CHARACTERISTIC DIRECTION FOR EQUATIONS OF MOTION OF NON-NEWTONIAN FLUIDS

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1. Introduction. According to the Reiner-Rivlin theory of non-
Newtonian fluids, the stress tensor $t_{ij}$ is given in terms of the rate of
strain tensor $d_{ij}$ by relations of the form

$$
t_{ij} = -p\delta_{ij} + \mathcal{F}_1 d_{ij} + \mathcal{F}_2 d_k d^k_j ,
$$

where $p$ is an arbitrary hydrostatic pressure, the $\mathcal{F}$'s are essentially
arbitrary differentiable functions of

$$
\Pi = -\frac{1}{2} d^i_j d_i , \quad \Pi = \det d_i^j ,
$$

and $d_{ij}$ satisfies the incompressibility condition

$$
d_i^i = 0 .
$$

The tensors $d_{ij}$ and $t_{ij}$ are both symmetric.

It is known [2] that the characteristic directions of the corresponding
equations of motion are the unit vectors $\nu_i$ satisfying

$$
F(\nu_i) = 2U^2 + 2UU_i + (U_i)^2 - U_j U_i = 0 ,
$$

where

$$
U = \mathcal{F}_1 + \mathcal{F}_2 \mu^i \nu_i ,
$$

$$
U_i = \mathcal{F}_2 (d_{ij} - \nu^i \nu_j) + 2(\mu^i - \nu^i \mu_k \nu_k)\left( \mu^m d_{mj} \frac{\partial \mathcal{F}_1}{\partial \Pi} - \mu_j \frac{\partial \mathcal{F}_1}{\partial \Pi} \right)
$$

$$
+ 2(d_{lm} \mu^m - \nu^l \mu_m \mu^m)\left( \mu^m d_{nj} \frac{\partial \mathcal{F}_2}{\partial \Pi} - \mu_j \frac{\partial \mathcal{F}_2}{\partial \Pi} \right) ,
$$

$$
\mu_i = d_{ij} \nu^j .
$$

Since $F(\nu_i)$ is a continuous function of $\nu_i$ on the compact set $\nu_i \nu^i = 1$, a
necessary and sufficient condition that no real characteristic directions
exist is that $F(\nu_i)$ be of one sign for all unit vectors. Using this fact,
we obtain simpler necessary conditions which are shown to be sufficient
when $\mathcal{F}_2 = 0$.

2. Necessary conditions. Let $d_1$, $d_2$ and $d_3$ denote the eigenvalues
of $d_{ij}$. From (3),

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1 This theory was proposed independently by Reiner [4] for compressible fluids, by
Rivlin [5] for incompressible materials. We treat the latter case.
(5) \[ d_1 + d_2 + d_3 = 0. \]

We restrict our attention to unit vectors \( \nu_i \) which are perpendicular to an eigenvector of \( d_j \) and note that \( F(\nu_i) \), being a continuous function of \( \nu_i \), must be of one sign for all unit vectors in order that no real characteristic directions exist. Given any unit vector \( \nu_i \) perpendicular to an eigenvector \( e_i \) corresponding to \( d_i \), we may introduce a rectangular Cartesian coordinate system such that, at a point, \( \nu_i \) is parallel to the positive \( x^1 \)-axis and \( e_i \) is parallel to the \( x^3 \)-axis. Then

\[
\nu_i = \delta_{i1}, \quad d_{13} = d_{23} = d_{31} = d_{32} = d_{33} = 0, \\
2d_{12} = (d_1 - d_2) \sin 2\phi, \quad d_{33} = d_3,
\]

where \( \phi \) is the angle between \( \nu_i \) and an eigenvector corresponding to \( d_i \). Making these substitutions in \( F(\nu_i) \), given by (4), we obtain, by a routine calculation,

\[
(6) \quad F(\nu_i) = 2\left[ \mathcal{F}_1 - \mathcal{F}_2 d_3 \right] \left\{ \mathcal{F}_1 - \mathcal{F}_2 d_3 - \frac{1}{2} (d_1 - d_2)^2 \sin^2 2\phi \left[ \frac{\partial \mathcal{F}_1}{\partial \Pi} - d_3 \frac{\partial \mathcal{F}_2}{\partial \Pi} - d_3 \frac{\partial \mathcal{F}_2}{\partial \Pi} \right] \right\},
\]

which must be of one sign for all real angles \( \phi \). This is clearly true if and only if it is of the same sign for \( \phi = 0 \) and \( \phi = \pi/4 \). That is, either

\[
(7) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_3][\mathcal{F}_1 - \mathcal{F}_2 d_3] > 0
\]

