

# Pacific Journal of Mathematics

**A PROPERTY OF DIFFERENTIAL FORMS IN THE CALCULUS  
OF VARIATIONS**

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# A PROPERTY OF DIFFERENTIAL FORMS IN THE CALCULUS OF VARIATIONS

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1. In the classical problems involving a simple integral

$$(1) \quad I_1 = \int L(t, q^i, \dot{q}^i) dt, \quad i=1, \dots, n,$$

one is led to the consideration of the Pfaffian form

$$(2) \quad \omega = L dt + \frac{\partial L}{\partial \dot{q}^i} \omega^i = \frac{\partial L}{\partial \dot{q}^i} dq^i - \left( \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L \right) dt$$

where

$$\omega^i = dq^i - \dot{q}^i dt.$$

For example this form  $\omega$  is the one which gives rise to the "relative integral invariant" of E. Cartan.

In a recent note [1] L. Auslander characterizes the form  $\omega$  by a theorem equivalent to the following one.

**THEOREM 1.** *Among all semi-basic forms  $\theta$  such that*

$$(3) \quad \theta \equiv L dt \pmod{\omega^i}$$

*the form  $\omega$  of (2) is the only one satisfying the condition*

$$(4) \quad d\theta \equiv 0 \pmod{\omega^i}.$$

In this, a *semi-basic form* is a form for which the local expression contains only the differentials of  $t, q^i$  (not of  $\dot{q}^i$ ). The integral  $I$  is defined over an arc  $\bar{c}$  of a space  $\mathscr{W}$  with local coordinates  $t, q^i, \dot{q}^i$  satisfying the equations  $\omega^i = 0$ : Therefore in (1) the form  $L dt$  may be replaced by any  $\theta$  satisfying (3).

Condition (4) is a special case of a congruence discovered by Lepage [5]. *The purpose of the present note is to give a natural reason for this congruence which goes beyond its nice algebraic expression.*

Let us observe that the space  $\mathscr{W}$  is the manifold of 1-dimensional contact elements of a manifold  $\mathscr{V}$  with local coordinates  $t, q^i$ . The map

$$(t, q^i, \dot{q}^i) \rightarrow (t, q^i)$$

is then the local expression of the natural projection  $\pi: \mathscr{W} \rightarrow \mathscr{V}$ . We

remark that we do not integrate (1) on any arc  $\bar{c}$  in  $\mathscr{W}$  satisfying  $\omega^i=0$  but on such an arc the projection  $c$  of which in  $\mathscr{V}$  is regular.

2. Let  $U$  be the domain in  $\mathscr{V}$  of the coordinates  $t, q^i$ ; then the  $t, q^i, \dot{q}^i$  are defined in an open subset  $W \subset \mathscr{W}$  of projection  $\pi(W)=U$ . If we denote by  $L_i$   $n$  real undetermined, we have coordinates  $t, q^i, \dot{q}^i, L_i$  in  $W \times R^n$ ; we then define in this product the Pfaffian form

$$(5) \quad \Omega_W = L dt + L_i \omega^i .$$

Now, let us cover  $\mathscr{W}$  with open sets  $W, W', \dots$ ; this way we get a family of products  $W \times R^n, W' \times R^n, \dots$  with forms  $\Omega_W, \Omega_{W'}, \dots$ . Using fibre bundle techniques, one proves that over a non-empty intersection  $W \cap W'$  the products  $W \times R^n$  and  $W' \times R^n$  can be glued together in such a way that the forms induced on  $W \cap W' \times R^n$  coincide. This yields a fibre bundle  $E(\mathscr{W}, R^n)$  over  $\mathscr{W}$  as base, with fibre  $R^n$ . This bundle is covered by open subsets isomorphic with the products  $W \times R^n$  and in which the  $t, q^i, \dot{q}^i, L_i$  are local coordinates; there is also on  $E$  a global Pfaffian form  $\Omega$  of local expression (5). Combining the projections  $E \rightarrow \mathscr{W}$  and  $\mathscr{W} \rightarrow \mathscr{V}$  we obtain a map  $E \rightarrow \mathscr{V}$  locally defined by

$$(t, q^i, \dot{q}^i, L_i) \rightarrow (t, q^i) .$$

We want to characterize in  $E$  the extremal arcs  $c^*$  of  $\int \Omega$  which have a regular projection in  $\mathscr{V}$ .

