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**A SUBSTITUTION THEOREM FOR THE LAPLACE
TRANSFORMATION AND ITS GENERALIZATION TO
TRANSFORMATIONS WITH SYMMETRIC KERNEL**

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A SUBSTITUTION THEOREM FOR THE LAPLACE TRANSFORMATION AND ITS GENERALIZATION TO TRANSFORMATIONS WITH SYMMETRIC KERNEL

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In the problem of the derivation of images of functions under the Laplace transformation, the question arises as to the type of image produced if t is replaced by $g(t)$ in the original. Specific examples have been given by Erdélyi [3, vol. I §§ 4.1, 5.1, 6.1], Doetsch [1, 75-80], McLachlan, Humbert, and Poli [6, pp. 11-13] and [7, pp. 15-20], and Labin [5, p. 41] and a general formula is also listed by Doetsch [1, 75-80].

The Laplace transformation will be taken as

$$f(s) = \int_0^{\infty} e^{-st} F(t) dt$$

in which the integral is taken in the Lebesgue sense and which, as suggested by Doetsch [2, vol. I, p. 44], will be denoted by

$$F(t) \underset{t}{\circ} \underset{s}{\bullet}^{\mathcal{L}} f(s).$$

(The symbol will be read “ $F(t)$ has a Laplace transform $f(s)$ ”.)

THEOREM 1. *If*

(i) $k, g,$ and the inverse function $h = g^{-1}$ are single-valued analytic functions, real on $(0, \infty)$, and such that $g(0) = 0$ and $g(\infty) = \infty$ (or $g(0) = \infty$ and $g(\infty) = 0$);

(ii) $F(t) \underset{t}{\circ} \underset{s}{\bullet}^{\mathcal{L}} f(s)$ and this Laplace integral converges for $0 < \Re s$;

(iii) there exists a function $\Phi(s, u)$, $\Phi(s, u) \underset{u}{\circ} \underset{p}{\bullet}^{\mathcal{L}} \phi(s, p)$ and this Laplace integral converges for $0 < \Re p$, and $\phi(s, p) = e^{-sh(p)} k[h(p)] |h'(p)|$; and

(iv) $\int_0^{\infty} \left[\int_0^{\infty} |e^{-uv} \Phi(s, u) F(p)| du \right] dp$ converges for $a < \Re s$;

then

$$k(t) F[g(t)] \underset{t}{\circ} \underset{s}{\bullet}^{\mathcal{L}} \int_0^{\infty} \Phi(s, u) f(u) du$$

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and this Laplace integral converges for $a < \Re s$.

Proof. From (iii) and (iv) it follows that

$$\int_0^{\infty} e^{-sh(p)} k[h(p)] |h'(p)| F(p) dp$$

is absolutely convergent for $a < \Re s$. There are two cases to be considered. Since from (i) both g and h are single valued, h is monotonic.

Case 1. If $g(0) = 0$ and $g(\infty) = \infty$, then $0 \leq h'(p)$.

Case 2. If $g(0) = \infty$ and $g(\infty) = 0$, then $h'(p) \leq 0$.

In either case, therefore, if the substitution $t = h(p)$ is made in the integral

$$\int_0^{\infty} e^{-st} k(t) F[g(t)] dt,$$

then

$$k(t) F[g(t)] \underset{t}{\overset{s}{\circ}} \int_0^{\infty} e^{-sh(p)} k[h(p)] |h'(p)| F(p) dp.$$

From (iii) $\Phi(s, u)$ can be introduced and by (iv) the order of integration changed so that

$$k(t) F[g(t)] \underset{t}{\overset{s}{\circ}} \int_0^{\infty} \left[\int_0^{\infty} e^{-up} F(p) dp \right] \Phi(s, u) du.$$

Finally, from (ii)

$$k(t) F[g(t)] \underset{t}{\overset{s}{\circ}} \int_0^{\infty} \Phi(s, u) f(u) du.$$

To show that there are functions $\phi(s, p)$ as assumed in (iii), let, for example, $g(t) = t^2$ and $k(t) = 1$ so that

$$\Phi(s, p) = (4p)^{-1/2} e^{-sp^{1/2}}$$

and

$$\phi(s, u) = (4\pi u)^{-1/2} e^{-s^2/4u}.$$

From this the known relation

$$F(t^2) \underset{t}{\overset{s}{\circ}} \int_0^{\infty} (4\pi u)^{-1/2} e^{-s^2/4u} f(u) du$$

is obtained.

