A SUBSTITUTION THEOREM FOR THE LAPLACE TRANSFORMATION AND ITS GENERALIZATION TO TRANSFORMATIONS WITH SYMMETRIC KERNEL

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In the problem of the derivation of images of functions under the Laplace transformation, the question arises as to the type of image produced if \( t \) is replaced by \( g(t) \) in the original. Specific examples have been given by Erdélyi [3, vol. I §§ 4.1, 5.1, 6.1], Doetsch [1, 75-80], McLachlan, Humbert, and Poli [6, pp. 11-13] and [7, pp. 15-20], and Labin [5, p. 41] and a general formula is also listed by Doetsch [1, 75-80].

The Laplace transformation will be taken as

\[ f(s) = \int_0^\infty e^{-st}F(t)\,dt \]

in which the integral is taken in the Lebesgue sense and which, as suggested by Doetsch [2, vol. I, p. 44], will be denoted by

\[ F(t) \overset{\mathcal{L}}{\rightarrow} f(s). \]

(The symbol will be read "\( F(t) \) has a Laplace transform \( f(s) \).")

**Theorem 1.** If

(1) \( k, g, \) and the inverse function \( h = g^{-1} \) are single-valued analytic functions, real on \((0, \infty)\), and such that \( g(0) = 0 \) and \( g(\infty) = \infty \) (or \( g(0) = \infty \) and \( g(\infty) = 0 \));

(2) \( F(t) \overset{\mathcal{L}}{\rightarrow} f(s) \) and this Laplace integral converges for \( 0 < \Re s \);

(3) there exists a function \( \Phi(s, u) \overset{\mathcal{L}}{\rightarrow} \phi(s, p) \) and this Laplace integral converges for \( 0 < \Re p \), and \( \phi(s, p) = e^{-sh(p)}k[h(p)]|h'(p)| \); and

(4) \( \int_0^a \left[ \int_0^\infty |e^{-u\Phi(s, u)}F(p)|\,dp \right] du \) converges for \( a < \Re s \);

then

\[ k(t) F[g(t)] \overset{\mathcal{L}}{\rightarrow} \int_0^\infty \Phi(s, u)f(u)\,du \]

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1529
and this Laplace integral converges for \( a < \Re s \).

**Proof.** From (iii) and (iv) it follows that

\[
\int_0^\infty e^{-st}k[t]F[g(t)]dt
\]

is absolutely convergent for \( a < \Re s \). There are two cases to be considered. Since from (i) both \( g \) and \( h \) are single valued, \( h \) is monotonic.

**Case 1.** If \( g(0) = 0 \) and \( g(\infty) = \infty \), then \( 0 \leq h'(p) \).

**Case 2.** If \( g(0) = \infty \) and \( g(\infty) = \infty \), then \( h'(p) \leq 0 \).

In either case, therefore, if the substitution \( t = h(p) \) is made in the integral

\[
\int_0^\infty e^{-st}k(t)F[g(t)]dt,
\]

then

\[
k(t)F[g(t)]\int_0^\infty e^{-sh(p)}k[h(p)]|h'(p)|F(p)dp.
\]

From (iii) \( \Phi(s, u) \) can be introduced and by (iv) the order of integration changed so that

\[
k(t)F[g(t)]\int_0^\infty \left[ \int_0^\infty e^{-up}F(p)dp \right]\Phi(s, u)du.
\]

Finally, from (ii)

\[
k(t)F[g(t)]\int_0^\infty \Phi(s, u)f(u)du.
\]

To show that there are functions \( \phi(s, p) \) as assumed in (iii), let, for example, \( g(t) = t^2 \) and \( k(t) = 1 \) so that

\[
\phi(s, p) = (4p)^{-1/2}e^{-sp^{1/2}}
\]

and

\[
\phi(s, u) = (4\pi u)^{-1/2}e^{-s\pi u}f(u).
\]

From this the known relation

\[
F(t^2)\int_0^\infty (4\pi u)^{-1/2}e^{-s\pi u}f(u)du.
\]
is obtained.

Special cases of \( k(t) \) will sometimes simplify the image of \( \Phi(s, u) \). If \( k(t) = |g'(t)| K[g(t)] \), then

\[
\Phi(s, u) \overset{\mathcal{L}}{\circ} \bullet \phi(s, u) - K(p) e^{-s h(p)}.
\]

If \( k(t) = |g'(t)| [g(t)]^c \), then

\[
\Phi(s, u) \overset{\mathcal{L}}{\circ} \bullet \phi^c(s, u) - K(p) e^{-s h(p)}.
\]

In the proof of Theorem 1 it is noted that the only important property required of the kernel is that it be symmetric. Therefore consider the transformation

\[
f(s) = \int_a^b K(s, t) F(t) \, dt
\]

in which the integral is taken in the Lebesgue sense and in which the interval \((a, b)\) may be unbounded. This transformation will be called the \( \mathcal{T} \)-transform and denoted by

\[
F(t) \overset{\mathcal{T}}{\circ} \bullet f(s),
\]

The following theorem is for this transformation with symmetric kernel.

**Theorem 2.** If

(i) \( k, g, \) and \( h = g^{-1} \) are single-valued analytic functions, real on \((a, b)\), and such that \( g(a) = a \) and \( g(b) = b \) (or \( g(a) = b \) and \( g(b) = a \));

(ii) \( F(t) \overset{\mathcal{T}}{\circ} \bullet f(s) \) and this transformation integral converges for \( a < s < b \);

(iii) there exists a function \( \Phi(s, u), \Phi(s, u) \overset{\mathcal{T}}{\circ} \bullet \phi(s, p) \), this transformation integral converges for \( a < s < b \), and

\[
\Phi(s, u) = K[s, h(p)] k[h(p)] |h'(p)|;
\]

(iv)

\[
\int_a^b \left[ \int_a^b |K(u, p) \phi(s, u) F(p)| \, du \right] \, dp
\]

converges for \( s = s_0 \); and

(v) \( K(u, p) = K(p, u) \); then \( k(t) F[g(t)] \overset{\mathcal{T}}{\circ} \bullet \int_a^b \phi(s, u) f(u) \, du \) and this
transformation integral converges for \( s = s_0 \).

