we find the functions which maximize or minimize $f(\eta)$ for real $\eta \in U_z$.

We observe that the existence and uniqueness of the solutions of these problems is assured because the families of functions belonging to the classes V, S and Σ are normal and compact.

2. Real univalent functions. Let D be the image in the W-plane

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$W=f(z)\in V$ of $|z|\leq 1$ and let us consider the Schiffer variation

$$(1)\qquad W_1^*=W+a\rho^2\frac{W}{W_0(W-W_0)}+\bar{a}\rho^2\frac{W}{W_0(W-W_0)},$$

where $\overline{W}_0$ is an interior point of D. It is easily seen that for small enough ρ say $\rho_1<\rho$, $|W-W_0|=\rho_1$ and $|W-\overline{W}_0|=\rho_1$ lie entirely in D and W_1^* is univalent on the boundary C of D and maps it univalently on to the boundary C^* of the new domain D^*. Further, we see that $W_1^*=0$ for $W=0$ and that W_1^* is real for real values of W. Thus if W is a symmetric univalent function which vanishes at the origin we have obtained another neighbouring function which also has the same properties. In order to be able to add some side conditions to the function W we consider the variation

$$(2)\qquad W^*=W+\rho^2\sum_{\nu=1}^{p}\left\{\frac{\alpha_\nu W}{(W-W_\nu)W_\nu}+\frac{\bar{\alpha}_\nu W}{(W-W_\nu)W_\nu}\right\},$$

where p is an integer ≥ 1. This variation is of the same type as (1) and has the independent constants $(\alpha_\nu)^p$ which can be used to satisfy the side conditions, if any.

The technique of getting the variation formula for $f(z)$ under the variation (2) is similar to that used in [2] in getting the variation formula for $f(z)$ under the variation $W^*=W+a\rho^2/(W-W_0)$. For the sake of completeness we mention that we first find the variation formula for the Green's function $G(W,0)$ of D under the variation (2). We thus have [3]

$$(3)\qquad \delta G(W,0)=\Re\left[\frac{\rho^2}{2\pi i}\int_\Gamma p'(\eta,0)p'(\eta,W)\varphi(\eta)d\eta\right]+O(\rho^3),$$

where

$$(4)\qquad \varphi(W)=\sum_{\nu=1}^{p}\left\{\frac{\alpha_\nu W}{(W-W_\nu)W_\nu}+\frac{\bar{\alpha}_\nu W}{(W-W_\nu)W_\nu}\right\},$$

and $p(W,\eta)$ is the analytic function whose real part is the Green's function $G(W,\eta)$ and Γ is a curve system in D which is homotopic to C and such that $\varphi(W)$ is analytic in the ring system bounded by C and Γ. If now $z=\varphi(W)$ is the inverse function of $W=f(z)$ then the relationship of the Green's function $G(W,\omega)$ to the function $\varphi(W)$ is given by

$$(5)\qquad G(W,\omega)=-\log\left|\frac{\varphi(W)-\varphi(\omega)}{1-\varphi(\omega)\varphi(W)}\right|;$$

and in particular

$$G(W, 0)=\log \frac{1}{|\varphi(W)|}.$$

Proceeding in this way we find that the variation formula for $f(z)$ is given by

$$(6) \quad f^*(z)=f(z)+\sum_{\nu=1}^{p}\alpha_\nu\rho^2\Big[\frac{z^2}{1-z\,t_\nu}\frac{f'(z)}{t_\nu f'^2(t_\nu)}+\frac{zf'(z)}{z_\nu f'^2(z_\nu)}\frac{1}{z_\nu-z}$$
$$+\frac{f(z)}{f(z_\nu)(f(z)-f(z_\nu))}\Big]+\Sigma\bar{\alpha}_\nu\rho^2\Big[\frac{z^2}{1-z\bar{z}_\nu}\frac{f'(z)}{\bar{z}_\nu f'^2(z_\nu)}+\frac{zf'(z)}{t_\nu f'^2(t^\nu)}\frac{1}{t_\nu-z}$$
$$+\frac{f(z)}{f(t_\nu)(f(z)-f(t_\nu))}\Big]+O(\rho^3),$$

where

$$t_\nu=\varphi(\overline{W}_\nu) \quad \text{or} \quad f(t_\nu)=f(\bar{z}_\nu).$$

Let $f(z)$ have the following power series expansion

$$(7) \qquad f(z)=z+\sum_{n=2}^{\infty}a_n z^n,$$

where the a_n are real. Then denoting by a_n^* the coefficient of z^n in $f^*(z)$ and substituting these expansions in (6) and equating the coefficients of z^n on both sides we get

$$(8) \qquad \frac{a_n^*}{a_1^*}=a_n+2\Re\Big\{\rho^2\sum_{\nu=1}^{p}\alpha_\nu\Big[A_\nu\sum_{m=1}^{n-1}(z_\nu^m+z_\nu^{-m}(n-m)a_{n-m}+n_{an}+z_\nu^{n-1}$$
$$+z_\nu^{-(n-1)}+T_n(f(z_\nu))\Big]\Big\}+O(\rho^3),$$

where

$$A_\nu=\frac{1}{z_\nu^2 f'^2(z_\nu)}$$

and $T_n(f(z))$ are given by the formula

$$(9) \qquad \frac{f(z)}{f(t_\nu)(f(z)-f(t_\nu))}=\sum_{n=1}^{\infty}z^n T_n(f(t_\nu)).$$

We remark that (8) is the variation formula for the coefficient a_n given by (7) and for $\nu=1$ it agrees with the variation formula obtained by Schiffer [2] for $|a_n|$ when a_n are complex. Further, if we put $\delta a_n=a_n^*/a_1^*-a_n$ then $da_n=\delta a_n+O(\rho^3)$ and we find that δa_n in the present case is twice $\delta\alpha_n$ in the general case $a_n=\alpha_n+i\beta_n$.

Now let us consider a function $\varphi(a_2, \cdots, a_n, \bar{a}_2, \cdots, \bar{a}_n)$ which is symmetric and analytic in a_ν and $\bar{a}_\nu$ and has real coefficients. Then, because $a_\nu = \alpha_\nu + i\beta_\nu$, we can write $F(\alpha_2, \cdots, \alpha_n; \beta_2, \cdots, \beta_n) = \varphi(a_2, \cdots, a_n; \bar{a}_2, \cdots, \bar{a}_n)$, and the function F will contain only even powers of β_ν's and

$$(10) \qquad \frac{\partial F}{\partial \beta_\nu} = 0 \quad \text{if} \quad \beta_2 = \beta_3 = \cdots = \beta_n = 0 .$$

Further, the condition for the extremum of F in the general case when a_ν are complex is

$$\sum_{\nu=2}^{n} \left[\frac{\partial F}{\partial \alpha_\nu} d\alpha_\nu + \frac{\partial F}{\partial \beta_\nu} d\beta_\nu \right] = 0 ,$$

which in the limit when $\rho \to 0$ can be written as

$$(11) \qquad \sum_{\nu=1}^{n} \left[\frac{\partial F}{\partial \alpha_\nu} \delta\alpha_\nu + \frac{\partial F}{\partial \beta_\nu} \delta\beta_\nu \right] = 0 .$$

If a_ν are real then in view of (10), (11) reduces to

$$(12) \qquad \sum_{\nu=2}^{n} \frac{\partial F}{\partial \alpha_\nu} \delta\alpha_\nu = 0 .$$

We will obtain the same equation if we look for the solution of the extremumproblem in the particular class V of real functions. Thus under the assumption that the extremum function is real the functional differential equation in the general case will coincide with the differential equation in the symmetric case. We know from compactness arguments that the problem $\varphi = \max$ has a solution in V. We also know that the same problem has a solution in the general class V_1. What we have shown is that both extremum functions satisfy the same functional differential equation. This implies that either there are many solutions of the problem in the class V_1 or that the solution lies in the class V. Thus in particular the coefficient problem $a_n\bar{a}_n = \max$ leads to the same functional differential equation for real univalent functions as for the general class.

3. Bounded univalent functions. We now consider a variation which transforms a function of the class S into another function of the class S. We will first obtain a variation which keeps the origin fixed and also keeps the unit circumference fixed. We will then add the side condition that for a fixed ζ, $f(\zeta) = \omega$. Let

$$W_2^* = W + a\rho^2 \frac{W}{W - W_0} - \frac{\bar{a}\rho^2 W^2}{1 - W_0 W} + O(\rho^4) , \tag{13}$$

where $O(\rho^4)$ can be suitably chosen and W_0 is an interior point of the image domain of $|z| \leq 1$ by $f(z)$. For small enough ρ, W_2^* is a univalent function of W outside $|W - W_0| = \rho_1 < \rho$ such that $W_2^* = 0$ for $W = 0$ and keeps the unit circumference $|W| = 1$ fixed. In fact, to prove the latter, we observe that when

$$|W| = 1, \quad W = \frac{1}{\bar{W}}$$

and

$$|W_2^*| = \left[1 + 4\rho^4 \left\{\Im\left(\frac{a}{W - W_0}\right)\right\}^2\right]^{1/2} + O(\rho^4) . \tag{14}$$

So to the order ρ^2 the unit circumference is kept fixed. By adding to (13) term of the order of ρ^4 one could make the unit circumference fixed. To see this let us denote by D^* the boundary of the domain to which W_2^* maps the unit circumference $|W| = 1$. Now by the Riemann mapping theorem there exists an analytic function $S(W_2^*)$ which vanishes at the origin and maps D^* univalently on the unit circumference $|S(W_2^*)| = 1$, $W_2^* \in D^*$. From the boundary behavior (14) of W_2^* when $W_2^* \in D^*$ we then conclude that

$$\left| \frac{S(W_2^*)}{W_2^*} \right| \simeq 1 + O(\rho^4) . \tag{15}$$

Since $\log |S(W_2^*)/W_2^*|$ is harmonic in $|W_2^*| < 1$, we conclude, by the maximum principle that (15) holds everywhere inside the unit circle. Thus

$$S(W_2^*) = S^*(W) = W_2^* + O(\rho^4)$$

for $|W| = 1$. We have thus obtained a function $S^*(W)$ which maps the unit circumference $|W| = 1$ onto itself and differs from W_2^* by $O(\rho^4)$.

A more general type of variation which can take care of some extra side conditions can be written in the form

$$W^* = W + \sum_{\nu=1}^{n} \left[\frac{a_\nu \rho^2 W}{W - W_\nu} - \frac{\bar{a}_\nu \rho^2 W^2}{1 - W_\nu W} \right] + O(\rho^4) . \tag{16}$$

Taking $n = 1$ we get according to the procedure outlined in § 2 the following variation formula

$$\varphi^*(W) = \varphi(W) - a_0 \rho^2 A(W, W_0) + \bar{a}_0 \rho^2 B(W, \overline{W}_0) \tag{17}$$

$$-a_1\rho^2 A(W,\ W_1)+\bar{a}_1\rho^2 B(W,\ \overline{W}_1)+O(\rho^4)$$

where

(18) $$A(W,\ W_0)=\frac{W\varphi'(W)}{W-W_0}+\frac{W_0\varphi'^2(W_0)\varphi(W)}{\varphi(W_0)(\varphi(W_0)-\varphi(W))},$$

and

(19) $$B(W,\ \overline{W}_0)=\frac{W^2\varphi'(W)}{1-W\overline{W}_0}-\frac{\varphi^2(W)\overline{W_0\varphi'^2(W_0)}}{\overline{\varphi(W_0)}(1-\varphi(W)\overline{\varphi(W_0)})}.$$

Since we require that for all $f(z)\in S$ and fixed ζ, $f(\zeta)=\omega$, we must have $\varphi^*(\omega)=\varphi(\omega)=\zeta$. Thus we obtain the determining relation between a_0 and a_1

(20) $$-a_0A(\omega,\ W_0)+\bar{a}_0B(\omega,\ \overline{W}_0)-a_1A(\omega,\ W_1)+\bar{a}_1B(\omega,\ \overline{W}_1)+O(\rho^2)=0\,.$$

We shall see that in general we can prescribe W_0, W_1 and a_0 arbitrarily and adjust a_1 such that (20) holds.

Again, as $\varphi'(W)=1/f'(z)$ we see that the minimum and maximum of $|f'(\zeta)|$ would be given by the maximum and minimum of $|\varphi'(\omega)|$. Thus the necessary condition for the extremum of $|f'(\zeta)|$ is

(21) $$\Re\left[a_0\frac{A'(\omega,\ W_0)}{\varphi'(\omega)}-\bar{a}_0\frac{B'(\omega,\ \overline{W}_0)}{\varphi'(\omega)}+a_1\frac{A'(\omega,\ W_1)}{\varphi'(\omega)}-\bar{a}_1\frac{B'(\omega,\ \overline{W}_1)}{\varphi'(\omega)}\right]$$
$$+O(\rho^2)=0\,,$$

where $A'(\omega,\ W_0)$ and $B'(\omega,\ \overline{W}_0)$ are, respectively, the derivatives of $A(\omega,\ W_0)$ and $B(\omega,\ \overline{W}_0)$ with respect to the first argument.

