A NOTE ON ADDITIVE FUNCTIONS

H. DELANGE AND HEINI HALBERSTAM
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1. A real valued function \( f(n) \), defined on the set of natural numbers, is called additive if \( f(mn) = f(m) + f(n) \) whenever \((m, n) = 1\), and strongly additive if also \( f(p^\alpha) = f(p) \) for \( p \) prime and \( \alpha = 2, 3, \ldots \). We define

\[
A_n = \sum_{p \leq n} f(p)/p, \quad B_n = \sum_{p \leq n} f^2(p)/p,
\]

and we assume throughout that

\[
B_n \to \infty, \quad n \to \infty.
\]

Additive functions for which \( B_n = O(1) \) have already been discussed thoroughly in Erdös and Wintner [4]. They proved the following theorem:

Define

\[
f'(p) = \begin{cases} 1 & \text{for } |f(p)| > 1, \\ f(p) & \text{for } |f(p)| \leq 1. \end{cases}
\]

Then the additive function \( f(n) \) possesses a distribution function if, and only if, the series

\[
\sum_p f'(p)/p \quad \text{and} \quad \sum_p \{f'(p)^2\}/p
\]

converge.

Moreover, it follows from a general result of P. Lévy [10] that this distribution function is continuous if, and only if, the series \( \sum_{f(p) \neq 0} f(p)/p \) diverges. Surveys of this subject are given in Kac [7] and Kubilyus [9]. A comprehensive account is being prepared by H. N. Shapiro.

Our knowledge of functions subject to (2) is not as complete. Outstanding is the result of Erdös and Kac [3] which states that if

\[
f(p) = O(1),
\]

the distribution of

\[
\frac{f(m) - A_n}{B_n^{1/2}}, \quad m \leq n,
\]

is asymptotically Gaussian. In a recent note H. N. Shapiro [11] has shown that the theorem of Erdös and Kac remains true even when (3) is replaced by

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1551


\[(4) \lim_{n \to \infty} \frac{1}{\log n} \sum_{p \leq n} f^2(p)/p = 0 \quad \text{for every } \varepsilon > 0.\]

Since (4) is essentially the Lindeberg condition which is necessary and sufficient for the central limit theorem to hold, one is led to conjecture that (4) is not only the sufficient but also the necessary condition for the truth of the theorem of Erdős and Kac. However, it seems very difficult to establish the necessity (see Kubilyus [8] and Tanaka [12]).

Associated with such questions about the distributions of additive arithmetic functions is a number of 'moment' problems, which, if solved, lead to results of independent interest. Thus, for example, the following result is suggested by, and includes, the theorem of Erdős and Kac.

**Theorem 1.** Let \( f(m) \) be strongly additive and subject to (2) and

\[(5) f(p) = o(B_p^{1/2}).\]

Then we have for each fixed \( k = 1, 2, 3, \ldots \)

\[
\lim_{n \to \infty} n^{-1/2} \sum_{m \leq n} \left( f(m) - A_n \right)^k = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \alpha^k e^{-\alpha^2/2} d\alpha.
\]

(For proofs see Delange [1], [2], Halberstam [5], [6].)

The purpose of the present communication is to indicate briefly a proof that Theorem 1 remains true even when (5) is replaced by the weaker pair of conditions (4) and

\[(5a) f(p) = O(B_p^{1/2}).\]

That (5a) alone does not suffice can be seen readily from the case \( f(p) = \log p \), which determines a very different kind of distribution. On the other hand, (4) alone would also be inadequate, as can be seen from the following example.

Let \( p_1, p_2, \ldots, p_j, \ldots \) be an increasing sequence of primes with the property that the number of primes which belong to this sequence and do not exceed \( x \) is \( o(\log \log x) \). Now take

\[
f(p) = \begin{cases} (p_j)^{1/2} & \text{if } p = p_j, \\ 1 & \text{if } p \text{ does not belong to the sequence.} \end{cases}
\]

Then \( B_n \sim (\log \log n) \) and condition (4) is satisfied. However,

\[
\sum_{m \leq p_j} (f(m) - A_{p_j})^i \leq (f(p_j) - A_{p_j})^i \sim p_j^i
\]

whereas, if Theorem 1 were true in this case, we should have
\[ \sum_{m \leq n} (f(m) - A_p)^i \sim 3p_j (\log \log p_j)^2. \]

The most general formulation of Theorem 1 remains an open question. The theorem shows, incidentally, that although the method of moments is in many ways more tractable for determining the distributions of given functions, it is not as wide in scope as the method evolved by Erdős and Kac.