The proof follows in a manner similar to that of Theorem 1.

Formulas which hold provided \( F(t) \) satisfies (ii) or (iv) of the theorem can be obtained for various transforms for specific \( k(s) \) and \( g(s) \) with the aid of tables [3, formulas 14.1(6), 8.12(10), 5.5(6)].

**Formula 1.** For the Stieltjes transformation \( K(s, t) = (s+t)^{-1} \).

\[
t^{b+1}F(at^b) = \mathcal{L}_t \int_0^\infty \left( \frac{u^2}{2\pi a} \right)^{b/2} \left[ \cos \frac{b\pi}{2} - s \sin \frac{b\pi}{2} \right] f(u) du
\]

for \( a \) positive.

**Formula 2.** For the Hankel transformation \( K(s, t) = J_\nu(st)(st)^{1/2} \).

\[
t^{-2}F(a/t) = \mathcal{H}_t \int_0^\infty \sqrt{aus}J_{\nu}(2\sqrt{aus}) f(u) du
\]

for \(-1/2 < \nu \) and \( a \) positive.

The Laplace transformation will be considered in the next two formulas.

**Formula 3.**

\[
(t+b/a)^a F(at^a + 2bt)
\]

\[
\mathcal{L}_t \int_0^\infty e^{-s^2/s/a} e^{-s^2/2au} (2\sqrt{2au})^{-a-1} D_d(s/\sqrt{2au}) f(u) du
\]

for \( a \) and \( b \) positive and in which \( D_d(z) \) is the parabolic cylinder function. The range of permissible values of \( d \) will depend, according to (iv), on the particular function \( F(u) \).

**Formula 4.**

\[
t^{d-1}F(at^{-b}) = \mathcal{L}_t \int_0^\infty (au)^{d/b} \phi[1/b, (d+b)/b; -s(au)^{1/b}] f(u) du
\]

for \( a \) and \( b \) positive and in which \( \phi(A, B; Z) \) is Wrights' function [4, vol. 3, §18.1]. The range of permissible values of \( d \) will depend, according to (iv), on the particular function \( F(u) \). In the special case \( b=1 \) the formula becomes

\[
t^{d-1}F(a/t) = \mathcal{L}_t \int_0^\infty (\sqrt{au}/s)^d J_d(2\sqrt{aus}) f(u) du
\]

for \( a \) and \( b \) positive and in which \( \phi(A, B; Z) \) is Wrights' function [4, vol. 3, §18.1]. The range of permissible values of \( d \) will depend, according to (iv), on the particular function \( F(u) \). In the special case \( b=1 \) the formula becomes

\[
t^{d-1}F(a/t) = \mathcal{L}_t \int_0^\infty (\sqrt{au}/s)^d J_d(2\sqrt{aus}) f(u) du
\]
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Robert George Buschman, *A substitution theorem for the Laplace transformation and its generalization to transformations with symmetric kernel* ......................................................... 1529

S. D. Conte, *Numerical solution of vibration problems in two space variables* ................................................................. 1535

Paul Dedecker, *A property of differential forms in the calculus of variations* ................................................................. 1545

H. Delange and Heini Halberstam, *A note on additive functions* ................................................................. 1551

Jerald L. Ericksen, *Characteristic direction for equations of motion of non-Newtonian fluids* ................................................................. 1557

Avner Friedman, *On two theorems of Phragmén-Lindelöf for linear elliptic and parabolic differential equations of the second order* ................................................................. 1563

Ronald Kay Getoor, *Additive functionals of a Markov process* ................................................................. 1577

U. C. Guha, *(γ, k)-summability of series* ................................................................. 1593

Alvin Hausner, *The tauberian theorem for group algebras of vector-valued functions* ................................................................. 1603

Lester J. Heider, *T-sets and abstract (L)-spaces* ................................................................. 1611

Melvin Henriksen, *Some remarks on a paper of Aronszajn and Panitchpakdi* ................................................................. 1619


Robert Osserman, *On the inequality Δu ≥ f (u)* ................................................................. 1641

Calvin R. Putnam, *On semi-normal operators* ................................................................. 1649

Binyamin Schwarz, *Bounds for the principal frequency of the non-homogeneous membrane and for the generalized Dirichlet integral* ................................................................. 1653

Edward Silverman, *Morrey’s representation theorem for surfaces in metric spaces* ................................................................. 1677

V. N. Singh, *Certain generalized hypergeometric identities of the Rogers-Ramanujan type. II* ................................................................. 1691

R. J. Smith, *A determinant in continuous rings* ................................................................. 1701

Drury William Wall, *Sub-quasigroups of finite quasigroups* ................................................................. 1711

Sadayuki Yamamuro, *Monotone completeness of normed semi-ordered linear spaces* ................................................................. 1715

C. T. Rajagopal, *Simplified proofs of “Some Tauberian theorems” of Jakimovski: Addendum and corrigendum* ................................................................. 1727

N. Aronszajn and Prom Panitchpakdi, *Correction to: “Extension of uniformly continuous transformations in hyperconvex metric spaces”* ................................................................. 1729

Alfred Huber, *Correction to: “The reflection principle for polyharmonic functions”* ................................................................. 1731