The extremum condition (21) can also be written in the form

(22) $$a_0C(\omega,\ W_0)-\bar{a}_0\overline{C(\omega,\ W_0)}+a_1C(\omega,\ W_1)-\bar{a}_1\overline{C(\omega,\ W_1)}+O(\rho^2)=0\,,$$

where

$$C(\omega,\ W_0)=\frac{A'(\omega,\ W_0)}{\varphi'(\omega)}-\frac{\overline{B'(\omega,\ W_0)}}{\overline{\varphi'(\omega)}}\,.$$

From (20) it is clear that for a fixed value of a_0, a_1 is a linear function of a_0 and $\bar{a}_0$ and can be written as

(23) $$a_1=\frac{a_0(\overline{B}_0B_1-A_0\overline{A}_1)+\bar{a}_0(B_0\overline{A}_1-\overline{A}_0B_1)}{|B_1|^2-|A_1|^2}+O(\rho^2),$$

if

(24) $$|B_1|^2\neq|A_1|^2\,,$$

where we have put $A_0=A(\omega,\ W_0)$, $A_1=A(\omega,\ W_1)$, and similarly for B_0 and B_1. We will show later that (24) can always be taken to be valid. Taking the case when (24) holds, we get on substituting this value of a_1 in (22) that

$$a_0[C(\omega,\ W_0)+\bar{\lambda}\bar{B}_0-\lambda A_0]+\bar{a}_0[\overline{C(\omega,\ W_0)}+\lambda B_0-\bar{\lambda}\bar{A}_0]+O(\rho^2)=0\,, \tag{25}$$

where

$$\lambda=\frac{\bar{B}_1\overline{C(\omega,\ W_1)}+\bar{A}_1 C(\omega,\ W_1)}{|B_1|^2-|A_1|^2}\,.$$

This holds for all sufficiently small values of ρ; hence because a_0 is arbitrary, in the limit $\rho\to 0$ the extremum function satisfies the equation

$$C(\omega,\ W_0)=\lambda A(\omega,\ W_0)-\bar{\lambda}\,\overline{B(\omega,\ \bar{W}_0)}\,, \tag{26}$$

where λ is independent of W_0. Again, because W_0 is an arbitrary point from (26), the equation satisfied by the extremum function can be written in the form

$$\begin{aligned}\frac{(1+\alpha)(\omega-W)-\omega}{(\omega-W)^2}-\frac{\bar{\omega}(2+\bar{\alpha})(1-\bar{\omega}W)+\bar{\omega}W}{(1-\bar{\omega}W)^2}-\frac{\lambda\omega\varphi'(\omega)}{\omega-W}+\frac{\overline{\lambda\omega^2\varphi'(\omega)}}{1-\bar{\omega}W}\\ =\frac{W\varphi'^2(W)}{\varphi(W)}\Bigg[-\frac{\varphi(W)}{(\varphi(W)-\varphi(\omega))^2}-\frac{\overline{\varphi(\omega)}(2-\overline{\varphi(\omega)}\varphi(W))}{(1-\overline{\varphi(\omega)}\varphi(W))^2}\\ +\frac{\lambda\varphi(\omega)}{\varphi(W)-\varphi(\omega)}+\frac{\bar{\lambda}\,\overline{\varphi^2(\omega)}}{1-\overline{\varphi(\omega)}\varphi(W)}\Bigg]\,,\end{aligned} \tag{27}$$

where

$$\alpha=\omega\varphi''(\omega)/\varphi'(\omega)=-f(\zeta)f''(\zeta)\,.$$

We now prove the following.

LEMMA. *For the extremum functions of the class S which satisfy the equation* (27), *we have*

$$\Im\{1+\omega\varphi''(\omega)/\varphi'(\omega)-\lambda\omega\varphi'(\omega)\}=0 \tag{28}$$

and

$$\Im(\lambda\varphi(\omega))=0\,. \tag{29}$$

Proof. Let us consider the variation

$$(30)\qquad W^* = W + \frac{a_0\rho^2 W}{W - W_0} - \frac{\bar{a}_0\rho^2 W^2}{1 - W_0 W} + \frac{a_1\rho^2 W}{W - W_1} - \frac{\bar{a}_1\rho^2 W^2}{1 - W_1 W} + iT\rho^2 W + O(\rho^4)\,,$$

where T is real. It is easily seen that this variation keeps the origin and the unit circumference fixed and, for small enough ρ, is univalent on the boundary. So this is an acceptable variation. Under this variation the variation formula (17) will have the additional term $iT\rho^2 W\varphi'(W)$ on the right-hand side. This will give rise to an additional term $iT\rho^2\omega\varphi'(\omega)$ in (19), and to $iT\{1+\omega\varphi''(\omega)/\varphi'(\omega)\}$ in (21). Then, because (26) holds, the equation corresponding to (26) in this case will give rise to (28).

To prove (29) we observe that the derivation of the variational equation (17) leaves an arbitrariness which permits us to add a term $ik\varphi(W)$, for k real, to the right hand side of (17). The addition of this term does affect the extremum condition (21), but it does appear as $ik\varphi(\omega)$ in the equation (20). The equation corresponding to (26) will then have an extra term $ik(\lambda\varphi(\omega) - \lambda\varphi(\omega)$, which must vanish since (26) has been proved to be the equation for the extremum function.

Transforming (27) in terms of $f(z)$ and using (28) and (29) we find that the extremum function statisfies the differential equation

$$(31)\qquad \frac{f'^2(z)(f(z)-\alpha_1)(f(z)-\alpha_2)}{f(z)(f(z)-\omega)^2(1-\bar{\omega}f(z))^2} = D\frac{(z-\beta_1)(z-\beta_2)}{z(z-\zeta)^2(1-\bar{\zeta}z)^2}\,,$$

where the constants α_1, α_2, D, β_1 and β_2 are obtained from (27) in the following form:

$$(32)\qquad D = \frac{\bar{\zeta}(-2|\zeta|^2 + \lambda\zeta(|\zeta|^2-1))}{\bar{\omega}(\beta(1-|\omega|^2)-(1+|\omega|^2))}\,,$$

$$(33)\qquad f^2(z) + \frac{f(z)}{\bar{\omega}}\,\frac{4|\omega|^2+\beta(-1+|\omega|^4)}{\beta(1-|\omega|^2)-(1+|\omega|^2)} + \frac{\omega}{\bar{\omega}} \equiv (f(z)-\alpha_1)(f(z)-\alpha_2)\,,$$

$$(34)\qquad z^2 + \frac{z}{\zeta}\,\frac{-1+4|\zeta|^2+|\zeta|^4+\lambda\zeta(1-|\zeta|^4)}{-2|\zeta|^2+A(|\zeta|^2-1)} + \frac{\zeta}{\bar{\zeta}} \equiv (z-\beta_1)(z-\beta_2)\,,$$

and

$$\beta = 1 + \omega\varphi''(\omega)/\varphi'(\omega) - \lambda\omega\varphi'(\omega)\,.$$

One further finds from (33) and (34) that $|\alpha_1\alpha_2| = 1$ and $|\beta_1\beta_2| = 1$. In order to fix λ which remains arbitrary, as yet, we need the geometry of the extremum domain. In particular we prove the following.

THEOREM. *If $f(z)$ is a function of the class S for which $|f'(\zeta)|$ is* *either a maximum or a minimum, then $f(z)$ maps the unit circle $|z|<1$ onto a slit domain.*

Proof. If the theorem were not true, then there would exist a point W_0, $|W_0|<1$ such that a neighborhood of W_0 is contained in $|W|<1$ and does not belong to the image domain. In the variation (16) taking W_0 and W_1 to be two such points we get the following variation formula for $f(z)$:

$$f^*(z)=f(z)+\rho^2\sum_{\nu=0}^{1}\left[\frac{a_\nu f(z)}{f(z)-f(z_\nu)}-\frac{\bar{a}_\nu f^2(z)}{1-f(z)f(z_\nu)}\right]+O(\rho^4)\,. \tag{35}$$

The requirement that for all $f(z)$, $f(\zeta)=f(\zeta)=\omega$, yields that

$$\sum_{\nu=0}^{1}\left[\frac{a_\nu\omega}{\omega-f(z_\nu)}-\frac{\bar{a}_\nu\omega^2}{1-\omega f(z_\nu)}\right]+O(\rho^2)=0\,, \tag{36}$$

and the condition for the extremum of $|f'(\zeta)|$ leads to

$$\Re\left\{\sum_{\nu=0}^{1}\left[\frac{-a_\nu f(z_\nu)}{(\omega-f(z_\nu))^2}-\frac{\bar{a}_\nu(2-\omega f(z_\nu))}{(1-\omega f(z_\nu))^2}\right]\right\}+O(\rho^2)=0\,. \tag{37}$$

Thus, because

$$\left|\frac{1}{\omega-f(z_\nu)}\right|^2>\left|\frac{\omega}{1-\omega f(z_\nu)}\right|^2,$$

we see that the extremal function satisfies the equation

$$\frac{f(z)}{(\omega-f(z))^2}+\frac{\bar{\omega}(2-\bar{\omega}f(z))}{(1-\bar{\omega}f(z))^2}=\frac{\lambda\omega}{\omega-f(z)}-\frac{\bar{\lambda}\,\bar{\omega}^2}{(1-\bar{\omega}f(z))^2}\,. \tag{38}$$

But this is impossible since the left hand side has a second order pole at $z=\zeta$, where as the right hand side has only a first order pole.

As a consequence of this theorem it follows that (24) can always be assumed to hold. Indeed, if it were not so, we could find no point z_1 in $|z|\leq 1$ such that $|A_1|\neq|B_1|$. Hence, because A_1 and $\bar{B}_1$ are analytic functions of z_1 and as equality is to hold for all z_1, we have

$$A(\omega,f(z))=\mu\overline{B(\omega,f(z))}\,,\qquad |\mu|=1\,. \tag{39}$$

(39) gives the following differential equation for $f(z)$:

$$\frac{f'^2(z)[(\omega f'^2(\zeta)-\mu\bar{\omega}^2f'(\zeta))-f(z)(|\omega|^2\bar{f}'^2(\zeta)-\mu\bar{\omega}^2f'(\zeta))]}{f'^2(\zeta)f(\zeta)f(z)(f(z)-\omega)(1-\bar{\omega}f(z))}=\frac{\zeta(1-\mu\bar{\zeta}^2)-z(|\zeta|^2-\mu\bar{\zeta}^2)}{z(z-\zeta)(1-\zeta z)}\,. \tag{40}$$

But the function given by the differential equation (40) does not map the unit circle onto a slit domain, because at one end of the slit $f'(z)$ will have a first order zero. Hence the right hand side of (40) should have a second order zero on the unit circumference. Since this is obviously not so, we have shown that (24) can always be taken to be valid.

We have thus shown that all extremal functions $f(z)$ which belong to S, and for which $|f'(\zeta)|$ is a maximum or minimum, satisfy the differential equation (31). As the extremal function $f(z)$ maps $|z| \leq 1$ onto a slit domain, at one end of the slit $f'(z)$ will have a first order zero. To this zero of $f'(z)$ there need to be a corresponding zero on the right-hand side of (31), and as it is on the unit circumference $|z|=1$, we must have $\beta_1=\beta_2=e^{i\varphi}$ in (31). Further, because the slit will make an angle θ with the unit circle such that $|\theta|<\pi$, we get from simple geometric considerations and the fact that the right hand side of (31) has no pole at any point on the unit circumference that $\alpha_1=\alpha_2=e^{i\theta}$. Geometrically this means that the slit starts from the unit circumference $|W|=1$, making an angle $\pi/2$ with it.

As a result of the equality of α_1, α_2 and β_1, β_2, we have from (33) and (34) that

$$f(\zeta)f''(\zeta)-\lambda\frac{f(\zeta)}{f'(\zeta)}=\pm\frac{2|\omega|}{1-|\omega|^2}-1\,, \tag{41}$$

and

$$\lambda\zeta=1\mp\frac{2|\zeta|}{1-|\zeta|^2}\,. \tag{42}$$

Eliminating λ from both these equations one finds that, at the fixed point ζ, the extremum function satisfies the equation

$$\zeta f''(\zeta)f'(\zeta)-\left(1\mp\frac{2|\zeta|}{1-|\zeta|^2}\right)=\frac{\zeta f'(\zeta)}{f(\zeta)}\left[-1\pm\frac{2|f(\zeta)|}{1-|f(\zeta)|^2}\right]. \tag{43}$$

The differential equation (31) now reduces to

$$\frac{f'^2(z)(f(z)\bar{\omega}\pm|\omega|)^2}{f(z)(f(z)-\omega)^2(1-\bar{\omega}f(z))^2}\,\frac{(1\mp|\omega|)^2}{\bar{\omega}}=\frac{(1\mp|\zeta|)^2}{\zeta}\,\frac{(z\bar{\zeta}\mp|\zeta|)^2}{z(z-\zeta)^2(1-\bar{\zeta}z)^2}\,, \tag{44}$$

where on each side either the upper or the lower sign is to be taken at one time.

From (44) one can get the information regarding the nature of the extremum domain. On account of the slit character of the extremum domain, the unit circumference $|W|=|f(z)|=1$ is definitely a part of the boundary. Further, if $z=e^{i\theta}$ we get from (44) that

$$\frac{z^2 f'^2(z)(f(z)\bar{\omega} \pm |\omega|)^2}{f(z)(f(z)-\omega)^2(1-\bar{\omega} f(z))^2} \frac{(1 \mp |\omega|)^2}{\bar{\omega}}$$

is real. Hence, writing $W(t)=f(z)$ and making a proper choice of the parameter t, we can put it in the form

$$\frac{W'^2}{W} \frac{(W\bar{\omega} \pm |\omega|)^2}{(W-\omega)^2(1-\bar{\omega} W)^2} \frac{(1 \mp |\omega|)^2}{\bar{\omega}} = C, \tag{45}$$

C being some real constant. We now observe that this is an ordinary differential equation of the first order and hence has only one solution. Further, the straight line $W=r|\omega|/\bar{\omega}$, where r is a real parameter, does satisfy the differential equation. Since there is only one slit, this line corresponds to the slit and we conclude that the boundary of the image domain consists of the unit circumference and a radial slit pointing inwards at the points $\pm|\omega|/\bar{\omega}$.