An extremal arc  $c^*$  of  $\int \Omega$  has to satisfy the local equations

$$\frac{\partial(d\Omega)}{\partial(dt)} = \frac{\partial(d\Omega)}{\partial(\omega^i)} = \frac{\partial(d\Omega)}{\partial(d\dot{q}^i)} = \frac{\partial(d\Omega)}{\partial(dL_i)} = 0 .$$

We have

$$d\Omega = \frac{\partial L}{\partial \dot{q}^i} \omega^i \wedge dt + \left( \frac{\partial L}{\partial \dot{q}^i} - L_i \right) d\dot{q}^i \wedge dt + dL_i \wedge \omega^i .$$

These equations are therefore

$$\omega^i = 0 , \quad \left( \frac{\partial L}{\partial \dot{q}^i} - L_i \right) dt = 0 , \quad \frac{\partial L}{\partial \dot{q}^i} dt - dL_i = 0 .$$

Since an arc  $c^*$  of regular projection in  $\mathscr{V}$  cannot satisfy simultaneously  $\omega^i=0$  and  $dt=0$  it has to lie in the submanifold  $F$  of  $E$  locally characterized by

$$\frac{\partial L}{\partial \dot{q}^i} = L_i$$

or equivalently by condition (4).

**THEOREM 2.** *Every arc  $c^*$  in  $E$  for which  $\int \Omega$  is stationary and the projection of which in  $\mathcal{V}$  is regular necessarily lies in the submanifold  $F$  of  $E$  locally defined by the congruence (4). Furthermore the projection  $c$  of  $c^*$  in  $\mathcal{V}$  extremizes in the classical sense the integral (1). Finally if  $c$  is a regular extremal arc of (1) in  $\mathcal{V}$  let  $c^*$  be the arc of  $F$  the projection  $\bar{c}$  of which in  $\mathcal{W}$  is the arc of tangent directions to  $c$ ; then  $c^*$  extremizes  $\int \Omega$ .*

**3. The submanifold  $F$**  can be identified with  $\mathcal{W}$  in an obvious way so that  $\mathcal{W}$  can be considered as a submanifold of  $E$ . Then clearly  $\Omega$  induces  $\omega$  on  $\mathcal{W}$ .

**THEOREM 3.** *If the integral (1) is regular there exists a (one-to-one) correspondence between the regular extremal arcs  $c$  in  $\mathcal{V}$  of (1) and the extremal arcs  $\bar{c}$  of  $\int \omega$  in  $\mathcal{W}$  which have a regular projection in  $\mathcal{V}$ . Starting from an extremal  $c$ , the corresponding  $\bar{c}$  is the arc the points of which are the tangent directions to  $c$ ; starting from  $\bar{c}$  the corresponding  $c$  is its projection in  $\mathcal{V}$ .*

In this statement, regularity of (1) means that the matrix  $(\partial^2 L / \partial \dot{q}^i \partial \dot{q}^j)$  is everywhere non singular.

Theorem 2 and 3 give a complete justification of condition (4). Theorem 3 was actually proved by E. Cartan [2]. These theorems are special cases of similar theorems involving multiple integrals and even those in which the function  $L$  depends on higher order contact elements. Theorem 2 was first proved by the author [3], as well as the alluded generalizations.

Combining Theorems 2 and 3 yields the following.

**THEOREM 4.** *In the regular case, every arc  $\bar{c}$  in  $\mathcal{W}$  of regular projection in  $\mathcal{V}$  which extremizes  $\int \omega$  with respect to variations confined to  $\mathcal{W}$  does also extremize  $\int \Omega$  with respect to variations in the larger space  $E$ .*

**4. There is a last question** to be answered: why in Theorem 1 restrict oneself to semi-basic forms?

We can only add to  $L.dt$  a linear combination of Pfaffian forms vanishing with  $\omega^i$ ; every such form is a linear combination of the  $\omega^i$

and is therefore semi-basic. Hence the restriction to semi-basic forms in Theorem 1 was actually redundant.

