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**CHARACTERISTIC DIRECTION FOR EQUATIONS OF
MOTION OF NON-NEWTONIAN FLUIDS**

JERALD L. ERICKSEN

CHARACTERISTIC DIRECTION FOR EQUATIONS OF MOTION OF NON-NEWTONIAN FLUIDS

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1. Introduction. According to the Reiner-Rivlin theory of non-Newtonian fluids,¹ the stress tensor t_j^i is given in terms of the rate of strain tensor d_j^i by relations of the form

$$(1) \quad t_j^i = -p\delta_j^i + \mathcal{F}_1 d_j^i + \mathcal{F}_2 d_k^i d_j^k,$$

where p is an arbitrary hydrostatic pressure, the \mathcal{F} 's are essentially arbitrary differentiable functions of

$$(2) \quad \text{II} = -\frac{1}{2} d_j^i d_i^j, \quad \text{III} = \det d_j^i,$$

and d_j^i satisfies the incompressibility condition

$$(3) \quad d_i^i = 0.$$

The tensors d_j^i and t_j^i are both symmetric.

It is known [2] that the characteristic directions of the corresponding equations of motion are the unit vectors ν_i satisfying

$$(4) \quad F(\nu_i) \equiv 2U^2 + 2UU_i^i + (U_i^i)^2 - U_j^i U_i^j = 0,$$

where

$$\begin{aligned} U &= \mathcal{F}_1 + \mathcal{F}_2 \mu^i \nu_i, \\ U_j^i &= \mathcal{F}_2 (d_j^i - \nu^i \mu_j) + 2(\mu^i - \nu^i \mu_k \nu^k) \left(\mu^m d_{mj} \frac{\partial \mathcal{F}_1}{\partial \text{III}} - \mu_j \frac{\partial \mathcal{F}_1}{\partial \text{II}} \right) \\ &\quad + 2(d_{mi} \mu^m - \nu^i \mu_m \mu^m) \left(\mu^n d_{nj} \frac{\partial \mathcal{F}_2}{\partial \text{III}} - \mu_j \frac{\partial \mathcal{F}_2}{\partial \text{II}} \right), \\ \mu_i &= d_{ij} \nu^j. \end{aligned}$$

Since $F(\nu_i)$ is a continuous function of ν_i on the compact set $\nu_i \nu^i = 1$, a necessary and sufficient condition that no real characteristic directions exist is that $F(\nu_i)$ be of one sign for all unit vectors. Using this fact, we obtain simpler necessary conditions which are shown to be sufficient when $\mathcal{F}_2 \equiv 0$.

2. Necessary conditions. Let d_1 , d_2 and d_3 denote the eigenvalues of d_j^i . From (3),

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¹ This theory was proposed independently by Reiner [4] for compressible fluids, by Rivlin [5] for incompressible materials. We treat the latter case.

$$(5) \quad d_1 + d_2 + d_3 = 0.$$

We restrict our attention to unit vectors ν_i which are perpendicular to an eigenvector of d_j^i and note that $F(\nu_i)$, being a continuous function of ν_i , must be of one sign for all unit vectors in order that no real characteristic directions exist. Given any unit vector ν_i perpendicular to an eigenvector e_i corresponding to d_3 , we may introduce a rectangular Cartesian coordinate system such that, at a point, ν_i is parallel to the positive x^1 -axis and e_i is parallel to the x^2 -axis. Then

$$\begin{aligned} \nu_i = \delta_{i1}, \quad d_{13} = d_{23} = d_{13}d_3^i = d_{21}d_3^i = 0, \\ 2d_{12} = (d_1 - d_2) \sin 2\phi, \quad d_{33} = d_3, \end{aligned}$$

where ϕ is the angle between ν_i and an eigenvector corresponding to d_1 . Making these substitutions in $F(\nu_i)$, given by (4), we obtain, by a routine calculation,