Taking the square root and integrating (44) we obtain

$$\left[\pm \log \frac{\sqrt{f(z)}-\sqrt{\omega}}{\sqrt{f(z)}+\sqrt{\omega}} + \log \frac{1+\sqrt{\bar{\omega} f(z)}}{1-\sqrt{\bar{\omega} f(z)}}\right] + \text{const.} \tag{46}$$

$$= \pm\left[\pm \log \frac{\sqrt{z}-\sqrt{\zeta}}{\sqrt{z}-\sqrt{\zeta}} + \log \frac{1+\sqrt{\zeta z}}{1-\sqrt{\zeta z}}\right].$$

The various possibilities arising from different combinations of signs on the two sides of (46) are to be taken in such a way that the singularities at $f(z)=\omega$ and $z=\zeta$ on the two sides of (46) balance each other. We are thus left with only four possible combinations which after some simple algebra give rise to the following equations for the extremal functions:

$$\frac{(1-|\omega|^2)\omega f(z)}{(\omega-|\omega| f(z))^2} = \frac{(1 \mp |\zeta|)^2 z\zeta}{(\zeta \mp |\zeta| z)^2}, \tag{47}$$

and

$$\frac{(1+|\omega|)^2 \omega f(z)}{(\omega+|\omega| f(z))^2} = \frac{(1 \pm |\zeta|)^2 z\zeta}{(\zeta \pm |\zeta| z)^2}. \tag{48}$$

Equations (47) and (48), respectively, give rise to the following values of $f'(\zeta)$:

$$f'(\zeta) = \frac{\omega}{\zeta} \frac{1 \pm |\zeta|}{1 \mp |\zeta|} \frac{1-|\omega|}{1+|\omega|}, \tag{49}$$

and

(50)
$$f'(\zeta)=\frac{\omega}{\zeta}\frac{1\mp|\zeta|}{1\pm|\zeta|}\frac{1+|\omega|}{1-|\omega|}.$$

As $|\omega|\leqq|\zeta|$, one easily sees that the maximum value of $|f'(\zeta)|$ is given by

(51)
$$f'(\zeta)=\frac{\omega}{\zeta}\frac{1+|\zeta|}{1-|\zeta|}\frac{1+|\omega|}{1-|\omega|},$$

and that the function $f(z)$ corresponding to it is given by

(52)
$$\frac{(1+|\omega|)^2\omega f(z)}{(\omega+|\omega|f(z))^2}=\frac{(1-|\zeta|)^2 z\zeta}{(\zeta-|\zeta|z)^2}.$$

Also, the minimum of $|f'(\zeta)|$ is given by

(53)
$$f'(\zeta)=\frac{\omega}{\zeta}\frac{1-|\zeta|}{1+|\zeta|}\frac{1-|\omega|}{1+|\omega|},$$

and the function corresponding to it is given by

(54)
$$\frac{(1-|\omega|)^2\omega f(z)}{(\omega-|\omega|f(z))^2}=\frac{(1+|\zeta|)^2 z\zeta}{(\zeta+|\zeta|z)^2}.$$

We have thus proven the following.

THEOREM. *Let S denote the family of regular univalent functions $f(z)$ defined in the unit circle $|z|\leqq 1$ such that $|f(z)|\leqq 1$, $f(0)=0$ and, for some fixed point ζ in $|z|<1$, $f(\zeta)=\omega$. Then the maximum and minimum values of $|f'(\zeta)|$ are given by*

(55)
$$f'(\zeta)=\frac{\omega}{\zeta}\frac{1\pm|\zeta|}{1\mp|\zeta|}\frac{1\pm|\omega|}{1\mp|\omega|},$$

and the corresponding maximizing and minimizing functions are, respectively, given by the equation

(56)
$$\frac{(1\pm|\omega|)^2\omega f(z)}{(\omega\pm|\omega|f(z))^2}=\frac{(1\mp|\zeta|)^2 z\zeta}{(\zeta\mp|\zeta|z)^2},$$

where the upper signs on both sides give the maximal function and the lower signs on both sides given the minimal function. The boundaries of the maximal and the minimal domains consist of the unit circumference together with radial slits starting, respectively, at the points $\pm|\omega|/\bar{\omega}$, the end points of the slits being the images of the points $\mp|\zeta|/\bar{\zeta}$ by the corresponding functions given by (56).

We now remark that we could as well have tried to solve the following problem:

In the class S of regular univalent functions $f(z)$ in $|z| \leqq 1$ statisfying the normalization $f(0)=0$, $|f(z)| \leqq 1$ and, $f(\zeta)=\omega$ at some fixed point ζ in $|z|<1$, to find the function which maximizes $|f(\eta)|$ at some $\eta \neq \zeta$ in $|z|<1$.

The existence and uniqueness of the solution is easily proven. It can also be readily shown that the extremal domain will be a slit damain. The variational equation for the extremum as obtained from (17) can be written as

$$\Re\left\{\left[\frac{a_0 A(\omega_1, f(z_0))}{\omega_1} - \frac{\bar{a}_0 B(\omega_1, f(z_0))}{\omega_1} + \frac{a_1 A(\omega_1, f(z_1))}{\omega_1} - \frac{\bar{a}_1 B(\omega_1, \overline{f(z_1)})}{\omega_1}\right]\right\}=0, \tag{57}$$

where $\omega_1=f(\eta)$, and we have replaced $\varphi(\omega)$ by ζ and W_0 by $f(z_0)$, $\varphi(W_0)=z_0$, $\varphi'(W_0)$ by $1/f'(z_0)$, and similarly for W_1.

By arguments similar to those which lead to the equation (26) we can again assert that if

$$C_0=\frac{A(\omega_1, f(z_0))}{\omega_1} - \frac{A(\bar{\omega}_1, f(z_0))}{\bar{\omega}_1}, \tag{58}$$

then the extremum function will satisfy the equation

$$C_0=\mu A(\omega, f(z_0)) - \bar{\mu}\overline{B(\omega, f(z_0))}, \tag{59}$$

where μ is independent of $f(z_0)$.

As in the lemma we can again show that $\mu f(\zeta)/f'(\zeta)$ and $\mu\zeta$ are real. The differential equation for the extremum function can now be written as

$$\frac{f'^2(z)(f(z)-c_1)(f(z)-d_1)}{f(z)(f(z)-\omega_1)(f(z)-\omega)(1-\bar{\omega}f(z))(1-\bar{\omega}_1 f(z))} = K\frac{(z-e_1)(z-e_2)}{z(z-\eta)(z-\zeta)(1-\zeta z)(1-\bar{\eta}z)}, \tag{60}$$

where $|c_1 d_1|=1$, and $|e_1 e_2|=1$ and c_1, d_1, e_1, e_2 and K can be determined by a comparison with (59).

From geometric considerations and the fact that the extremal domain will be a slight domain, one easily deduces that $c_1=d_1=e^{i\theta}$ and $e_1=e_2=e^{i\gamma}$. These conditions lead to

(61) $[(1-|\omega_1|^2)(1+|\omega|^2)-\mu_1(1-|\omega|^2)(1+|\omega_1|^2)]^2=4|(1-|\omega_1|^2)\omega-\mu_1(1-|\omega|^2)\omega_1|^2,$

and

(62) $[(1-|\eta|^2)(1+|\zeta|^2)\alpha_1-\mu\zeta(1-|\zeta|^2)(1+|\eta|^2)]^2=4|(1-|\eta|^2)\zeta\alpha_1-\mu\zeta\eta(1-|\zeta|^2)|^2,$

where

$$\mu_1=\mu f(\zeta)/f'(\zeta)$$

and

$$\alpha_1=\eta f'(\eta)/f(\eta)\,.$$

Eliminating μ from (61) and (62), we get

(63) $$\frac{\zeta f'(\zeta)}{f(\zeta)}\frac{[|\omega-\omega_1|\pm|1-\bar{\omega}\omega_1|]^2}{(1-|\omega|^2)(1-|\omega_1|^2)}=\frac{\eta f'(\eta)}{f(\eta)}\frac{[|\zeta-\eta|\pm|1-\bar{\zeta}\eta|]^2}{(1-|\zeta|^2)(1-\eta|^2)}\,.$$

From the slit character of the extremal domain and geometric considerations we prove that the boundary of the extremal domain consists of the unit circumference with a slit that starts at right angles to the circumference. But we can no longer claim that this slit is radial. Also, because the integration of (60) involves hyperelliptic integrals it is not possible to get analytically any further information about the nature of the image domain. However, if one could show that the image domain is symmetric one could obtain an explicit result at least when ζ and η are real. We are thus lead to reformulate the problem for bounded symmetric univalent functions.

4. **Symmetrtc bounded univalent functions.** We now want to construct a variation which keeps the unit circumference and the real axis fixed and which maps the origin into the origin. Evidently such a variation will be a combination of the variations considered in §§ 2 and 3. One easily deduces that any such variation will be of the form

(64) $$W_1^*=W+\frac{a_0\rho^2W}{W-W_0}-\frac{\bar{a}_0\rho^2W^2}{1-\bar{W}_0W}+\frac{\bar{a}_0\rho^2W}{W-\bar{W}_0}-\frac{a_0\rho^2W^2}{1-W_0W}+O(\rho^4)\,,$$

where $O(\rho^4)$ can be suitably chosen.

In order to be able to get a variation which can take account of some side conditions we need to take a linear combination of the variational terms in (64) with different a_0 and W_0. Thus, we get the variation

(65) $$W^*=W+\sum_{\nu=0}^{n}\left[\frac{a_\nu\rho^2W(1-W^2)}{(W-W_\nu)(1-W_\nu W)}+\frac{\bar{a}_\nu\rho^2W(1-W^2)}{(W-\bar{W}_\nu)(1-\bar{W}_\nu W)}\right]+O(\rho^4)\,.$$

The variation formula for $f(z)$ in this case is

$$(66)\quad f^*(z)=f(z)+\rho^2\sum_{\nu=0}^{n}\Big[\frac{a_\nu f(z)(1-f^2(z))}{(f(z)-f(z_\nu))(1-f(z)f(z_\nu))}$$

$$+\frac{\bar{a}_\nu f(z)(1-f^2(z))}{(f(z)-f(z_\nu))(1-f(z)f(z_\nu))}+\frac{a_\nu f'(z)z(1-z^2)}{(z_\nu-z)(1-z_\nu z)}\frac{f(z_\nu)}{z_\nu f'^2(z_\nu)}$$

$$+\frac{\bar{a}_\nu f'(z)z(1-z^2)}{(1-\bar{z}_\nu z)(\bar{z}_\nu-z)}\frac{f(z_\nu)}{\bar{z}_\nu f'^2(z_\nu)}\Big]+O(\rho^4)\,.$$

If we require $f^*(\zeta)=f(\zeta)=\omega$, ζ real, we get, using that $f(z)$ is symmetric,

$$(67)\quad \Re\Big[\sum a_\nu\Big\{\frac{\omega(1-\omega^2)}{(\omega-f(z_\nu))(1-\omega f(z_\nu))}-\frac{\zeta(1-\zeta^2)f'(\zeta)}{(\zeta-z_\nu)(1-\zeta z_\nu)}\frac{f(z_\nu)}{z_\nu f'^2(z_\nu)}\Big\}\Big]+O(\rho^2)=0\,.$$

Also, if η is real, then the condition for the extremum of $f(\eta)=\omega_1$ is

$$(68)\quad \Re\Big[\sum a_\nu\Big\{\frac{\omega_1(1-\omega_1^2)}{(\omega_1-f(z_\nu))(1-\omega_1 f(z_\nu))}-\frac{\eta(1-\eta^2)f'(\eta)}{(\eta-z_\nu)(1-\eta z_\nu)}\frac{f(z_\nu)}{z_\nu f'^2(z_\nu)}\Big\}\Big]+O(\rho^2)=0\,.$$

Thus the extremum function satisfies the equations (67) and (68). By Lagrange's method of multipliers we see that the equation satisfied by the extremum function is

$$(69)\quad \frac{zf'^2(z)}{f(z)}\Big[\frac{(1-\omega_1^2)}{(\omega_1-f(z))(1-\omega_1 f(z))}-\frac{\lambda(1-\omega^2)}{(\omega-f(z))(1-\omega f(z))}\Big]$$

$$=\frac{\alpha(1-\eta^2)}{(\eta-z)(1-\eta z)}-\frac{\alpha_1\lambda(1-\zeta^2)}{(\zeta-z)(1-\zeta z)}\,,$$

where

$$\alpha=\eta f'(\eta)/f(\eta)$$

and

$$\alpha_1=\zeta f'/(\zeta)'f(\zeta)\,.$$

It is easily proven in this case that the image domain is a slit domain. Thus, as in § 3, from geometric considerations and the fact that the image domain is a slit domain we conclude that numerators on both the sides of (69) should be perfect squares. We thus have that either

$$(70)\quad \alpha(1+\eta)(1-\zeta)=\alpha_1\lambda(1+\zeta)(1-\eta)\,,$$

or

$$\alpha(1-\eta)(1+\zeta)=\alpha_1\lambda(1-\zeta)(1+\eta)\,; \tag{71}$$

and either

$$(1+\omega_1)(1-\omega)=\lambda(1+\omega)(1-\omega_1) \tag{72}$$

or

$$(1-\omega_1)(1+\omega)=\lambda(1-\omega)(1+\omega_1)\,. \tag{73}$$

The differential equation finally reduces to the form

$$\frac{(1\pm\omega_1)(\omega-\omega_1)(1-\omega\omega_1)f'^2(z)(f(z)\mp 1)^2}{(1\mp\omega_1 f(z))(\omega-f(z))(\omega_1-f(z))(1-\omega f(z))(1-\omega_1 f(z))} = \frac{\alpha(1\pm\eta)(\zeta-\eta)(1-\zeta\eta)(z\mp 1)^2}{(1\mp\eta)z(\zeta-z)(\eta-z)(1-\zeta z)(1-\eta z)}\,, \tag{74}$$

where the upper or the lower sign on each side is to be taken at one time.

The alternative in (70), (71) and (72), (73) arises on account of the ambiguity of sign of the root in (74).

In the left hand side of (74) let us make the transformation

$$f(z)=\mp\frac{W+1}{W-1}\,, \tag{75}$$

according as we take the upper or the lower sign in (74). Then the left hand side transforms either to

$$-\frac{16W'^2W^2(1+\omega_1)(\omega-\omega_1)(1-\omega\omega_1)}{(1+\omega)^2(1+\omega_1)^2(W^2-1)(W^2-\beta_1^2)(W^2-\beta_2^2)}\,, \tag{76}$$

or

$$\frac{16W'^2W^2(1-\omega_1)(\omega-\omega_1)(1-\omega\omega_1)}{(1-\omega)^2(1-\omega_1)^2(W^2-1)(W^2-\beta_1^{-2})(W^2-\beta_2^{-2})}\,, \tag{77}$$

respectively, where $\beta_1=(1+\omega)/(1-\omega)$ and $\beta_2=(1+\omega_1)/(1-\omega_1)$.