Special cases of $k(t)$ will sometimes simplify the image of $\Phi(s, u)$. If $k(t) = |g'(t)|K[g(t)]$, then

$$\Phi(s, u) \underset{u}{\circ} \underset{p}{\bullet} \mathcal{L} K(p) e^{-s h(p)} .$$

If $k(t) = |g'(t)|[g(t)]^c$, then

$$\Phi(s, u) \underset{u}{\circ} \underset{p}{\bullet} p^c e^{-s h(p)} .$$

In the proof of Theorem 1 it is noted that the only important property required of the kernel is that it be symmetric. Therefore consider the transformation

$$f(s) = \int_a^b K(s, t) F(t) dt$$

in which the integral is taken in the Lebesgue sense and in which the interval (a, b) may be unbounded. This transformation will be called the \mathcal{J} -transform and denoted by

$$F(t) \underset{t}{\circ} \underset{s}{\bullet} \mathcal{J} f(s) ,$$

The following theorem is for this transformation with symmetric kernel.

THEOREM 2. *If*

(i) k, g , and $h = g^{-1}$ are single-valued analytic functions, real on (a, b) , and such that $g(a) = a$ and $g(b) = b$ (or $g(a) = b$ and $g(b) = a$);

(ii) $F(t) \underset{t}{\circ} \underset{s}{\bullet} \mathcal{J} f(s)$ and this transformation integral converges for $a < s < b$;

(iii) there exists a function $\Phi(s, u)$, $\Phi(s, u) \underset{u}{\circ} \underset{p}{\bullet} \mathcal{J} \phi(s, p)$, this transformation integral converges for $a < s < b$, and

$$\phi(s, p) = K[s, h(p)] k[h(p)] |h'(p)| ;$$

(iv)

$$\int_a^b \left[\int_a^b |K(u, p) \Phi(s, u) F(p)| du \right] dp$$

converges for $s = s_0$; and

(v) $K(u, p) \equiv K(p, u)$; then $k(t) F[g(t)] \underset{t}{\circ} \underset{s}{\bullet} \mathcal{J} \Phi(s, u) f(u) du$ and this

transformation integral converges for $s=s_0$.

The proof follows in a manner similar to that of Theorem 1.

Formulas which hold provided $F(t)$ satisfies (ii) or (iv) of the theorem can be obtained for various transforms for specific $k(s)$ and $g(s)$ with the aid of tables [3, formulas 14.1(6), 8.12(10), 5.5(6)].

Formula 1. For the Stieltjes transformation $K(s, t)=(s+t)^{-1}$.

$$t^{b+1}F(at^2) \underset{t}{\circ} \underset{s}{-\bullet} \overset{\mathcal{L}}{\int_0^\infty} \frac{(u/a)^{b/2} \left[(u/a) \cos b\pi/2 - s \sin b\pi/2 \right]}{2\pi a [s^2 + u/a]} f(u) du$$

for a positive.

Formula 2. For the Hankel transformation $K(s, t)=J_\nu(st)(st)^{1/2}$

$$t^{-2}F(a/t) \underset{t}{\circ} \underset{s}{-\bullet} a^{-1} \overset{\mathcal{H}}{\int_0^\infty} \sqrt{aus} J_{2\nu}(2\sqrt{aus}) f(u) du$$

for $-1/2 < \nu$ and a positive.

The Laplace transformation will be considered in the next two formulas.

Formula 3.

$$(t+b/a)^d F(at^2+2bt)$$

$$\underset{t}{\circ} \underset{s}{-\bullet} (1/2\pi) e^{bs/a} \overset{\mathcal{L}}{\int_0^\infty} e^{-b^2s/a} e^{-s^2/2au} (\sqrt{2au})^{-d-1} D_d(s/\sqrt{2au}) f(u) du$$

for a and b positive and in which $D_d(z)$ is the parabolic cylinder function. The range of permissible values of d will depend, according to (iv), on the particular function $F(u)$.

Formula 4.

$$t^{a-1}F(at^{-b}) \underset{t}{\circ} \underset{s}{-\bullet} b^{-1} \overset{\mathcal{L}}{\int_0^\infty} (au)^{a/b} \phi[1/b, (d+b)/b; -s(au)^{1/b}] f(u) du$$

for a and b positive and in which $\phi(A, B; Z)$ is Wright's function [4, vol. 3, §18.1]. The range of permissible values of d will depend, according to (iv), on the particular function $F(u)$. In the special case $b=1$ the formula becomes

$$t^{a-1}F(a/t) \underset{t}{\circ} \underset{s}{-\bullet} \overset{\mathcal{L}}{\int_0^\infty} (\sqrt{au/s})^d J_d(2\sqrt{aus}) f(u) du .$$

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