2. We suppose throughout this section that (4) and (5a) hold. First of all, we rewrite (4) as

\[ \lim_{n \to \infty} \phi(n, \epsilon) = 0 \quad \text{for every } \epsilon > 0, \]

where

\[ \phi(n, \epsilon) = B_n^{-1} \sum_{p \leq n, |f(p)| \geq B_n^{1/2}} f^i(p)/p. \]

To simplify subsequent arithmetic we choose \( \epsilon < 1/2 \) and keep it fixed; then we choose \( n \) so large that

\[ \phi(n, \epsilon) < \frac{1}{2} \]

as is possible by (6). We set

\[ \alpha_n = n^{1/(3k)} \]

and observe that in view of (9) and the well-known relation

\[ \sum_{p < y} p^{-1} = \log \log y + c + o(1) \]

where \( c \) is an absolute constant,

\[ \sum_{a_{n,p} \leq n} p^{-1} = O(1). \]

We define

\[ A_y^* = \sum_{p \leq y, |f(p)| \geq B_n^{1/2}} f(p)/p, \quad B_y^* = \sum_{p \leq y, |f(p)| \geq B_n^{1/2}} f^i(p)/p \]

and

\[ f^*(m) = \sum_{p \leq a_{n,p} |m|, |f(p)| \leq B_n^{1/2}} f(p). \]

By (7) and (12)

\[ \frac{1}{2} \quad \text{The constants implied by the use of the O-notation depend throughout on at most } k. \]
$$B_n^* = B_n(1 - \phi(n, \varepsilon))$$

and this combines with (11) to give

(14)  $$B_n^* = B_n(1 + O(\varepsilon^2 + \phi(n, \varepsilon))).$$

**Lemma 1.**  $$A_n = A_n^* + O(B_n^{1/2} \{\varepsilon + \varepsilon^{-1}\phi(n, \varepsilon)\}).$$

**Proof.**  By (1)

$$A_n = \sum_{p < \alpha_n} f(p)/p + \sum_{\alpha_n \leq p < n} f(p)/p + \sum_{p \geq n} f(p)/p.$$  The first sum on the right is $$A_n^*$$ by (12) with $$y = a_n$$, the second sum is $$O(\varepsilon B_n)$$ by (11), and the third is less than

$$\varepsilon^{-1} B_n^{1/2} \sum_{p \leq n} f^*(p)/p = B_n^{1/2} \varepsilon^{-1}\phi(n, \varepsilon)$$

by (7). Hence the result.

**Lemma 2.**  If $$r \leq k$$, then

$$\sum_{m=1}^{n} (f(m) - f^*(m)) \varphi r = O(n B_n^\varepsilon \{\varepsilon + \varepsilon^{-1}\phi(n, \varepsilon)\}).$$

**Proof.**  By (13) and the definition of $$f(m)

$$f(m) - f^*(m) = \sum_{p < n, p|m} f(p) + \sum_{\alpha_n \leq p < n, p|m} f(p) + \sum_{p \geq n} f(p)$$

where $$\mathcal{E}_n$$ is the set of those primes less than $$n$$ which satisfy either

(i)  $$|f(p)| > \varepsilon B_n^{1/2}$$

or

(ii)  $$|f(p)| \leq \varepsilon B_n^{1/2}, \quad p \geq \alpha_n.$$

Then the sum of Lemma 2 is

$$O\left( \sum_{r=1}^{2r} \sum_{r_{i+1} \cdots r_{i+j} = p_{i+1} \cdots p_{j}} \sum_{r_{i+1} \cdots r_{i+j} = r_{i+1} \cdots r_{i+j}} |f(r_{i+1} \cdots r_{i+j})| \sum_{m=1}^{n} 1 \right)$$

$$= O\left( \sum_{r=1}^{2r} \{\max_{p \leq n} |f(p)| r_{i+1} \cdots r_{i+j} \sum_{p_{i+1} \cdots p_{j}} \left[ \frac{n}{p_{i+1} \cdots p_{j}} \right] |f(p_{i+1} \cdots f(p_{j})|) \right)$$