Similarly, making the transformation

$$z=\mp\frac{y+1}{y-1}\,, \tag{78}$$

according as we take the upper or the lower sign in the right hand side of (74), we get either

(79) $$-\frac{16\alpha y'^2 y^2(1+\eta)(1-\zeta\eta)(\zeta-\eta)}{(y^2-1)(y^2-\gamma_1^2)(y^2-\gamma_2^2)(1-\eta)},$$

or

(80) $$-\frac{16\alpha y'^2 y^2(1-\eta)(1-\zeta\eta)(\zeta-\eta)}{(y^2-1)(y^2-\gamma_1^{-2})(y^2-\gamma_2^{-2})(1+\eta)},$$

respectively, where $\gamma_1=(1+\zeta)/(1-\zeta)$ and $\gamma_2=(1+\eta)/(1-\eta)$. We note that (77) is obtained from (76) by changing the signs of ω and ω_1, and similarly (80) is obtained from (79) by changing the signs of ζ and η. Thus it is enough to consider the cases

(81) $$c_1\frac{W'W}{\sqrt{(W^2-1)(W^2-\beta_1^2)(W^2-\beta_2^2)}}=\pm c_2\frac{y'y}{\sqrt{(y^2-1)(y^2-\gamma_1^2)(y^2-\gamma_2^2)}},$$

where

$$c_2=\left[\frac{\alpha(1+\eta)(\zeta-\eta)(1-\zeta\eta)}{(1-\eta)(1+\zeta)^2(1+\eta)^2}\right]^{1/2},$$

and

$$c_1=\left[\frac{(1+\omega_1)(\omega-\omega_1)(1-\omega\omega_1)}{(1+\omega)^2(1+\omega_1)^2}\right]^{1/2}.$$

Putting

$$W_2=W^2-\frac{1}{3}(1+\beta_1^2+\beta_2^2),$$

and

$$X=y^2-\frac{1}{3}(1+\gamma_1^2+\gamma_2^2),$$

we have from (81), on integration,

$$c_1 v=c_2 u+\text{const.},$$

where $p(v)=W_2$ and $p^*(u)=X$, p and p^* being the Weierstrass's p-functions.

Since $f(0)=0$, $f(\zeta)=\omega$ and $f(\eta)=\omega_1$, we get, using the periodicity and homogeneity property of the p-functions,

(82) $$W_2=\frac{c_1^2}{c_2^2}X.$$

Transforming back to z and $f(z)$, we can write (82) in the form

$$(83)\qquad \left[\frac{f(z)-1}{f(z)+1}\right]^2-\frac{1}{3}\left\{1+\left(\frac{1-\omega}{1+\omega}\right)^2+\left(\frac{1-\omega_1}{1+\omega_1}\right)^2\right\}$$

$$=\frac{c_1^2}{c_2^2}\left[\left(\frac{z-1}{z+1}\right)^2-\frac{1}{3}\left\{1+\left(\frac{1-\zeta}{1+\zeta}\right)^2+\left(\frac{1-\eta}{1+\eta}\right)^2\right\}\right].$$

Since $f(0)=0$ we have from (83) that

$$(84)\qquad 1-\frac{1}{3}\left\{1+\left(\frac{1-\omega}{1+\omega}\right)^2+\left(\frac{1-\omega_1}{1+\omega_1}\right)^2\right\}=\frac{c_1^2}{c_2^2}\left[1-\frac{1}{3}\left\{1+\left(\frac{1-\zeta}{1+\zeta}\right)^2+\left(\frac{1-\eta}{1+\eta}\right)^2\right\}\right].$$

This gives us ω_1, but it involves α which is not yet known in terms of ζ, ω and η. Towards this we observe that on subtracting (84) from (83) we have

$$\frac{4f(z)}{(f(z)+1)^2}=\frac{c_1^2}{c_2^2}\frac{4z}{(1+z)^2},$$

and because $f(\zeta)=\omega$ we have

$$(85)\qquad \frac{\omega}{(1+\omega)^2}=\frac{c_1^2}{c_2^2}\frac{4\zeta}{(1+\zeta)^2},$$

and finally

$$(86)\qquad \frac{f(z)(1+\omega)^2}{\omega(1+f(z))^2}=\frac{z(1+\zeta)^2}{(1+z)^2\zeta}.$$

Now, putting $z=\eta$ and $f(\eta)=\omega_1$, we get for ω_1 the required equation

$$(87)\qquad \frac{(1+\omega)^2\omega_1}{\omega(1+\omega_1)^2}=\frac{\eta}{(1+\eta)^2}\frac{(1+\zeta)^2}{\zeta}.$$

Observing that all the possible extremal functions could be obtained by changing the signs of ω, ω_1, ζ and η and taking all the combinations, we see that

$$(88)\qquad \frac{(1\pm\omega)^2}{\omega}\frac{f(z)}{(1\pm f(z))^2}=\pm\frac{z}{(1\pm z)^2}\frac{(1\pm\zeta)^2}{\zeta},$$

gives all the extremal functions and

$$(89)\qquad \frac{(1\pm\omega)^2\omega_1}{\omega(1\pm\omega_1)^2}=\pm\frac{\eta}{(1\pm\eta)^2}\frac{(1\pm\zeta)^2}{\zeta},$$

the corresponding values of ω_1, where at one time either the upper or the lower sign is to be taken on each side of (88) and (89).

Further, on account of the continuity and univalence of $f(z)$ on the real axis, $f(z)$ will have the same sign or different sign as z according as ω has the same or different sign as ζ. Thus (88) and (89), respectively, reduce to

$$\frac{(1\pm\omega)^2}{\omega}\frac{f(z)}{(1\pm f(z))^2}=\frac{z}{(1\pm z)^2}\frac{(1\pm\zeta)^2}{\zeta}, \tag{90}$$

and

$$\frac{(1\pm\omega)^2}{\omega}\frac{\omega_1}{(1\pm\omega_1)^2}=\frac{\eta}{(1\pm\eta)^2}\frac{(1\pm\zeta)^2}{\zeta}. \tag{91}$$

The following different cases need to be considered: (i) $\omega>0$, $\zeta>0$ and $\eta>0$ (ii) $\omega>0$, $\zeta>0$ and $\eta<0$ (iii) $\omega>0$, $\zeta<0$ and $\eta>0$ and (iv) $\omega>0$, $\zeta<0$ and $\eta<0$. We observe that (ii) can be easily deduced from (i) for in this case $\omega_1<0$, and the maximum and the minimum of ω_1 in this case will be the same as the minimum and maximum of ω_1 in (i). A similar relationship exists between (iii) and (iv). So we need to consider only the two cases (i) and (iii).

We now observe that if $|x|<1$ and x is real then $x+1/x$ is a monotonic decreasing function of x, and also that

$$\frac{\eta+1/\eta-2}{\zeta+1/\zeta-2}\lessgtr\frac{\eta+1/\eta+2}{\zeta+1/\zeta+2}, \tag{92}$$

according as $\eta\gtrless\zeta>0$.

With these considerations we can prove the following.

THEOREM. *Let Σ be the class of bounded, symmetric univalent functions $f(z)$ which are normalized so that $|f(z)|\leq 1$, $|z|\leq 1$, $f(0)=0$ and $f(\zeta)=\omega$ where ζ is a fixed real point in $|z|<1$. Further let $\omega>0$, $\zeta>0$ and for some real point η let $f(\eta)=\omega_1$. Then the maximum and the minimum values of ω_1, when $\eta>\zeta$, are given by*

$$\frac{\omega_1+1/\omega_1\pm2}{\omega+1/\omega\pm2}=\frac{\eta+1/\eta\mp2}{\zeta+1/\zeta\mp2}, \tag{93}$$

and the corresponding maximizing and minimizing functions are respectively given by

$$\frac{(1\pm\omega)^2}{\omega}\frac{f(z)}{(1\pm f(z))^2}=\frac{z}{(1+z)^2}\frac{(1\mp\zeta)^2}{\zeta}, \tag{94}$$

where the upper signs on both sides give the maximum and the lower signs the minimum. However, if $\eta<\zeta$ then the maximum and minimum values given by (93) *and the corresponding function given by* (94) *are interchanged.*

If $\omega < 0$, $\zeta < 0$ *and* $\eta > 0$ *then the maximum and the minimum values of* ω_1 *are given by*

$$\frac{\omega_1 + 1/\omega_1 \pm 2}{\omega + 1/\omega \pm 2} = \frac{\eta + 1/\eta \pm 2}{\zeta + 1/\zeta \pm 2}, \tag{95}$$

and the corresponding maximizing and minimizing functions are given by

$$\frac{(1 \pm \omega)^2}{\omega} \frac{f(z)}{(1 \pm f(z))^2} = \frac{z}{(1 \pm z)^2} \frac{(1 \pm \zeta)^2}{\zeta}, \tag{96}$$

where, as before, the upper sign on both sides gives the maximum and the lower sign gives the minimum.

The boundary of the extremal domain in each case consists of the unit circumference with a radial slit starting either at $W = 1$ *or* $W = -1$. *The length of the slit differs in various cases.*

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ON GENERALIZED EUCLIDEAN AND NON-EUCLIDEAN SPACES

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Introduction. The present paper develops necessary and sufficient conditions that a complete, convex, metric space with extendible segments shall be generalized euclidean, r-hyperbolic, r-spherical, or r-elliptic. Blumenthal and others have given four-point conditions which characterize these generalized spaces among certain classes of spaces, and the results of this paper follow the general plan of these earlier works.

1. Definitions, notation and previous results. Unless otherwise noted all terms used have the same meanings as those given in [1]. The distance between two points p and q of a semi-metric space is denoted by pq, a point s distinct from p and from q is *between p and q*, denoted by psq, provided $ps+sq=pq$, and a triple of points (not necessarily distinct) is a *mid-point triple*, denoted by (psq), provided $ps=sq=pq/2$. A metric space is said to be generalized {euclidean, r-hyperbolic, r-spherical, r-elliptic} provided each of its n-dimensional subspaces is congruent with $\{E_n, H_{n,r}, S_{n,r}, \mathscr{E}_{n,r}\}$, where these four symbols represent n-dimensional euclidean, hyperbolic, spherical, elliptic space respectively, the last three of space constant $r>0$. A metric space is said to have the weak {euclidean, r-hyperbolic, r-spherical, r-elliptic} four-point property provided each of its quadruples containing a triple of points congruent to three points of $\{E_1, H_{1,r}, S_{1,r}, \mathscr{E}_{1,r}\}$ is itself congruent to four points of $\{E_2, H_{2,r}, S_{2,r}, \mathscr{E}_{2,r}\}$. A space has the feeble {euclidean, r-hyperbolic, r-spherical, r-elliptic} four-point property provided each quadruple containing a mid-point triple is congruently imbeddable in $\{E_2, H_{2,r}, S_{2,r}, \mathscr{E}_{2,r}\}$. The weak property obviously implies the feeble property.

THEOREM 1 (Blumenthal [2]). *A complete, convex, externally convex metric space is generalized euclidean if and only if it has the feeble euclidean four-point property.*

Defining a conjugate space as one with finite metric diameter $\delta>0$ and having the further property that corresponding to each pair of points p, q of the space with $0<pq<\delta$ there exist points p^*, q^* of the space with pqp^*, qpq^*, and $pp^*=qq^*=\delta$ all holding, Hankins [4] has shown the following.

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THEOREM 2. *If a complete, convex, conjugate, metric space M has diameter $\pi r/2$, $r>0$, and if M possesses the feeble r-elliptic four-point property, then M is generalized r-elliptic.*

2. Metric characterizations. Throughout the remainder of the paper Σ will denote a space which satisfies:

(i) Σ is metric,

(ii) Σ is complete,

(iii) Σ is metrically convex,

(iv) if $T_{p,q}$ is a segment with end points p, q, there exists $\delta(p)>0$ such that if $s\in T_{p,q}$ with $0<ps<\delta$, then there exists a point $t\in\Sigma$ with (spt) holding.

LEMMA 1. *If Σ has the feeble euclidean, r-hyperbolic, r-elliptic, or r-spherical four point property, and if (pqs), (pqt), $qs=qt$, then $s=t$.*

Proof. Let R represent any one of the spaces E_2, $H_{2,r}$, $S_{2,r}$, $\mathscr{E}_{2,r}$. Then $p, q, s, t\approx p_1, q_1, s_1, t_1\in R$ and $(p_1q_1s_1)$, $(p_1q_1t_1)$, $q_1s_1=q_1t_1$ imply that $s_1=t_1$, so that $s=t$.

REMARK. If in condition (iv) on Σ the quantity $\delta(p)$ is unbounded for all $p\in\Sigma$, then Σ is externally convex.

THEOREM 1. *If Σ is externally convex then each two points of Σ lie on a unique metric line if and only if pqs, pqt, and $ps=pt$ imply $s=t$.*

Proof. The necessity is obvious. The sufficiency is proved by noting that each two points are joined by at least one metric line. Then if there are two distinct segments joining p and q, each may be prolonged beyond q along the same segment $T_{q,s}$ to a point s, but this implies that $T_{q,s}$ may be prolonged in two distinct ways beyond q to p, contrary to hypotheses. Thus p and q must determine a unique segment, and this segment can be prolonged to a metric line in exactly one way.

THEOREM 2. *If Σ has the feeble euclidean or feeble r-hyperbolic four-point property then Σ is externally convex.*

Proof. Let $p, q\in\Sigma$ with $p\neq q$. Then on a segment $T_{p,q}$ joining p and q choose a point s with $qs>0$ and such that there exists a point t with (sqt). Then denoting either E_2 or $H_{2,r}$ by R_2, the hypotheses guarantee that $p, q, s, t\approx R_2$. This together with psq and (sqt) implies that pqt holds.

THEOREM 3. *If Σ has the feeble $\{$euclidean, r-hyperbolic$\}$ four-point* *property, then* Σ *is generalized* {*euclidean*, r-*hyperbolic*}.

Proof. By Theorem 2 Σ is externally convex and by Lemma 1 (along with the completeness and convexity of Σ) pqs, pqt, $qs=qt$ imply that $s=t$. Thus (Theorem 1) each two points of Σ lie on a unique metric line. Then the theorem in the euclidean case is identical with theorem 4.1 in [2]. The r-hyperbolic case is handled in the same manner as the euclidean case.