However, as mentioned above and as I have proved in various papers (e.g. [3, 4]), the above properties generalize to a multiple integral

$$(6) \quad I_p = \int L(t^\alpha, q^i, q_\alpha^i) dt, \\ dt = dt^1 \wedge \dots \wedge dt^p, \quad \alpha = 1, 2, \dots, p; \quad i = 1, 2, \dots, n,$$

to be integrated over a  $p$ -surface  $c$  defined by  $q^i = q^i(t^\alpha)$  and where  $q_\alpha^i$  stands for  $\partial q^i / \partial t^\alpha$ . Then  $\mathcal{V}$  is of dimension  $n+p$  and  $\mathcal{W}$  (which is geometrically the manifold of  $p$ -dimensional contact elements of  $\mathcal{V}$ ) is of dimension  $n+p+np$ . We can consider that we integrate (6) in  $\mathcal{W}$  over a  $p$ -surface  $\bar{c}$  of regular projection in  $\mathcal{V}$  and solution of the Pfaffian equations

$$\omega^i = dq^i - \sum q_\alpha^i dt^\alpha = 0.$$

Such a  $p$ -surface  $\bar{c}$  is formed of the contact elements of dimension  $p$  to a regular  $p$ -surface in  $\mathcal{V}$  and will be called a  $p$ -multiplicity.

Now in (6) we can add to  $L \cdot dt$  any  $p$ -form vanishing on all  $p$ -multiplicities and all such forms are no longer semi-basic if  $p > 1$ : for example  $d\omega^i \wedge dt^1 \wedge \dots \wedge dt^p$  is such one. Nevertheless, the semi-basic forms satisfying the Lepage congruences [5]:

$$(7) \quad \theta \equiv L dt \quad \text{mod } \omega^i,$$

$$(8) \quad d\theta \equiv 0 \quad \text{mod } \omega^i.$$

play an important role for a deeper reason which is actually a *transversality condition*. We briefly discuss this below referring the reader to my memoir [4] for further details.

5. Let  $\mathcal{K}$  be a  $p$ -dimensional manifold and  $K$  a domain of  $\mathcal{K}$  with regular boundary  $K$ . A map

$$c: K \rightarrow \mathcal{V}$$

is a domain of integration of (6); it gives rise canonically to a map

$$\bar{c}: K \rightarrow \mathcal{W}$$

such that for  $k \in K$ ,  $\bar{c}(k)$  is the contact element of dimension  $p$  to  $c$  at  $k$ . A *variation* (or *homotopy*) of  $c$  is a family of maps

$$c_t: K \rightarrow \mathcal{V}, \quad t \in R, \quad c_0 = c;$$

this yields a variation of  $\bar{c}$ :

$$\bar{c}_i: K \rightarrow \mathscr{W}.$$

We also define  $C: K \times R \rightarrow \mathscr{V}$ ,  $\bar{C}: K \times R \rightarrow \mathscr{W}$  by

$$C(k, t) = c_i(k), \quad \bar{C}(k, t) = \bar{c}_i(k).$$

The corresponding variation of  $\int \theta$  is then

$$\Delta = \int_{\bar{c}_i} \theta - \int_{\bar{c}_0} \theta$$

which may be expressed as a sum of two terms:

$$(9) \quad \Delta = \int_{\bar{c}_{0t}} d\theta + \int_{\lambda_{0t}\bar{c}} \theta.$$

The domains of integration  $\bar{C}_{0t}$  and  $\lambda_{0t}\bar{C}$  are the restrictions of  $\bar{C}$  to  $K \times I_{0t}$  and  $\dot{K} \times I_{0t}$  respectively (where  $I_{0t} = [0, t] \subset R$ ). We say that the variation  $\bar{C}$  is *transversal* to  $\theta$  if this form vanishes on  $\lambda\bar{C}$  (restriction of  $\bar{C}$  to  $\dot{K} \times R$ ). This being the case, the last integral (or boundary term) in (9) is zero.

Now the variations usually considered are those for which the restriction of  $C$  to  $\dot{K}$  is constant (fixed boundary variations): for those,  $\lambda\bar{C}$  has an everywhere non-regular projection in  $\mathscr{V}$ , so that every semi-basic form vanishes on  $\lambda\bar{C}$ . Therefore if we replace in (6)  $L \cdot dt$  by a semi-basic  $p$ -form  $\theta$  satisfying (7), all variations with fixed boundary are transversal to it. This would of course not be the case, should we add to  $L \cdot dt$  a non-semi-basic  $p$ -form vanishing on all  $p$ -multiplicities.

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