$$(6) \quad F(\nu_i) = 2[\mathcal{F}_1 - \mathcal{F}_2 d_2] \left\{ \mathcal{F}_1 - \mathcal{F}_2 d_3 - \frac{1}{2}(d_1 - d_2)^2 \sin^2 2\phi \left[\frac{\partial \mathcal{F}_1}{\partial \text{II}} \right. \right. \\ \left. \left. - d_3 \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_3 \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_3^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] \right\},$$

which must be of one sign for all real angles ϕ . This is clearly true if and only if it is of the same sign for $\phi=0$ and $\phi=\pi/4$. That is, either

$$(7) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_2][\mathcal{F}_1 - \mathcal{F}_2 d_3] > 0$$

and

$$(8) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_2] \left\{ \mathcal{F}_1 - \mathcal{F}_2 d_3 - \frac{1}{2}(d_1 - d_2)^2 \left[\frac{\partial \mathcal{F}_1}{\partial \text{II}} \right. \right. \\ \left. \left. - d_3 \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_3 \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_3^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] \right\} > 0,$$

or (7) and (8) hold simultaneously with the inequalities reversed. By similarly analyzing the cases where ν_i is perpendicular to eigenvectors of d_j^i corresponding to d_1 and d_2 , we conclude that either

$$(9) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_i][\mathcal{F}_1 - \mathcal{F}_2 d_j] > 0 \quad (i \neq j),$$

and

$$(10) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_j] \left\{ \mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[\frac{\partial \mathcal{F}_1}{\partial \text{II}} \right. \right. \\ \left. \left. - d_k \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_k \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_k^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] \right\} > 0 \quad (i, j, k \neq),$$

or

$$(11) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_i][\mathcal{F}_1 - \mathcal{F}_2 d_j] < 0 \quad (i \neq j),$$

and (10) holds with the inequality reversed. Now (11) cannot hold for all i and j , so this possibility is ruled out. We thus have

THEOREM 1. *A necessary and sufficient condition that no real characteristic directions exist is that $F(\nu_i) > 0$; in order that there exist no real characteristic directions perpendicular to an eigenvector of d_j^i , it is necessary and sufficient that the inequalities (9) and (10) hold.*

For (9) and (10) to hold, it is necessary and sufficient that either

$$(12) \quad \mathcal{F}_1 - \mathcal{F}_2 d_i > 0$$

and

$$(13) \quad \mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[\frac{\partial \mathcal{F}_1}{\partial \text{II}} - d_k \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_k \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_k^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] > 0 \quad (i, j, k \neq),$$

or

$$(14) \quad \mathcal{F}_1 - \mathcal{F}_2 d_i < 0$$

and

$$(15) \quad \mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[\frac{\partial \mathcal{F}_1}{\partial \text{II}} - d_k \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_k \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_k^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] < 0 \quad (i, j, k \neq).$$

3. Equivalent conditions. Let t_i denote the eigenvalues of the stress tensor corresponding to the eigenvalue d_i of d_{mn} so that from (1),

$$t_i = -p + \mathcal{F}_1 d_i + \mathcal{F}_2 d_i^2.$$

Using (5),

$$(16) \quad \begin{aligned} t_i - t_j &= [\mathcal{F}_1 + \mathcal{F}_2 (d_i + d_j)](d_i - d_j) \\ &= [\mathcal{F}_1 - \mathcal{F}_2 d_k](d_i - d_j) \end{aligned} \quad (i, j, k \neq).$$

From (2) and (5),

$$(17) \quad \begin{aligned} \text{II} &= -\frac{1}{2}(d_1^2 + d_2^2 + d_3^2) = -\frac{1}{4}(d_i - d_j)^2 - \frac{3}{4}d_k^2, \\ \text{III} &= d_1 d_2 d_3 = \frac{1}{4}d_k [d_k^2 - (d_i - d_j)^2] \end{aligned} \quad (i, j, k \neq).$$