THEOREM 4. *If* Σ *has the feeble* r-*spherical four-point property, then* Σ *is a conjugate metric space with metric diameter equal to* πr, *and each point* $p \in \Sigma$ *determines a unique point* p^* *such that* $pp^* = \pi r$.

Proof. Since Σ has the feeble r-spherical four-point property, the metric diameter of Σ is at most πr. If $p, q \in \Sigma$ with $0 < pq < \pi r$ there exist points $t, v \in \Sigma$ such that ptq and (tqv) hold and $pt + tq + qv < \pi r$. The feeble r-spherical four-point property then implies that $p, t, q, v \approx\!\!\!\!< S_{2,r}$, and this can be strengthened to $p, t, q, v \approx\!\!\!\!< S_{1,r}$ because ptq and tqv hold.

The feeble r-spherical four-point property implies that each pair of points of Σ with distance less than πr have a unique mid point. This then implies that each two such points are joined by a unique segment. Let $T_{p,q}$ be the segment joining p and q, and let E be the set of points x of Σ such that pqx holds. All $x \in E$ such that $px < \pi r$ lie on a unique segment since repeated application of Lemma 1 will show that if pqx_1, pqx_2, $px_1 = px_2$, then $x_1 = x_2$. If for $x \in E$, $\alpha = \operatorname{lub} px$, then there exists a point $\bar{x} \in E$ such that $p\bar{x} = \alpha$. If $p\bar{x} < \pi r$ there exists a point $y \in E$ such that $py > \alpha$, so $p\bar{x} = \pi r$.

If there exist two points p^*, p^{**} in Σ with $pp^* = pp^{**} = \pi r$, let q be a mid-point of p^* and p^{**}. Then $p, p^*, p^{**}, q \approx p_1, p_1^*, p_1^{**}, q_1 \in S_{2,r}$ and $p_1p_1^* = p_1p_1^{**} = \pi r$ gives $p_1^* = p_1^{**}$ so that $p^* = p^{**}$.

THEOREM 5. *If* Σ *has the feeble* r-*spherical four-point property, then* Σ *has the weak* r-*spherical four-point property.*

Proof. Let p, q, s, t be four points of Σ with $p, q, s \approx\!\!\!\!< S_{1,r}$, to show that $p, q, s, t \approx\!\!\!\!< S_{2,r}$. If two of the points p, q, s coincide, then $p, q, s, t \approx\!\!\!\!< S_{2,r}$, so let it be assumed that p, q, s are pairwise distinct. Then because of the feeble r-spherical four-point property some pair, say p and q, have distance less than πr and determine a unique segment $T_{p,q}$. Let $p, q, s \approx p_1, q_1, s_1 \in S_{2,r}$ and let $S_{1,r}(p_1, q_1)$ be the unique $S_{1,r}$ determined by p_1 and q_1. If v and v_1 are the unique mid-points of p, q and p_1, q_1 respectively, the congruence $p, q, v, t \approx p_1, q_1, v_1, t_1$ can be extended to $t+T_{p,q} \approx t_1 + T_{p_1,q_1}$. If $s \in T_{p,q}$, $p, q, s, t \approxeq S_{2,r}$. If not, suppose the labelling is such that $qs \leqq ps$, and consider the congruence $q, w, s, q^* \approx q_1, w_1, s_1, q_1^*$, where w is the mid-point of the unique segment T_{q,q^*} joining q, q^* and containing s. This congruence follows from the feeble r-spherical four-point property and the free movability of $S_{2,r}$. Then this congruence can be extended to $t+T_{q,q^*} \approx t_1 + T_{q_1,q_1^*}$ and $p, q, s, t \approxeq S_{2,r}$.

THEOREM 6. *If Σ has the feeble r-elliptic four-point property, then Σ has metric diameter $\pi r/2$ and Σ is a conjugate space.*

Proof. Because of the feeble r-elliptic four-point property Σ has diameter at most $\pi r/2$. Let $p, q \in \Sigma$ with $pq < \pi r/2$. Then there exist points $t, v \in \Sigma$ with ptq, (tqv) holding and $pt+tq+qv < \pi r/2$. By the feeble r-elliptic four-point property $p, t, q, v \approxeq \mathscr{E}_{2,r}$ and this can be strengthened to $p, t, q, v \approxeq \mathscr{E}_{1,r}$ because of ptq and (tqv).

Let $p, t, q, v \approx p_1, t_1, q_1, v_1 \in \mathscr{E}_{2,r}$ and let $x \neq v$ and w be points of Σ with pwq and (wqx) holding and $px < \pi r$. Then $p, w, q, x \approx p_2, w_2, q_2, x_2 \in \mathscr{E}_{2,r}$ and p_2, w_2, q_2, x_2 lie on an $\mathscr{E}_{1,r}$. Then there exists a motion of $\mathscr{E}_{2,r}$ sending p_2, q_2 into p_1, q_1 respectively and sending w_2 and x_2 into uniquely determined points w_1 and x_1 on the $\mathscr{E}_{1,r}$ determined by p_1 and q_1. Thus if M is the set of $x \in \Sigma$ with pqx and $px < \pi r$ holding, the unique segment $T_{p,q}$ can be uniquely extended to $T_{p,q} \cup T_{q,x} = T_{p,x}$ for $x \in M$.

Let now $\alpha = \operatorname{lub} px$ for $x \in M$ and let $\{x_i\}$ be a sequence of points of M such that $\lim px = \alpha$ and if $i < j$, px_ix_j holds. Then since p and all of the x_i lie on the same metric segment, as $i, j \to \infty$, $x_ix_j \to 0$. The completeness of Σ then implies the existence of a point y such that $y = \lim x_i$ and $py = \alpha \leqq \pi r/2$. Furthermore, since pqx_i holds for $i=1, 2, \cdots$ and Σ is metric, pqy holds. If $py < \pi r/2$, then $y \in M$ and there exists $\bar{y} \in \Sigma$ such that $p\bar{y} > py = \alpha$, and this is impossible.

Finally the uniqueness of extensions of segments insures that if $pp^* = pp^{**} = \pi r/2$ and pqp^*, pqp^{**} hold, then $p^* = p^{**}$.

THEOREM 7. *If Σ has the feeble $\{r$-spherical, r-elliptic$\}$ four-point property, then Σ is generalized $\{r$-spherical, r-elliptic$\}$.*

Proof. The theorem follows in the spherical case from Theorem 5 upon application of Theorem 66.5 of [1] and in the elliptic case from Theorem 6 and Theorem 4.4 of [3].

M. M. Day[1] [3] has defined another four-point property which he calls the "queasy euclidean four-point property" and has shown that a

[1] The author is indebted to the referee for calling his attention to Day's work and for suggesting the possibility of the extension of Day's work.

complete, externally convex semimetric space possessing this property is generalized euclidean. The remainder of this paper is devoted to extending Day's work.

A semimetric space M will be said to have the *queasy* {euclidean, r-hyperbolic, r-spherical} four-point property provided that corresponding to each pair of points $p, s \in M$ there exists $q \in M$ such that pqs holds and for each $t \in M$, the quadruple $p, q, s, t \lessapprox \{E_2, H_{2,r}, S_{2,r}\}$.

LEMMA 2. *If Σ has the {euclidean, r-hyperbolic} four-point property, then each two distinct points of Σ are joined by a unique metric segment.*

Proof. Since Σ is complete, convex and metric each two points are joined by at least one segment. It will be sufficient then to show that each pair of points of Σ have just one mid-point. Let $p, q_1, q_2, s \in \Sigma$ with (pq_1s), (pq_2s) and $p \neq s$ holding, and let R represent either of the spaces E_2 or $H_{2,r}$. If there exists a sequence of points $t_i \in \Sigma$, $i=1, 2, \cdots$, with $\lim t_i = q_1$, pt_iq_1, pt_iq_2 holding, then $\lim t_i = q_2$ and $q_1 = q_2$.

If $q_1 \neq q_2$, then there exists a positive number $\bar{\rho}_1$ such that if $pt + tq_1 = pq_1$ and $pt + tq_2 = pq_2$ then $tq_1 = tq_2 > \bar{\rho}_1$. Also there exists $\bar{\rho}_2 > 0$ such that if $q_1t + ts = q_1s$ and $q_2t + ts = q_2s$, then $tq_1 = tq_2 > \bar{\rho}_2$. Let ρ_1 be the least upper bound of the numbers $\bar{\rho}_1$ and ρ_2 that of the numbers $\bar{\rho}_2$. Let $\bar{p}$ and p^* be points of Σ with $p\bar{p} + \bar{p}q_1 = pq_1$, $pp^* + p^*q_2 = pq_2$ and $\bar{p}q_1 = p^*q_2 = \rho_1$. Then either $\bar{p} = p^*$ or there is a sequence p_i with pp_iq_1, pp_iq_2 holding and $\lim p_i = \bar{p}$, $\lim p_i = p^*$ so that $\bar{p} = p^*$. Thus there exist two points of Σ with q_1 and q_2 each metrically between these points but such that any segment joining the points and containing q_1 has only end points in common with a segment joining these points and containing q_2. There will be no loss of generality if these points are taken to be p and s and if q_1 and q_2 are assumed to be distinct middle points of p and s.

The queasy four-point property of Σ implies that there exist $x \in \Sigma$, $\bar{p}, \bar{x}, \bar{q}, \bar{s}, p^*, x^*, q^*, s^* \in R$ with pxs holding and

$$p, x, q_1, s \approx \bar{p}, \bar{x}, \bar{q}, \bar{s}$$
$$p, x, q_2, s \approx p^*, x^*, q^*, s^*.$$

Then since $p^*x^*s^*$ and $p^*q^*s^*$ hold, there is a motion sending the "starred" points into the corresponding "barred" ones, and $\bar{p}, \bar{x}, \bar{q}, \bar{s}$ all lie on one metric segment of R. Thus either $\bar{x} = \bar{q}$ and $x = q_1 = q_2$ or there is a metric segment joining p, q_1, s and one joining p, q_2, s with these two segments having interior point x in common. This contradiction completes the proof.

LEMMA 3. *If Σ has the queasy r-spherical four-point property, then each two distinct points having distance less than πr are joined by a unique segment.*

Proof. The proof is identical with that of the preceeding lemma if distance ps is restricted to be less than πr.

THEOREM 8. *If Σ has the queasy {euclidean, r-hyperbolic, r-spherical} four point property, then Σ is generalized {euclidean, r-hyperbolic, r-spherical}.*

Proof. It will be sufficient to show that if $p, q, s, t \in \Sigma$ with (pqs) holding, then $p, q, s, t \approxeq R$, where R represents any one of the spaces $E_2, H_{2,r}, S_{2,r}$. Assume for the present that if R is spherical, $ps < \pi r$. Then let $x \in R$ with pxs holding and

$$p, x, q, s \approx \bar{p}, \bar{x}, \bar{q}, \bar{s} \in R$$
$$p, x, s, t \approx p^*, x^*, s^*, t^* \in R$$

Then there exists a motion of R sending p^*, s^* into $\bar{p}$, $\bar{s}$ respectively and t^* into a point $\bar{t}$. If $qt = \overline{qt}$, then $p, q, s, t \approx \bar{p}, \bar{q}, \bar{s}, \bar{t}$.

If $qt \neq \overline{qt}$, let a congruence f between the segments $T_{p,s}$ and $T_{\bar{p},\bar{s}}$ be established so that $f(p) = \bar{p}$, $f(s) = \bar{s}$. Let Q represent the set of points $x \in T_{p,s}$ such that $tx = \bar{t}f(x)$. Then the continuity of the metric in Σ implies that in traversing $T_{p,s}$ from p to q there is a last point of Q encountered. Let this point be u, and let w be the last point of Q encountered in traversing $T_{p,s}$ from s toward q. Denote $\bar{u} = f(u)$, $\bar{w} = f(w)$.

Then there exists by the queasy property a point $y \in \Sigma$ with uyw holding and $u, y, w, t \approx u', y', w', t' \in R$. A motion of R sends u', w', t' into $\bar{u}, \bar{w}, \bar{t}$ and y' into a unique point $\bar{y}$ with $\overline{uyw}$ holding and $\overline{yt} = y't' = yt$. This contradicts the property used to pick out u and w so that $qt = \overline{qt}$ and $p, q, s, t \approx R$.

Finally if R represents $S_{2,r}$ and $ps = \pi r$, there is a point $x \in \Sigma$ with pxs holding and

$$p, x, s, q \approx \bar{p}, \bar{x}, \bar{s}, \bar{q} \in S_{2,r}$$
$$p, x, s, t \approx p^*, x^*, s^*, t^* \in S_{2,r}\,.$$

Let a motion be performed sending p^*, s^* into $\bar{p}$, $\bar{s}$ respectively and t^* into a point $\bar{t}$. Consider the set of distances $\bar{t}x$ where x belongs to the $S_{1,r}$ at distance $\pi r/2$ from p. This set of numbers has a minimum m and a maximum M. Let the labelling be taken so that $pt \leqq \pi r/2$. Then it is necessary that $m \leqq tq \leqq M$. For if $tq < m$, then $pt + m = \pi r/2$ and $pt + tq < \pi r/2 = pq$. Also if $tq > M$, $tq > tp + pq$.

Now on the $S_{1,r}$ at distance $\pi r/2$ from p there is a point $\bar{\bar{q}}$ so that $\bar{t}\bar{\bar{q}} = tq$. Then $p, q, s, t \approx \bar{p}, \bar{\bar{q}}, \bar{s}, \bar{t}$, and this completes the proof.

Of course the proof of Theorem 8 is not valid for $\mathscr{E}_{2,r}$ because of the strong use made of free movability. It should also be noted that when the queasy four-point property is assumed for a semi-metric space, it is unnecessary to assume convexity and metricity since the queasy property implies these.