where $$\sum'$$ indicates that the summation is carried out over all sets of distinct prime numbers $$p_{i+1}, p_{i+2}, \ldots, p_{j}$$ with $$p_{i} \in \mathcal{E} (i = 1, 2, \ldots, v)$$, and $$[y]$$ stands for the integer part of $$y$$. Using (5a), (i) and (ii) this expression is
which, as in the proof of Lemma 1, becomes
\[
O\left(n \sum_{\gamma=1}^{2r} B_n^{-\frac{1}{2} \gamma} \sum_{s=0}^{\gamma} \{ f(p) \}^s \right) \leq \left( \sum_{\gamma=0}^{2r} \{ f(p) \}^s \right)^{\gamma-s},
\]
which, as in the proof of Lemma 1, becomes
\[
O\left(n \sum_{\gamma=1}^{2r} B_n^{-\frac{1}{2} \gamma} \sum_{s=0}^{\gamma} \{ B_n^{1/2} (\epsilon^{-1} \phi) \}^s \right) = O\left(n B_n^{-\frac{1}{2} \gamma} \sum_{s=0}^{\gamma} (\epsilon^{-1} \phi)^s e^{\gamma-s} \right)
\]
\[
= O(n B_n^{-\frac{1}{2} \gamma} (\epsilon^{-1} \phi + \epsilon) )
\]
here we have used the restrictions on the magnitudes of \( \epsilon \) and \( \phi \) imposed at the beginning of § 2 (see inequality (8)).

Next we set
\[
M_k(n) = \sum_{m=1}^{n} (f(m) - A_n)^k, \quad M_r^*(n) = \sum_{m=1}^{n} (f^*(m) - A_n^*)^r.
\]
Then
\[
M_k(n) = \sum_{m=1}^{n} \{ (A_n - A_n) + (f(m) - f^*(m)) + (f^*(m) - A_n^*) \}^k,
\]
so that by Lemmas 1 and 2 and Cauchy’s inequality
\[
M_k(n) - M_k^*(n)
\]
\[
= O\left( \sum_{\gamma=1}^{2r} B_n^{-\frac{1}{2} \gamma} \sum_{s=0}^{\gamma} |f^*(m)|^s |f^*(m) - A_n^*|^3 \right)
\]
\[
= O\left( \sum_{\gamma=1}^{2r} B_n^{-\frac{1}{2} \gamma} \sum_{s=0}^{\gamma} (\epsilon^{-1} \phi)^s \right) \sum_{m=1}^{n} \left( f(m) - f^*(m) \right)^{2r_s} \left( M_{2r_0}(n) \right)^{1/2}
\]
\[
= O\left( n^{1/2} \sum_{r=2k} \sum_{s=0}^{\gamma} B_n^{(2-k)/2} (\epsilon^{-1} \phi)^s \left( M_{2r_0}(n) \right)^{1/2} \right).
\]
But by the methods of Halberstam [5] or Delange [2] it is a straightforward matter to confirm that for \( n \) sufficiently large
\[
M^*_k(n) = n B_n^{1/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega e^{-\omega^2 n} d\omega \{ 1 + O(\epsilon) \}, \quad l \leq 2k,
\]
so that by (14) and (8)
\[
M^*_k(n) = n B_n^{1/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \omega e^{-\omega^2 n} d\omega \{ 1 + O(\epsilon) \}, \quad l \leq 2k,
\]
and, in particular
\[
M^*_r(n) = O(n B_n^r), \quad r \leq k.
\]
Hence
now, whilst still keeping $\epsilon$ fixed, we let $n$ tend to infinity, and obtain
\[
\lim_{n \to \infty} \left| \frac{M_k(n)}{n^{k/2}} - \frac{M_k^*(n)}{n^{k/2}} \right| = O(\epsilon^{1/2}).
\]
Thus, by (15) with $l = k$,
\[
\lim_{n \to \infty} \left| \frac{M_k(n)}{n^{k/2}} - (2\pi)^{-1} \int_{-\infty}^{\infty} \omega^{k} e^{-\omega^2/2} d\omega \right| = O(\epsilon^{1/2}).
\]
Since the left side is entirely independent of $\epsilon$, and yet the relation is true for every $\epsilon < 1/2$, we have now proved that
\[
\lim_{n \to \infty} \frac{M_k(n)}{n^{k/2}} = (2\pi)^{-1} \int_{-\infty}^{\infty} \omega^{k} e^{-\omega^2/2} d\omega
\]
for every fixed $k = 1, 2, 3, \ldots$.

