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A SUBSTITUTION THEOREM FOR THE LAPLACE TRANSFORMATION AND ITS GENERALIZATION TO TRANSFORMATIONS WITH SYMMETRIC KERNEL

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In the problem of the derivation of images of functions under the Laplace transformation, the question arises as to the type of image produced if t is replaced by $g(t)$ in the original. Specific examples have been given by Erdélyi [3, vol. I §§ 4.1, 5.1, 6.1], Doetsch [1, 75–80], McLachlan, Humbert, and Poli [6, pp. 11–13] and [7, pp. 15–20], and Labin [5, p. 41] and a general formula is also listed by Doetsch [1, 75–80].

The Laplace transformation will be taken as

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

in which the integral is taken in the Lebesgue sense and which, as suggested by Doetsch [2, vol. I, p. 44], will be denoted by

$$F(t) \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} f(s)\,.$$

(The symbol will be read " $F(t)$ has a Laplace transform $f(s)$ ".)

THEOREM 1. *If*

(i) *k, g, and the inverse function $h = g^{-1}$ are single-valued analytic functins, real on $(0, \infty)$, and such that $g(0) = 0$ and $g(\infty) = \infty$ (or $g(0) = \infty$ and $g(\infty) = 0$)*;

(ii) $F(t) \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} f(s)$ *and this Laplace integral converges for* $0 < \Re s$;

(iii) *there exists a function $\Phi(s, u)$, $\Phi(s, u) \underset{u\quad p}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} \phi(s, p)$ and this Laplace integral converges for $0 < \Re p$, and $\phi(s, p) = e^{-sh(p)} k[h(p)]\, |h'(p)|$; and*

(iv) $\displaystyle\int_0^\infty \left[\int_0^\infty |e^{-up}\Phi(s, u)F(p)|\, du\right] dp$ *converges for* $a < \Re s$;

then

$$k(t)\, F[g(t)] \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} \int_0^\infty \Phi(s, u) f(u)\, du$$

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and this Laplace integral converges for $a < \Re s$.

Proof. From (iii) and (iv) it follows that

$$\int_0^\infty e^{-sh(p)} k[h(p)]\,|h'(p)|\,F(p)dp$$

is absolutely convergent for $a < \Re s$. There are two cases to be considered. Since from (i) both g and h are single valued, h is monotonic.

Case 1. If $g(0)=0$ and $g(\infty)=\infty$, then $0 \leqq h'(p)$.

Case 2. If $g(0)=\infty$ and $g(\infty)=0$, then $h'(p) \leqq 0$.
In either case, therefore, if the substitution $t=h(p)$ is made in the integral

$$\int_0^\infty e^{-st} k(t) F[g(t)] dt\,,$$

then

$$k(t)\,F[g(t)] \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} \int_0^\infty e^{-sh(p)} k[h(p)]\,|h'(p)|\,F(p)\,dp\,.$$

From (iii) $\Phi(s, u)$ can be introduced and by (iv) the order of integration changed so that

$$k(t)\,F[g(t)] \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} \int_0^\infty \left[\int_0^\infty e^{-up} F(p)\,dp\right] \Phi(s, u)\,du\,.$$

Finally, from (ii)

$$k(t)\,F[g(t)] \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} \int_0^\infty \Phi(s, u) f(u)\,du\,.$$

To show that there are functions $\phi(s, p)$ as assumed in (iii), let, for example, $g(t)=t^2$ and $k(t)=1$ so that

$$\Phi(s, p)=(4p)^{-1/2} e^{-sp^{1/2}}$$

and

$$\phi(s, u)=(4\pi u)^{-1/2} e^{-s^2/4u}.$$

From this the known relation

$$F(t^2) \underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}} \int_0^\infty (4\pi u)^{-1/2} e^{-s^2/4u} f(u) du$$

is obtained.