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RETRACTIONS IN SEMIGROUPS

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Let S be a semigroup (that is, a Hausdorff space together with a continuous associative multiplication) and let E denote the set of idempotents of S. If $x \in S$ let

$$L_x = \{y | y \cup Sy = x \cup Sx\}$$

and

$$R_x = \{y | y \cup yS = x \cup xS\} \ .$$

Put $H_x = L_x \cap R_x$ and for $e \in E$ let

$$H = \bigcup \{H_e | e \in E\} \ ,$$

$$M_e = \{x | ex \in H \text{ and } xe \in H\} \ ,$$

$$Z_e = H_e \times (R_e \cap E) \times (L_e \cap E)$$

and

$$K_e = (L_e \cap E) \cdot H_e \cdot (R_e \cap E) \ .$$

Under the assumption that S is compact we shall prove that K_e is a retract of M_e and that K_e and Z_e are equivalent, both algebraically and topologically. This latter fact sharpens a result announced in [6] and the former settles several questions raised in [7].

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LEMMA 1. *Let $Z = S \times S \times S$ and define a multiplication in Z by*

$$(t, x, y) \cdot (t', x', y') = (txy't', x', y) \ ;$$

then Z is a semigroup and, with this multiplication, the function f: $Z \to S$ defined by $f(t, x, y) = ytx$ is a continuous homomorphism.

The proof of this is immediate. *We use only the above defined multiplication in Z and not coordinatewise multiplication.* It is clear that $f(Z_e) = K_e$.

Since the sets H_e, $e \in E$, are pairwise disjoint groups [1] it is legitimate to define functions

$$\eta : H \to E \ , \qquad \theta : H \to H$$

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by "$\eta(x)$ is the unit of the group H_e which contains x" and "$\theta(x)$ is the inverse of x in the group H_e which contains x". If $x \in M_e$ then ex, $xe \in H$ so that $\eta(ex)$, $\eta(xe)$ are defined. Define $g: M_e \to Z$ by

$$g(x)=(exe,\ \eta(ex),\ \eta(xe))$$

and note that the continuity of η implies the continuity of g. For $x \in M_e$ let

$$\rho(x)=\eta(xe)x\eta(ex)$$

so that ρ is continuous if η is continuous.

LEMMA 2. *For any $x \in K_e$ we have $fg(x)=x=\rho(x)$ and $g(K_e)=Z_e$. The function $f|Z_e$ takes Z_e onto K_e in a one-to-one way and is a homeomorphism if η is continuous. If η is continuous then ρ retracts M_e onto K_e.*

Proof. Let $t \in H_e, e_1 \in R_e \cap E$ and $e_2 \in L_e \cap E$. Since $L_{e_2}=L_e$ it is immediate that $ee_2=e$ and since t is an element of the group H_e whose unit is e (Green [3]) we also have $et=t=te$. Similarly we see that $e_1e=e$. It is important to observe that the sets $\{L_x|x \in S\}$, $\{R_x|x \in S\}$ and $\{H_x|x \in S\}$ are disjointed covers of S so that, for example $L_x \cap L_y \neq \square$ implies $L_x=L_y$. We see that $ee_2te_1=te_1$ and $e_2te_1e=e_2t$ so that $ee_2te_1e=t$. We note next that $te_1 \in H_{e_1}$ and thus $\eta(te_1)=e_1$. For $e \in R_e \cap L_e=R_{e_1} \cap L_t$ and $e^2=e$, give $te_1 \in R_t \cap L_{e_1}$ in view of Theorem 3 of [2]. But

$$R_t \cap L_{e_1}=R_e \cap L_{e_1}=R_{e_1} \cap L_{e_1}=H_{e_1}$$

and H_{e_1} being a group with unit e_1 we have, from the definition of η, $\eta(te_1)=e_1$. In a similar fashion we show that $\eta(e_2t)=e_2$. If $x \in K_e$ then we have $x=e_2te_1$ with the above notation and

$$\begin{aligned} fg(x)&=f(exe,\ \eta(ex),\ \eta(xe))=\eta(xe)exe\,\eta(ex)\\ &=\eta(e_2t)t\,\eta(te_1)=e_2te_1=x\ . \end{aligned}$$

It will suffice to show in addition that $gf(z)=z$ for $z \in Z$ since $fg(x)=x$ gives $x=\rho(x)$. Now let $z=(t,\ e_1,\ e_2) \in Z_e$ so that $f(z)=e_2te_1 \in K_e$ and

$$g(f(z))=(ef(z)e,\ \eta(ef(z)),\ \eta(f(z)e))=(t,\ e_1,\ e_2)$$

in virtue of the computation given earlier.

It remains to prove the continuity of η when S is compact. This was announced in [7] but no proof of this fact has been published. Let

$$\mathscr{L}=\{(x,\ y)\,|\,L_x=L_y\}\ , \qquad \mathscr{R}=\{(x,\ y)\,|\,R_x=R_y\}$$

and let $\mathscr{H}=\mathscr{L} \cap \mathscr{R}$.

LEMMA 3. *If S is compact then $\mathscr{H}$, $\mathscr{L}$ and $\mathscr{R}$ are closed.*

Proof. Let

$$\mathscr{L}'=\{(x, y)\,|\, Sx\subset Sy\}$$

and assume that $(a, b)\in S\times S\diagdown\mathscr{L}'$. Then $Sb\subset S\diagdown a$ and hence $Sb\subset S\diagdown U^*$ for some open set U about a since Sb is closed and S is regular. Again from the compactness of S we can find an open set V about b such that $SV\subset S\diagdown U^*$. Hence $(U\times V)\cap\mathscr{L}'=\square$ and we may infer that $\mathscr{L}'$ is closed. There is no loss of generality in assuming that S has a unit [3]. Hence if $h: S\times S\to S\times S$ is defined by $h(x, y)=(y, x)$ then $h(\mathscr{L}')$ is closed and thus $\mathscr{L}=\mathscr{L}'\cap h(\mathscr{L}')$ is closed. In a similar way it may be shown that $\mathscr{R}$ is closed. Moreover, $\mathscr{H}$ is closed because $\mathscr{H}=\mathscr{L}\cap\mathscr{R}$.

THEOREM 1 [7]. *If S is compact then H is closed, $\eta: H\to E$ is a retraction and $\theta: H\to H$ is a homeomorphism.*

Proof. Define $p: S\times S\to S$ by $p(x, y)=x$. Then

$$H=\bigcup\{H_e\,|\, e\in E\}=p(\mathscr{H}\cap(S\times E))$$

is closed since $\mathscr{H}$ and E are closed. We show next that θ is continuous and to this end it is enough to prove that $G=\{(x, \theta(x))|x\in H\}$ in virtue of the fact that H is compact Hausdorff. If $m: S\times S\to S$ is defined by $m(x, y)=xy$ then $\mathscr{H}\cap(H\times H)\cap m^{-1}(E)$ is closed and we will show that this set is the same as G. For $(x, \theta(x))$ in G implies $m(x, \theta(x))=x\theta(x)\in E$ in virtue of the definition of θ. Since x and $\theta(x)$ are in the same set H_e, $e\in E$, it is clear that $(x, \theta(x))\in H\times H$ and it is easily seen from the definition of $H_x=L_x\cap R_x$, and $\mathscr{H}=\mathscr{L}\cap\mathscr{R}$ that also $(x, \theta(x))\in\mathscr{H}$. Now take x, y such that $xy=e\in E$, $x, y\in H$ and $(x, y)\in\mathscr{H}$. The last fact shows that $H_x=H_y$ and the penultimate condition, together with this shows that $x, y\in H_{e_1}$ for some $e_1\in E$. But $e=xy\in H_{e_1}$ and the fact that H_{e_1} is a group implies that $e=e_1$. Now the uniqueness of inversion in the group H_e shows that $y=\theta(x)$. Hence θ is continuous and η is continuous because $\eta(x)=x\theta(x)$ from the definition of η and θ.

G. B. Preston raised the question as to the continuity of a certain generalized "inversion"—Suppose that there is a unique function $\alpha: S\to S$ such that $x\alpha(x)x=x$ and $\alpha(x)x\alpha(x)=\alpha(x)$ for each $x\in S$. If S is compact then α is continuous. To see this let $\mathscr{N}$ be the set of all $(x, y)\in S\times S$ such that $xyx=x$ and $yxy=y$ and define $\varphi: S\times S\to S\times S$ by $\varphi(x, y)=(xyx, x)$. If D is the diagonal of $S\times S$ then $\varphi^{-1}(D)$ is closed. Similarly $\psi^{-1}(D)$ is closed where $\psi(x, y)=(y, yxy)$ and $\mathscr{N}=\varphi^{-1}(D)\cap\psi^{-1}(D)$ is therefore closed. The uniqueness of α implies that $\{(x, \alpha(x))|x\in S\}=\mathscr{N}$

so that α is continuous if S is compact. For a discussion of the existence and uniqueness of such functions as α, see [2, pp. 273–274] as well as references therein to Liber, Munn and Penrose, Thierrin, Vagner and the papers of Preston in London Math. Soc., 1954.

From Theorem 1 and Lemma 2 we obtain at once

THEOREM 3. *Let S be compact and let $e \in E$; then K_e is topologically isomorphic with*

$$Z_e = H_e \times (L_e \cap E) \times (R_e \cap E)$$

and K_e is a retract of M_e.

It is not asserted that K_e is a subsemigroup of S. The first corollary is a topologized form of the Rees-Suschkewitsch theorem, see [6], [7] and [2] for a bibliography of relevant algebraic results.

COROLLARY 1. *If S is compact, if K is the minimal ideal of S and if $e \in E \cap K$ then K is topologically isomorphic with $eSe \times (Se \cap E) \times (es \cap E)$ and K and each "factor" of K is a retract of S.*

Proof. We rely, without explicit citation, on the results of [1]. It is immediate that $M_e = S$. Now $L_e = Se$, $Re = eS$ and $H_e = eSe$ so that (by definition and [1]) $K_e = Se \cdot eSe \cdot eS \subset K$ and, being an ideal, $K_e = K$. Clearly $x \rightarrow exe$ retracts S onto eSe. Now $Se \subset K \subset H$ and $\eta|Se$ retracts Se onto $Se \cap E$.

It is clear, when S is compact, that K enjoys all the retraction invariants of S, for example, if S is locally connected so is K. We do not list these nor do we give here the applications of Corollary 1 that were mentioned in [6].

COROLLARY 2. *If S is a clan [7], if $K \subset E$ and if $H^n(S) \neq 0$ for some $n > 0$ and some coefficient group, then* $\dim K \geqq 2$.

Proof. If $K \subset E$ then $H_e = \{e\}$ and K is thus topologically the product $Se \times eS$ since $Se, eS \subset K$. Now $H^n(Se) \approx H^n(S) \approx H^n(eS)$ [9] and hence Se, eS are non-degenerate continua. It follows that $\dim K \geqq 2$.

It is possible to put some of the above in a more general framework. Let T be a closed subsemigroup of S and let

$$L_x = \{y | x \cup Tx = y \cup Ty\} ,$$

with similar definitions for R_x and H_x. If $e \in E$ then H_e is a semigroup and H_e is a group if $eT \cup Te \subset T$. If $\mathscr{H}, \mathscr{L}, \mathscr{R}$ are defined analogously then $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. Moreover we have $\mathscr{L} \circ \mathscr{R} = \mathscr{J}$, where

$$\mathscr{J} = \{(x, y) | x \cup Tx \cup xT \cup TxT = y \cup Ty \cup yT \cup TyT\} ,$$

when S is compact [5]. In this case $\mathscr{H}$, $\mathscr{L}$, $\mathscr{R}$, $\mathscr{L} \circ \mathscr{R}$ and $\mathscr{J}$ are closed. It is easy to see that many of the results of [3] and [2] are valid in this setting. If we define a left T-ideal as a non-void set A such that $TA \subset A$, then the basic propositions about ideals are also available. Many of these results follow from general theorems on structs [8].

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MONOTONE MAPPINGS OF MANIFOLDS

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1. Introduction. Mappings of the 2-sphere, and more generally of the 2-manifolds, have been studied by various authors. (See, for instance, [9] and references therein, [7].) Generally, these mappings have been subjected to certain " monotoneity " conditions on the counter-images of points. Thus, in Moore's first paper [8] on the 2-sphere, it was required not only that counter-images be connected, but that they not separate the sphere. In terms of homology, then, he required of a counter-image C that $p_a^r(C)=0$ for $r=0,1$. Later studies of Moore and others usually omitted the requirement that $p^1(C)=0$, thus increasing the possible number of topological types of images. With the condition $p^1(C)=0$ imposed, the image of the 2-sphere is a 2-sphere, and of a 2-manifold is a 2-manifold of the same type. Without this condition, the various types of " cactoids " are obtained.

In the present paper we consider some higher dimensional cases. As might be expected, we impose higher dimensional "monotoneity" conditions.

DEFINITION 1. A mapping f: $A \to B$ is called *n-monotone* if $H^r(f^{-1}(b))=0$ for all $b \in B$ and $r \leq n$. (See [10; p. 904].)

EXAMPLE. Let us consider the mapping induced by decomposing the 3-sphere into disjoint closed sets each of which is a point, except that all points on some suitable " wild " arc [5 ; Ex. 1.1] A are identified. This mapping is r-monotone for all r, but the image-space is no longer a 3-sphere; indeed, it is not a 3-manifold in the classical sense at all, since the point corresponding to A does not have a 3-cell neighborhood.

This example makes it at first appear that because of such "homotopy" difficulties, it may be useless to look for any well-defined class of configurations in higher dimensions. However, as we show below, the class of configurations obtained is precisely that of the generalized manifolds. Moreover, we need not restrict the mappings to the mappings of 3-manifolds in the classical sense, since the generalized manifolds turn out to form a class which is closed relative to the mappings considered. This result forms, then, a new justification for the study of generalized manifolds.

2. Preliminary theorems and lemmas. In general, spaces are

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Hausdorff, but no conditions of metrizibility or separability are assumed. Except where noted to the contrary, we use augmented Čech homology with an algebraic field as coefficient domain. We recall the following definition [11; p. 237].

DEFINITION 2. If S is a locally compact space, such that for every pair of open sets P, Q for which $P \supset \overline{Q}$ and $\overline{Q}$ is compact, the group $H^n(S:\overline{Q}, 0; \overline{P}, 0)$ (cf. [11; 166, Def. 18.28]) is of finite dimension, then S is said to have property $(P, Q)^n$.