Using (16) and (17) to express $t_i - t_j$ as a function of $d_i - d_j$ and d_k ($i, j, k \neq$), we calculate

$$(18) \quad \frac{\partial(t_i - t_j)}{\partial(d_i - d_j)} \Big|_{d_k = \text{const.}} = \mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2} (d_i - d_j)^2 \left[\frac{\partial \mathcal{F}_1}{\partial \text{II}} - d_k \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_k \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_k^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right].$$

From (12), (13), (14), (15), (16), (18) and Theorem 1, we have

THEOREM 2. *When the eigenvalues of d_j^i are all unequal, a necessary and sufficient condition that there exist no real characteristic direction perpendicular to an eigenvector of d_j^i is that either*

$$(t_i - t_j)/(d_i - d_j) > 0 \quad \text{and} \quad \partial(t_i - t_j)/\partial(d_i - d_j) \Big|_{d_k = \text{const.}} > 0,$$

or

$$(t_i - t_j)/(d_i - d_j) < 0 \quad \text{and} \quad \partial(t_i - t_j)/\partial(d_i - d_j) \Big|_{d_k = \text{const.}} < 0 \quad (i, j, k \neq).$$

When (12) holds, the stress power Φ , given by

$$3\Phi = 3t_j^i d_j^i = (t_1 - t_2)(d_1 - d_2) + (t_2 - t_3)(d_2 - d_3) + (t_3 - t_1)(d_3 - d_1)$$

is negative, a possibility which many writers exclude on thermodynamic grounds.

4. The case $\mathcal{F}_2 \equiv 0$. When $\mathcal{F}_2 \equiv 0$, $\mathcal{F}_1 \neq 0$, the characteristic equation (4) has been shown [2] to reduce to

$$(19) \quad G(\nu_i) \equiv \mathcal{F}_1 + A^i B_i = 0,$$

where

$$A^i = 2(\mu^i - \nu^i \mu_k \nu^k),$$

$$B_i = \mu^m d_{mi} \frac{\partial \mathcal{F}_1}{\partial \text{III}} - \mu_i \frac{\partial \mathcal{F}_1}{\partial \text{II}}.$$

In fact, $F(\nu_i) = 2\mathcal{F}_1 G(\nu_i)$. When $\mathcal{F}_2 = 0$, $\mathcal{F}_1 = 0$, every direction is characteristic, a case which we exclude. Using the Hamilton-Cayley theorem,

$$d_j^i d_k^j d_m^k = \text{III} \delta_m^i - \text{II} d_m^i,$$

we can reduce (19) to the form

$$(20) \quad G(\alpha, \beta) \equiv \mathcal{F}_1 + 2(\text{III} - \text{II}\alpha - \beta\alpha) \frac{\partial \mathcal{F}_1}{\partial \text{III}} + 2(\alpha^2 - \beta) \frac{\partial \mathcal{F}_1}{\partial \text{II}} = 0,$$

where

$$(21) \quad \alpha = \mu_i \nu^i = d_{ij} \nu^i \nu^j, \quad \beta = \mu^i \mu_i = d_k^i d_{im} \nu^k \nu^m.$$

Now (21) is a mapping of the unit sphere $\nu_i\nu^i=1$ onto a region R in the $\alpha-\beta$ plane. The conditions

$$\begin{aligned} \frac{\partial G}{\partial \alpha} &= -2(\text{II} + \beta) \frac{\partial \mathcal{F}_1}{\partial \text{III}} + 4\alpha \frac{\partial \mathcal{F}_1}{\partial \text{II}} = 0, \\ \frac{\partial G}{\partial \beta} &= -2\alpha \frac{\partial \mathcal{F}_1}{\partial \text{III}} - 2 \frac{\partial \mathcal{F}_1}{\partial \text{II}} = 0, \\ \pm d^2 G &= \pm 4 \left[\frac{\partial \mathcal{F}_1}{\partial \text{II}} d\alpha^2 - \frac{\partial \mathcal{F}_1}{\partial \text{III}} d\alpha d\beta \right] \geq 0 \text{ for all } d\alpha, d\beta, \end{aligned}$$