Special cases of $k(t)$ will sometimes simplify the image of $\Phi(s, u)$. If $k(t)=|g'(t)|K[g(t)]$, then

$$\Phi(s, u) \underset{u\ \ p}{\overset{\mathscr{L}}{\circ\!\!-\!\!\bullet}} K(p)\, e^{-s\,h(p)}\ .$$

If $k(t)=|g'(t)|[g(t)]^c$, then

$$\Phi(s, u) \underset{u\ \ p}{\overset{\mathscr{L}}{\circ\!\!-\!\!\bullet}} p^c\, e^{-s\,h(p)}\ .$$

In the proof of Theorem 1 it is noted that the only important property required of the kernel is that it be symmetric. Therefore consider the transformation

$$f(s)=\int_a^b K(s, t)\, F(t)\, dt$$

in which the integral is taken in the Lebesgue sense and in which the interval (a, b) may be unbounded. This transformation will be called the $\mathscr{T}$-transform and denoted by

$$F(t) \underset{t\ \ s}{\overset{\mathscr{T}}{\circ\!\!-\!\!\bullet}} f(s)\ ,$$

The following theorem is for this transformation with symmetric kernel.

THEOREM 2. *If*

(i) *k, g, and $h=g^{-1}$ are single-valued analytic functions, real on (a, b), and such that $g(a)=a$ and $g(b)=b$ (or $g(a)=b$ and $g(b)=a$);*

(ii) *$F(t) \underset{t\ \ s}{\overset{\mathscr{T}}{\circ\!\!-\!\!\bullet}} f(s)$ and this transformation integral converges for $a<s<b$;*

(iii) *there exists a function $\Phi(s, u)$, $\Phi(s, u) \underset{u\ \ p}{\overset{\mathscr{T}}{\circ\!\!-\!\!\bullet}} \phi(s, p)$, this transformation integral converges for $a<s<b$, and*

$$\phi(s, p)=K[s, h(p)]\, k[h(p)]\, |h'(p)|\ ;$$

(iv)

$$\int_a^b \left[\int_a^b |K(u, p)\, \Phi(s, u)\, F(p)|\, du \right] dp$$

converges for $s=s_0$; and

(v) *$K(u, p)\equiv K(p, u)$; then $k(t)\, F[g(t)] \underset{t\ \ s}{\overset{\mathscr{T}}{\circ\!\!-\!\!\bullet}} \int_a^b \Phi(s, u)\, f(u)\, du$ and this* *transformation integral converges for* $s=s_0$.

The proof follows in a manner similar to that of Theorem 1.

Formulas which hold provided $F(t)$ satisfies (ii) or (iv) of the theorem can be obtained for various transforms for specific $k(s)$ and $g(s)$ with the aid of tables [3, formulas 14.1(6), 8.12(10), 5.5(6)].

Formula 1. For the Stieltjes transformation $K(s, t)=(s+t)^{-1}$.

$$t^{b+1}F(at^2)\underset{t\quad s}{\overset{\mathscr{S}}{\circ\!-\!\bullet}}\int_0^\infty \frac{(u/a)^{b/2}}{2\pi a}\left[\frac{(u/a)\cos b\pi/2 - s\sin b\pi/2}{s^2+u/a}\right]f(u)\,du$$

for a positive.

Formula 2. For the Hankel transformation $K(s, t)=J_\nu(st)(st)^{1/2}$

$$t^{-2}F(a/t)\underset{t\quad s}{\overset{\mathscr{H}}{\circ\!-\!\bullet}}a^{-1}\int_0^\infty \sqrt{aus}\,J_{2\nu}(2\sqrt{aus})f(u)\,du$$

for $-1/2<\nu$ and a positive.

The Laplace transformation will be considered in the next two formulas.

Formula 3.

$$(t+b/a)^d F(at^2+2bt)$$

$$\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}(1/2\pi)\,e^{bs/a}\int_0^\infty e^{-b^2s/a}\,e^{-s^2/8au}(\sqrt{2au})^{-d-1}D_d(s/\sqrt{2au})f(u)\,du$$

for a and b positive and in which $D_d(z)$ is the parabolic cylinder function. The range of permissible values of d will depend, according to (iv), on the particular function $F(u)$.

Formula 4.

$$t^{d-1}F(at^{-b})\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}b^{-1}\int_0^\infty (au)^{d/b}\phi[1/b,\,(d+b)/b;\,-s(au)^{1/b}]f(u)\,du$$

for a and b positive and in which $\phi(A, B; Z)$ is Wrights' function [4, vol. 3, § 18.1]. The range of permissible values of d will depend, according to (iv), on the particular function $F(u)$. In the special case $b=1$ the formula becomes

$$t^{d-1}F(a/t)\underset{t\quad s}{\overset{\mathscr{L}}{\circ\!-\!\bullet}}\int_0^\infty (\sqrt{au/s})^d\,J_d(2\sqrt{aus})f(u)\,du\,.$$

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