This concludes the proof of Theorem 1 with condition (5) replaced by the pair of conditions (5a) and (4).

References

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<table>
<thead>
<tr>
<th>Author</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robert Gero George Buschman</td>
<td>A substitution theorem for the Laplace transformation and its generalization to transformations with symmetric kernel</td>
<td>1529</td>
</tr>
<tr>
<td>S. D. Conte</td>
<td>Numerical solution of vibration problems in two space variables</td>
<td>1535</td>
</tr>
<tr>
<td>Paul Dedecker</td>
<td>A property of differential forms in the calculus of variations</td>
<td>1545</td>
</tr>
<tr>
<td>H. Delange and Heini Halberstam</td>
<td>A note on additive functions</td>
<td>1551</td>
</tr>
<tr>
<td>Jerald L. Ericksen</td>
<td>Characteristic direction for equations of motion of non-Newtonian fluids</td>
<td>1557</td>
</tr>
<tr>
<td>Avner Friedman</td>
<td>On two theorems of Phragmén-Lindelöf for linear elliptic and parabolic differential equations of the second order</td>
<td>1563</td>
</tr>
<tr>
<td>Ronald Kay Getoor</td>
<td>Additive functionals of a Markov process</td>
<td>1577</td>
</tr>
<tr>
<td>U. C. Guha</td>
<td>$(\gamma, k)$-summability of series</td>
<td>1593</td>
</tr>
<tr>
<td>Alvin Hausner</td>
<td>The tauberian theorem for group algebras of vector-valued functions</td>
<td>1603</td>
</tr>
<tr>
<td>Lester J. Heider</td>
<td>T-sets and abstract (L)-spaces</td>
<td>1611</td>
</tr>
<tr>
<td>Melvin Henriksen</td>
<td>Some remarks on a paper of Aronszajn and Panitchpakdi</td>
<td>1619</td>
</tr>
<tr>
<td>H. M. Lieberstein</td>
<td>On the generalized radiation problem of A. Weinstein</td>
<td>1623</td>
</tr>
<tr>
<td>Robert Osserman</td>
<td>On the inequality $\Delta u \geq f(u)$</td>
<td>1641</td>
</tr>
<tr>
<td>Calvin R. Putnam</td>
<td>On semi-normal operators</td>
<td>1649</td>
</tr>
<tr>
<td>Binyamin Schwarz</td>
<td>Bounds for the principal frequency of the non-homogeneous membrane and for the generalized Dirichlet integral</td>
<td>1653</td>
</tr>
<tr>
<td>Edward Silverman</td>
<td>Morrey’s representation theorem for surfaces in metric spaces</td>
<td>1677</td>
</tr>
<tr>
<td>V. N. Singh</td>
<td>Certain generalized hypergeometric identities of the Rogers-Ramanujan type. II</td>
<td>1691</td>
</tr>
<tr>
<td>R. J. Smith</td>
<td>A determinant in continuous rings</td>
<td>1701</td>
</tr>
<tr>
<td>Drury William Wall</td>
<td>Sub-quasigroups of finite quasigroups</td>
<td>1711</td>
</tr>
<tr>
<td>Sadayuki Yamamuro</td>
<td>Monotone completeness of normed semi-ordered linear spaces</td>
<td>1715</td>
</tr>
<tr>
<td>C. T. Rajagopal</td>
<td>Simplified proofs of “Some Tauberian theorems” of Jakimovski: Addendum and corrigendum</td>
<td>1727</td>
</tr>
<tr>
<td>N. Aronszajn and Prom Panitchpakdi</td>
<td>Correction to: “Extension of uniformly continuous transformations in hyperconvex metric spaces”</td>
<td>1729</td>
</tr>
<tr>
<td>Alfred Huber</td>
<td>Correction to: “The reflection principle for polyharmonic functions”</td>
<td>1731</td>
</tr>
</tbody>
</table>