must be satisfied at any interior point of R at which G is a maximum or minimum. These conditions cannot be satisfied unless $\partial \mathcal{F}_1 / \partial \text{II} = \partial \mathcal{F}_1 / \partial \text{III} = 0$, in which case $G(\nu_i)$ is independent of ν_i , and $\mathcal{F}_1 \neq 0$ is then necessary and sufficient that there exist no real characteristics. From the implicit function theorem, values of ν_i corresponding to boundary points of R are such that the equations

$$d\alpha = 2d_{i,j}\nu^i d\nu^j, \quad d\beta = 2d_{i,k}^i d_{i,m}\nu^k d\nu^m, \quad 0 = \nu_i d\nu^i$$

do not admit a unique solution for $d\nu^i$ in terms of $d\alpha$ and $d\beta$. We thus have

THEOREM 3. *Maximum and minimum values of $G(\nu_i)$, hence of $F(\nu_i)$, hence of $F(\nu_i)$, occur only at values of ν_i such that the vectors ν_i , $d_{i,j}\nu^j$ and $d_i^k d_{km}\nu^m$ are linearly dependent or, equivalently, at values such that the determinant D of these three vectors vanishes.*

Whatever be the unit vector ν_i , we can always choose rectangular Cartesian coordinates such that, at a point, $\nu_i = \delta_{i1}$, $d_{23} = 0$. The condition $D=0$ then reduces to

$$\begin{aligned} 0 &= \begin{vmatrix} 1 & 0 & 0 \\ d_{11} & d_{21} & d_{31} \\ d_{11}^2 + d_{12}^2 + d_{13}^2 & d_{21}(d_{11} + d_{22}) & d_{31}(d_{11} + d_{33}) \end{vmatrix} \\ &= d_{21}d_{31}(d_{33} - d_{22}). \end{aligned}$$

If $d_{21} = 0$ ($d_{31} = 0$), $\delta_{i2}(\delta_{i3})$ is an eigenvector of $d_{i,j}$. If $d_{21}d_{31} \neq 0$, $d_{33} = d_{22}$, the vector with components $(0, d_{31}, -d_{21})$ is an eigenvector of $d_{i,j}$, whence follows

THEOREM 4. *The vectors ν_i , $d_{i,j}\nu^j$, $d_i^k d_{km}\nu^m$ can be linearly dependent only when ν_i is perpendicular to an eigenvector of $d_{i,j}$.*

Theorems 3 and 4 imply that, when $\mathcal{F}_2 \equiv 0$, we will have $F(\nu_i) > 0$ for all unit vectors ν_i if and only if $F(\nu_i) > 0$ for each unit vector ν_i which is perpendicular to an eigenvector of $d_{i,j}$. From Theorem 1, we then deduce

THEOREM 5. *When $\mathcal{F}_2 \equiv 0$, a necessary and sufficient condition that there exist no real characteristic directions is that the inequalities (9) and (10) hold.*

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Pacific Journal of Mathematics

Vol. 7, No. 4

April, 1957

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Paul Dedecker, <i>A property of differential forms in the calculus of variations</i>	1545
H. Delange and Heini Halberstam, <i>A note on additive functions</i>	1551
Jerald L. Ericksen, <i>Characteristic direction for equations of motion of non-Newtonian fluids</i>	1557
Avner Friedman, <i>On two theorems of Phragmén-Lindelöf for linear elliptic and parabolic differential equations of the second order</i>	1563
Ronald Kay Gettoor, <i>Additive functionals of a Markov process</i>	1577
U. C. Guha, <i>(γ, k)-summability of series</i>	1593
Alvin Hausner, <i>The tauberian theorem for group algebras of vector-valued functions</i>	1603
Lester J. Heider, <i>T-sets and abstract (L)-spaces</i>	1611
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N. Aronszajn and Prom Panitchpakdi, <i>Correction to: "Extension of uniformly continuous transformations in hyperconvex metric spaces"</i>	1729
Alfred Huber, <i>Correction to: "The reflection principle for polyharmonic functions"</i>	1731