REMARK. Since the space is assumed locally compact, the above definition can be stated in a number of different but equivalent forms. Thus, Q may be replaced in the definition by any compact set M; that is, S has property $(P, Q)^n$ if for every pair of sets P, M such that P is open, M is compact, and $P \supset M$, then $H^n(S: M, 0; \overline{P}, 0)$ is of finite dimension. Another variant, but equivalent form of the definition, is obtained if in either of the above definitions it be required only that there exist at most a finite number of n-cycles on $\overline{Q}(M)$ which are lirh on compact subsets of P (that is, *in* P).

Another variant would be to require that there exist at most a finite number of cycles on compact subsets of Q (that is, *in* Q) that are lirh on $\overline{P}$ (or, that are lirh in P). Each of the equivalent forms of the definition may be found particularly adapted to a given situation.

We express the fact that S has property $(P, Q)^r$ for $r=0, 1, \cdots, n$ by stating that S has property $(P, Q)_0^n$.

THEOREM 1. *If S is a compact space having property $(P, Q)^n$, and $f: S \to S'$ is a continuous $(n-1)$-monotone mapping of S onto a Hausdorff space S', then S' has property $(P, Q)^n$.*

Proof. Let U', V' be open subsets of S' such that $U' \supset \overline{V}'$ and $\overline{V}'$ is compact. The sets $U = f^{-1}(U')$, $V^* = f^{-1}(\overline{V}')$ are open and closed subsets, respectively, of S, such that $U \supset V^*$.

In the mapping $f(V^*) = \overline{V}'$, counter-images of points are all r-acyclic for $r=0, 1, \cdots, n-1$. Hence [3] for any cycle γ^n on $\overline{V}'$, there exists a cycle Z^n on V^* such that

$$(1) \qquad f(Z^n) \sim \gamma^n \text{ on } \overline{V}' .$$

Since S has property $(P, Q)^n$, there exist cycles Z_i^n, $i=1, \cdots, m$ of V^* such that if Z^n is any cycle of V^*, then

$$(2) \qquad Z^n \sim \sum_{i=1}^{m} a^i Z_i^n \text{ in } U .$$

Consequently, since (2) implies

$$(3) \qquad f(Z^n) \sim \sum_{i=1}^{m} a^i f(Z_i^n) \text{ in } U' ,$$

we have, combining (1) and (3), that

$$\gamma^n \sim \sum_{i=1}^{m} a^i f(Z_i^n) \text{ in } U' .$$

It follows that at most m cycles on $\overline{V}'$ are lirh in U and hence that S' has property $(P, Q)^n$.

REMARK. It is worthwhile noting that the above proof gives the following: If $f: S \rightarrow S'$ is a continuous $(n-1)$-monotone proper mapping of a locally compact space S onto a Hausdorff space S', and P', F' are open and compact subsets of S', respectively, such that $P' \supset F'$, then

$$f \mid P_*: H^n(S: F, P) \rightarrow H^n(S': F', P')$$

is a homomorphism onto, where $F = f^{-1}(F')$, $P = f^{-1}(P')$, and $H^n(S: F, P)$ denotes the group of n-cycles on F reduced modulo the subgroup of n-cycles that bound *in* P. A similar argument shows that $f \mid P_*$: $H^{n-1}(S: F, P) \rightarrow H^{n-1}(S': F', P')$ is an isomorphism onto. These are generalizations of the Vietoris mapping theorem [2], [3].

THEOREM 2. *If S is an lc^n compact space, $n > 0$, and $f: S \rightarrow S'$ is a continuous $(n-1)$-monotone mapping of S onto a Hausdorff space S', then S' is lc^n.*

Proof. By [**11**; p. 70, Th. 1.6], S' is 0-lc. And since S' is a compact 0-lc space, it has property $(P, Q)^0$. (See [**11**; p. 106, 3.7]). That S' has property $(P, Q)^r$ for $r = 1, 2, \cdots, n$ follows from Theorem 1. Since, for compact spaces, lc^n and $(P, Q)_0^n$ are equivalent, we conclude that S' is lc^n (see **11**; p. 238, 7.17]).

LEMMA 1. *In a locally compact space S, let P and Q be open sets such that $\overline{P}$ is compact and $P \supset \overline{Q}$; and let M be a closed subset of Q such that for any open set Q_ν for which $M \subset Q_\nu \subset Q$, the dimension of $H_r(S: S, S-\overline{P}; S, S-\overline{Q}_\nu)$* [**11**; 166, Def. 18.29] *is the same finite number k. If $Z_r^1, \cdots, Z_r^k$ form a base for r-cocycles mod $S-\overline{P}$ relative to cohomologies mod $S-\overline{Q}$, then for every open set Q_ν such that $M \subset Q_\nu \subset Q$,* *the cocycles* Z_r^i *form a base for* r*-cocycles mod* $S-\overline{P}$ *relative to cohomologies mod* $S-\overline{Q}_\nu$.

Proof. Let $\gamma_r^1, \cdots, \gamma_r^k$ be a base for cocycles mod $S-\overline{P}$ relative to cohomologies mod $S-\overline{Q}_\nu$. Then there exist cohomologies:

$$(1) \qquad \gamma_r^j \sim \sum_{i=1}^{k} c_i^j Z_r^i \bmod S-\overline{Q}\,, \qquad j=1, \cdots, k\,.$$

Relations (1) hold a fortiori mod $S-\overline{Q}_\nu$.

The matrix $\|c_i^j\|$ is of rank k, since otherwise there would exist a cohomology relation between the γ_r^j's, mod $S-\overline{Q}_\nu$.

Suppose the Z_r^i's are not lircoh mod $S-\overline{Q}_\nu$. Then there exists a relation

$$\sum a_i Z_r^i \sim 0 \qquad \bmod S-\overline{Q}_\nu\,.$$

But the system of equations

$$c_i^1 x_1 + \cdots + c_i^j x_j + \cdots + c_i^k x_k = a_i\,, \qquad i=1, \cdots, k$$

has a non-trivial solution in the x_j's. Hence, multiplying the relations (1) by $x_1, \cdots, x_k$, respectively, we get

$$\sum x_j \gamma_r^j \sim \sum a_i Z_r^i \sim 0 \qquad \bmod S-\overline{Q}_\nu\,.$$

Thus, the assumption that the Z_r^i are not liroch mod $S-\overline{Q}_\nu$ leads to contradiction; and since the dimension of $H_r(S\colon S, S-\overline{P}; S, S-\overline{Q}_\nu)=k$, we conclude that the Z_r^i's form a base for cohomologies mod $S-\overline{Q}_\nu$.

LEMMA 2. *In a locally compact space* S, *let* M *be a compact set such that* $H^r(M)=0$; *and suppose that there exist open sets* P, Q *such that* $M\subset Q\subset P$ *and such that* $H^r(S\colon \overline{Q}, 0; \overline{P}, 0)$ *has finite dimension. Then there exists an open set* Q_ν *such that* $M\subset Q_\nu \subset Q$ *and* $H^r(S\colon \overline{Q}_\nu, 0; \overline{P}, 0)=0$.

Proof. Suppose, on the contrary, that for all such Q_ν, $H^r(S\colon \overline{Q}_\nu, 0; \overline{P}, 0) \neq 0$. Since $H^r(S\colon \overline{Q}, 0; \overline{P}, 0)$ is of finite dimension, we may assume Q shrunk so that all dimensions of groups $H^r(S\colon \overline{Q}_\nu, 0; \overline{P}, 0)$ are equal to the same positive integer k for all Q_ν such that $M\subset Q_\nu \subset Q$.

Since

$$H_r(S\colon S, S-\overline{P}; S, S-\overline{Q}_\nu) \cong H^r(S\colon \overline{Q}_\nu, 0; \overline{P}, 0)$$

[**11**; 166, 18.30], there exist, by Lemma 2, cocycles Z_r^i, $i=1, \cdots, k$, mod $S-\overline{P}$, that form a base for cocycles mod $S-\overline{P}$ relative to cohomologies mod $S-\overline{Q}_\nu$ for all Q_ν such that $M\subset Q_\nu\subset Q$. Consider Z^1_r, and $\mathfrak{U}$ a fcos of $\overline{P}$ such that $Z^1_r(\mathfrak{U})$ exists. Let $\mathfrak{V}>\mathfrak{U}$ be a normal refinement of $\mathfrak{U}$ rel. M [**11**; 140], and let Q_ν be such that if a simplex of $\mathfrak{V}$ meets Q_ν, then it meets M. Since $Z^1_r\not\sim 0 \bmod S-\overline{Q}_\nu$, there exists on $\overline{Q}_\nu$ a cycle Z^r such that $Z^1_r\cdot Z^r=1$. And by the choice of $\mathfrak{V}$, the coordinate $Z^r(\mathfrak{V})$ is on M. Hence $\pi_{\mathfrak{U}\mathfrak{V}}Z^r(\mathfrak{V})$ is the coordinate on M of a Čech cycle γ^r.

But $H^r(M)=0$ and consequently $\gamma^r\sim 0$ on M, and a fortiori, $\gamma^r(\mathfrak{U})\sim 0$ on $\overline{Q}$; and since $Z^r(\mathfrak{U})\sim\pi_{\mathfrak{U}\mathfrak{V}}Z^r(\mathfrak{V})$ on $\overline{Q}$, it follows that $Z^r(\mathfrak{U})\sim 0$ on $\overline{Q}$. But then $Z^r(\mathfrak{U})\cdot Z^1_r(\mathfrak{U})=0$, in contradiction to the choice of $Z^r(\mathfrak{U})$. We conclude, then, that for some Q_ν, $H^r(S:\overline{Q}_\nu, 0; \overline{P}, 0)=0$.

THEOREM 3. *A necessary and sufficient condition that a locally compact space S be lc^n is that if M is any compact subset of S such that $H^r(M)=0$, for some $r\leqq n$, then for any open set P containing M there exist an open set Q such that $M\subset Q\subset\overline{Q}\subset P$ and such that $H^r(S:\overline{Q}, 0; \overline{P}, 0)=0$.*

Proof of sufficiency. Trivial. (See [**11**; 193, 6.14]).

Proof of necessity. With M and P as in the hypothesis, and any open set Q such that $\overline{Q}$ is compact and $M\subset Q\subset\overline{Q}\subset P$, the dimension of $H^r(S:\overline{Q}, 0; \overline{P}, 0)$ is finite [**11**; 193, 6.16]. Lemma 2 now gives the desired result.

LEMMA 3. *If S is an orientable n-gm and M a compact subset of S which is r- and $(n-r-1)$-acyclic for some r such that $r\leqq n-2$, then for any open set P containing M, there exists an open set Q such that $M\subset Q\subset\overline{Q}\subset P$ and such that all compact r-cycles in $Q-M$ bound in $P-M$.*

Proof. Since S is lc^n [**11**; 244], there exists by Theorem 3 an open set Q containing M such that $\overline{Q}\subset P$ and such that all r- and $(n-r-1)$-cycles in Q bound in P. Suppose there exists a cycle Z^r in $Q-M$ that does not bound in $P-M$.

By Lemma VIII 5.4 of [**11**; 255] there exists a cocycle $Z_{n-r}=\tau^*Z^r$ in $Q-M$ such that $Z_{n-r}\smile\Gamma^n\sim Z^r$ in $Q-M$, where Γ^n is the fundamental n-cycle of S. And since $Z^r\sim 0$ in P, we may assume that $Z_{n-r}\sim 0$ in P. There exists a covering $\mathfrak{U}$ and a relation.

$$\delta C_{n-r-1}(\mathfrak{U})=Z_{n-r}(\mathfrak{U}) \quad \text{in } P. \tag{1}$$

The chain $C_{n-r-1}(\mathfrak{U})$ is clearly a cocycle mod $P-M=S-[(\text{Ext } P)\cup M]$.

And if $C_{n-r-1} \not\sim 0$ mod $S-[(\text{Ext } P)\cup M]$, then by [11; 164, 18.19] there exists a cycle Z^{n-r-1} on $(\text{Ext } P)\cup M$ such that $C_{n-r-1}\cdot Z^{n-r-1}=1$. Since $Z^{n-r-1}=Z_1+Z_2$, where Z_1 is on Ext P and Z_2 on M, we may neglect Z_1 (as $C_{n-r-1}(\mathfrak{U})$ is in P) and write $C_{n-r-1}\cdot Z_2=1$. But $Z_2\sim 0$ on M since M is $(n-r-1)$-acyclic, implying $C_{n-r-1}\cdot Z_2=0$. We conclude, then, that $C_{n-r-1}\sim 0$ mod $P-M$. There exists, therefore, a covering $\mathfrak{V}>\mathfrak{U}$ and a relation

$$\delta C_{n-r-2}(\mathfrak{V})=\pi^*_{\mathfrak{U}\mathfrak{V}}C_{n-r-1}(\mathfrak{U})-L_{n-r-1}(\mathfrak{V})\ , \tag{2}$$

where $L_{n-r-1}(\mathfrak{V})$ is in $P-M$.

Applying δ to (2) and utilizing (1), we get

$$\delta L_{n-r-1}(\mathfrak{V})=\pi^*_{\mathfrak{U}\mathfrak{V}}Z_{n-r}(\mathfrak{U})\ .$$

That is, $Z_{n-r}\sim 0$ in $P-M$. But this implies $Z^r\sim 0$ in $P-M$, contrary to supposition.

REMARK. In the hypothesis of Lemma 3 it was assumed that $r\leqq n-2$, that is, $n-r-2\geqq 0$. The necessity for this is shown by the following example: Let S be the 2-sphere, S^2, and in S let M be a circular disk, and U and V open circular disks concentric with M and such that $M\subset V\subset \bar{V}\subset U$. Then in $V-M$ an S^1 which encloses M carries a Z^1 which fails to bound in $U-M$.

Note also that if M is an S^1 in S^4, then M is 2-acyclic but in any open set P containing M there exist 2-dimensional cycles linking M. This shows the necessity for the assumption that M be $(n-r-1)$-acyclic in the hypothesis.

LEMMA 4. *Let Z^{n-1} be a cycle carried by a closed subset K of an orientable n-gcm S, and M a connected subset of $S-K$. If $Z^{n-1}\sim 0$ on S, then must $Z^{n-1}\sim 0$ on a compact subset of $S-M$.*

Proof. This is analogous to that of Lemma XII 3.12, p. 375 of [11].

For the purposes of the proof of the next theorem, let us recall the following form of the definition of an orientable *n-gcm*: An n-dimensional compact space S such that (1) $p^n(S)=1$ and all n-cycles on closed proper subsets of S bound on S; (2) S is semi-r-connected for all r such that $1\leqq r\leqq n-1$; (3) S is completely r-avoidable at all points for all $r\leqq n-2$; (4) S is n-extendible at all points. (This is IX 3.6, p. 273, of [11]). (By Lemmas VII 5.2, 5.3, p. 224 of [11], condition (4) may be replaced by the requirement that S is locally $(n-1)$-avoidable at all points; this fact will be utilized in the proof of Main Theorem A below.)

3. Main theorems.

MAIN THEOREM A. *Let S be an orientable n-gcm and $f: S \to S'$ an $(n-1)$-monotone continuous mapping of S onto an at most n-dimensional nondegenerate Hausdorff space S'. Then S' is an orientable n-gcm of the same homology type as S.*

Proof. Since S' is nondegenerate, f is n-monotone and therefore by the Vietoris-Begle Theorem [2], $p^n(S')=p^n(S)=1$. And since $p^n(S')>0$, S' is at least n-dimensional, and therefore, by the dimensionality assumption of the hypothesis, is exactly n-dimensional. And if F' is a proper closed subset of S', and Z^n a cycle on F', there exists on the set $F=f^{-1}(F')$ a cycle γ^n such that $f(\gamma^n)\sim Z^n$ on F' (see [2; § 5]). As F is a proper closed subset of $S, \gamma^n \sim 0$ on S and therefore $f(\gamma^n)\sim 0$ on S'—implying that $Z^n \sim 0$ on S'. Thus S' satisfies condition (1) above.

That condition (2) is satisfied, follows from the fact that S' is $\bar{l}c^n$ by Theorem 2.

Let $p' \in S'$, and U' an open set containing p'. Then $U=f^{-1}(U')$ is an open set containing the set $M=f^{-1}(p')$. Let r be any integer such that $1 \leqq r \leqq n-2$. Since $H^r(M)=H^{n-r-1}(M)=0$, there exists by Lemma 3 an open set P such that $M \subset P \subset \bar{P} \subset U$ and such that all r-cycles in $P-M$ bound in $U-M$. Let W' be an open set such that $p' \in W' \subset \overline{W'} \subset U'$, and such that $f^{-1}(\overline{W'}) \subset P$. Let Q' be an open set such that $p' \in Q' \subset \overline{Q'} \subset W'$. As S' is $\bar{l}c^n$, there exists a finite base $Z_1^r, \cdots, Z_k^r$ of r-cycles of $F(W')$ relative to homologies in $U'-\overline{Q'}$. Let $W=f^{-1}(W')$, $Q=f^{-1}(Q)$, and consider any cycle Z_i^r. There exists a cycle γ_i^r on $f^{-1}(F(W'))$ such that $f(\gamma_i^r)\sim Z_i^r$ on $F(W')$. And as $\gamma_i^r \sim 0$ in $U-M$, Z_i^r must bound in $U'-P'$. Finally, since there are only a finite number of the r-cycles Z_i^r, there must exist an open set R' such that $p' \in R' \subset \overline{R'} \subset Q'$ and such that all r-cycles on $F(W')$ bound in $U'-\overline{R'}$. Thus S' satisfies condition (3).

To show that S' satisfies condition (4), let p', U', U and M be as before. Since by hypothesis $p^{n-1}(M)=0$, there exists by Theorem 3 an open set V such that $M \subset V \subset \bar{V} \subset U$ such that all $(n-1)$-cycles of $\bar{V}$ bound on $\bar{U}$. Let P' be an open set such that $p' \in P' \subset \overline{P'} \subset U'$ and such that if $F'=F(P')$, then the set $F=f^{-1}(F')$ lies in V. Let Q' be an open set such that $p' \in Q' \subset \overline{Q'} \subset P'$. As above, there exist cycles Z_i^{n-1}, $i=1, \cdots, k$, of F' forming a base for $(n-1)$-cycles of F' relative to homologies in $S'-\overline{Q'}$. And for each Z_i^{n-1} there exists a cycle γ_i^{n-1} on F such that $f(\gamma_i^{n-1})\sim Z_i^{n-1}$ on F'. But since $\gamma_i^{n-1}\sim 0$ on $\bar{U}$, hence on S, it follows from Lemma 4 that $\gamma_i^{n-1}\sim 0$ in $S-M$. Therefore each $Z_i^{n-1}\sim 0$

in $S'-p'$, and it follows that, as above, an open set R' exists such that $p' \in R' \subset Q'$ and all Z_i^{n-1} bound in $S'-\overline{R}'$. Thus S' is locally $(n-1)$-avoidable.

The necessity for assuming that S' is at most n-dimensional above may be avoided if the monotoneity condition on f is strengthened. We recall that for the Vietoris Mapping Theorem to hold when the coefficient group is not a field or an elementary compact topological group, it is necessary to phrase the monotoneity condition in terms of the individual coordinates of cycles (just as, for example, may be done with the r-*lc* condition; compare [**11**; 176, Defs. 1.1, 1.2]). In terms of the generalized Vietoris cycles such as Begle employed [**2**], the condition is defined as follows:

DEFINITION 3. A mapping f of a space X onto a space Y is a *Vietoris mapping of order* n if for each covering $\mathfrak{U}$ of X and $y \in Y$ there exists a refinement $\mathfrak{B}=\mathfrak{B}(\mathfrak{U}, y)$ of $\mathfrak{U}$ such that every r-cycle of $X(\mathfrak{B}) \wedge f^{-1}(y)$ [**11**; 131], $r \leqq n$, bounds on $X(\mathfrak{U}) \wedge f^{-1}(y)$. (By $X(\mathfrak{U})$ is denoted the complex consisting of all simplexes σ such that the vertices of σ are points of X and diameter of $\sigma < \mathfrak{U}$.)

When the coefficient group is a field or elementary compact group, this definition is equivalent to that of n-monotone. It will be convenient, then, to retain the term "n-monotone" with, however, a qualification regarding the coefficient group employed. Also, for working with Čech cycles, the definition is more suitable in the following form:

DEFINITION 3′. A mapping f of a space X onto a space Y is n-monotone over (an abelian group) G if for each covering $\mathfrak{U}$ of X, $y \in Y$ and $M=f^{-1}(y)$, there exists a refinement $\mathfrak{B}$ of $\mathfrak{U}$ such that for every r-cycle $Z^r(\mathfrak{B})$ over G, $r \leqq n$, on $\mathfrak{B} \wedge M$ the projection $\pi_{\mathfrak{U}\mathfrak{B}} Z^r(\mathfrak{B})$ bounds on $\mathfrak{U} \wedge M$.

A routine argument shows that the two Definitions 3 and 3′ are equivalent.

LEMMA 4. *If f is an n-monotone mapping over the additive group I of integers of a compact space S onto a Hausdorff space S', then f is n-monotone over every abelian group G.*

(Remark. As will be seen from the proof below, it is sufficient to assume the condition of the Definition 3′ only for $r=n$ and $n-1$.)

Proof. For $n=0$ the lemma follows at once since, as is easily shown, 0-monotone over any group G is equivalent to the connectedness of $f^{-1}(x)$ for all $x \in S'$.

For $n>0$ we proceed as follows (see Čech [4; 11–13], where a similar type of argument is employed for quite different purposes): Given a covering $\mathfrak{U}_1$ of S and $x \in S'$, $M-f^{-1}(x)$, we choose $\mathfrak{U}_2 > \mathfrak{U}_1$ such that for every n-cycle $Z^n(\mathfrak{U}_2)$ over I on $\mathfrak{U}_2 \wedge M$, the projection $\pi_{12}Z^n(\mathfrak{U}_2)$ thereof from $\mathfrak{U}_2$ to $\mathfrak{U}_1$ bounds on $\mathfrak{U}_1 \wedge M$; and $\mathfrak{U}_3 > \mathfrak{U}_2$ such that for every $(n-1)$-cycle $Z^{n-1}(\mathfrak{U}_3)$ over I on $\mathfrak{U}_3 \wedge M$, the projection $\pi_{23}Z^{n-1}(\mathfrak{U}_3)$ thereof bounds on $\mathfrak{U}_2 \wedge M$.

There exists a base for n-chains over I for the complex $\mathfrak{U}_3 \wedge M$ consisting of chains $C_i^n(\mathfrak{U}_3)$, $i=1, \cdots, \alpha_n$, such that

$$\partial C_i^n(\mathfrak{U}_3) = \eta_i^n C_i^{n-1}(\mathfrak{U}_3) , \qquad i=1, \cdots, \beta_n ,$$
$$\partial C_i^n(\mathfrak{U}_3) = 0 , \qquad i=\beta_n+1, \cdots, \alpha_n ,$$

where $0 \leqq \beta_n \leqq \min(\alpha_n, \alpha_{n-1})$.

Consider any cycle $Z^n(\mathfrak{U}_3)$ over G of $\mathfrak{U}_3 \wedge M$. Then

$$Z^n(\mathfrak{U}_3) = \sum_{i=1}^{\alpha_n} g_i C_i^n(\mathfrak{U}_3) , \qquad g_i \in G .$$

And since $Z^n(\mathfrak{U}_3)$ is a cycle,

$$\sum_{i=1}^{\beta_n} g_i \eta_i^n C_i^{n-1}(\mathfrak{U}_3) = 0 ,$$

implying that

$$(1) \qquad g_i \eta_i^n = 0 \qquad \text{for } 1 \leqq i \leqq \beta_n .$$

Also, since for $\beta_n+1 \leqq i \leqq \alpha_n$ the chain $C_i^n(\mathfrak{U}_3)$ is a cycle, there exist chains $H_i^{n+1}(\mathfrak{U}_1)$ over I of $\mathfrak{U}_1 \wedge M$ such that

$$(2) \qquad \partial H_i^{n+1}(\mathfrak{U}_1) = \pi_{12}\pi_{23} C_i^n(\mathfrak{U}_3) , \qquad \beta_n+1 \leqq i \leqq \alpha_n .$$

Furthermore there exist chains $D_i^n(\mathfrak{U}_2)$ over I of $\mathfrak{U}_2 \wedge M$ such that

$$\partial D_i^n(\mathfrak{U}_2) = \pi_{23} C_i^{n-1}(\mathfrak{U}_3) , \qquad 1 \leqq i \leqq \beta_n .$$

And since the chains $\pi_{23}C_i^n(\mathfrak{U}_3) - \eta_i^n D_i^n(\mathfrak{U}_2)$ are cycles over I, we also have relations

$$(3) \qquad \partial H_i^{n+1}(\mathfrak{U}_1) = \pi_{12}\pi_{23} C_i^n(\mathfrak{U}_3) - \pi_{12}\eta_i^n D_i^n(\mathfrak{U}_2) , \qquad 1 \leqq i \leqq \beta_n$$

on $\mathfrak{U}_1 \wedge M$. From (1), (2), and (3) we get

$$\begin{aligned} \partial \sum_{i=1}^{\alpha_n} g_i H_i^{n+1}(\mathfrak{U}_1) &= \sum_{i=1}^{\alpha_n} \pi_{12}\pi_{23} g_i C_i^n(\mathfrak{U}_3) \\ &= \pi_{12}\pi_{23} Z^n(\mathfrak{U}_3) \\ &\sim \pi_{13} Z^n(\mathfrak{U}_3) . \end{aligned}$$

on $\mathfrak{U}_1 \wedge M$.

MAIN THEOREM B. *Let S be an orientable n-gcm and $f\colon S\to S'$ a continuous mapping of S, $(n-1)$-monotone over the integers, onto a finite-dimensional nondegenerate Hausdorff space S'. Then S' is an orientable n-gcm of the some homology type as S.*

Proof. The defining properties of an orientable n-*gcm* S utilize an algebraic field $\mathscr{F}$ as coefficient domain, and in particular specify that if F is a proper closed subset of S, then $H^n(F;\mathscr{F})=0$. It follows that since S' is nondegenerate, f is n-monotone as defined in Definition 1, and consequently [2; 542–3] is n-monotone over $\mathscr{F}$ as defined in Definition 3′. Furthermore, f is n-monotone over I. For it is trivial that n-monotoneity over a cofinal system of coverings of a space is sufficient for n-monotoneity, and S has a cofinal system Σ of coverings of dimension n; and since a cycle $Z^n(\mathfrak{B})$, $\mathfrak{B}\in\Sigma$, over I is a fortiori a cycle over $\mathscr{F}$, for a projection $\pi_{\mathfrak{U}\mathfrak{B}}Z^n(\mathfrak{B})$, $\mathfrak{U}\in\Sigma$, to bound implies $\pi_{\mathfrak{U}\mathfrak{B}}Z^n(\mathfrak{B})=0$. We conclude then that f is n-monotone over I.

Now suppose the dimension, dim $S',>n$. Then ([6]; [1]) there exists a closed set $C\subset S'$ and cycle Z^n over R_1 (the additive group of the reals mod 1) such that $Z^n\sim 0$ on S' but $Z^n\nsim 0$ on C. As f is n-monotone over R_1 by Lemma 4, there exists [2; § 5] a cycle γ^n on $f^{-1}(C)$ such that $f(\gamma^n)\sim Z^n$ on C. But since $Z^n\sim 0$ on S', it follows [2; 542] that $\gamma^n\sim 0$ on S. As S is n-dimensional, this implies $\gamma^n=0$ and a fortiori that $\gamma^n\sim 0$ on C and consequently $f(\gamma^n)\sim 0$ on C, implying $Z_n\sim 0$ on C, contrary to the choice of Z^n.

The theorem now follows from Main Theorem A, since by Lemma 4, γ is $(n-1)$-monotone over $\mathscr{F}$.

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