and

\[
(8) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_3] \left\{ \mathcal{F}_1 - \mathcal{F}_2 d_3 - \frac{1}{2} (d_1 - d_2)^2 \left[ \frac{\partial \mathcal{F}_1}{\partial \Pi} - d_3 \frac{\partial \mathcal{F}_2}{\partial \Pi} - d_3 \frac{\partial \mathcal{F}_2}{\partial \Pi} \right] \right\} > 0,
\]

or (7) and (8) hold simultaneously with the inequalities reversed. By similarly analyzing the cases where \( \nu_i \) is perpendicular to eigenvectors of \( d_j \) corresponding to \( d_i \) and \( d_2 \), we conclude that either

\[
(9) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_i][\mathcal{F}_1 - \mathcal{F}_2 d_j] > 0 \quad (i \neq j),
\]

and

\[
(10) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_i] \left\{ \mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2} (d_i - d_j)^2 \left[ \frac{\partial \mathcal{F}_1}{\partial \Pi} - d_3 \frac{\partial \mathcal{F}_2}{\partial \Pi} - d_3 \frac{\partial \mathcal{F}_2}{\partial \Pi} \right] \right\} > 0 \quad (i, j, k \neq),
\]
and (10) holds with the inequality reversed. Now (11) cannot hold for all $i$ and $j$, so this possibility is ruled out. We thus have

**Theorem 1.** A necessary and sufficient condition that no real characteristic directions exist is that $F(\nu_i) > 0$; in order that there exist no real characteristic directions perpendicular to an eigenvector of $d_j$, it is necessary and sufficient that the inequalities (9) and (10) hold.

For (9) and (10) to hold, it is necessary and sufficient that either

\begin{equation}
\mathcal{F}_1 - \mathcal{F}_2 d_i > 0
\end{equation}

and

\begin{equation}
\mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[ \frac{\partial \mathcal{F}_1}{\partial \Pi} - \frac{\partial \mathcal{F}_2}{\partial \Pi} + d_k \frac{\partial \mathcal{F}_1}{\partial \Pi} - d_k \frac{\partial \mathcal{F}_2}{\partial \Pi} \right] > 0
\end{equation}

\begin{equation}
(i, j, k \neq) ,
\end{equation}

or

\begin{equation}
\mathcal{F}_1 - \mathcal{F}_2 d_i < 0
\end{equation}

and

\begin{equation}
\mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[ \frac{\partial \mathcal{F}_1}{\partial \Pi} - \frac{\partial \mathcal{F}_2}{\partial \Pi} + d_k \frac{\partial \mathcal{F}_1}{\partial \Pi} - d_k \frac{\partial \mathcal{F}_2}{\partial \Pi} \right] < 0
\end{equation}

\begin{equation}
(i, j, k \neq) .
\end{equation}

3. **Equivalent conditions.** Let $t_i$ denote the eigenvalues of the stress tensor corresponding to the eigenvalue $d_i$ of $d_{mn}$ so that from (1),

\begin{equation}
t_i = -p + \mathcal{F}_1 d_i + \mathcal{F}_2 d_i^2 .
\end{equation}

Using (5),

\begin{equation}
t_i - t_j = [\mathcal{F}_1 + \mathcal{F}_2 (d_i + d_j)](d_i - d_j)
\end{equation}

\begin{equation}
= [\mathcal{F}_1 - \mathcal{F}_2 d_k](d_i - d_j) \quad (i, j, k \neq) .
\end{equation}

From (2) and (5),

\begin{equation}
\Pi = -\frac{1}{2}(d_i^2 + d_j^2 + d_k^2) = -\frac{1}{4}(d_i - d_j)^2 - \frac{3}{4} d_k^2 ,
\end{equation}

\begin{equation}
\Pi = d_i d_j d_k = \frac{1}{4} d_k [d_k^2 - (d_i - d_j)^2] \quad (i, j, k \neq) .
\end{equation}

Using (16) and (17) to express $t_i - t_j$ as a function of $d_i - d_j$ and $d_k(i, j, k \neq)$, we calculate
From (12), (13), (14), (15), (16), (18) and Theorem 1, we have

**THEOREM 2.** When the eigenvalues of $d^i_j$ are all unequal, a necessary and sufficient condition that there exist no real characteristic direction perpendicular to an eigenvector of $d^i_j$ is that either

$$(t_i - t_j)(d_i - d_j) > 0 \quad \text{and} \quad \frac{\partial (t_i - t_j)}{\partial (d_i - d_j)} \bigg|_{d_j = \text{const.}} > 0 ,$$

or

$$(t_i - t_j)(d_i - d_j) < 0 \quad \text{and} \quad \frac{\partial (t_i - t_j)}{\partial (d_i - d_j)} \bigg|_{d_j = \text{const.}} < 0 \quad (i, j, k \neq).$$

When (12) holds, the stress power $\Phi$, given by

$$3\Phi = 3t_i d_i = (t_1 - t_2)(d_1 - d_2) + (t_2 - t_3)(d_2 - d_3) + (t_3 - t_1)(d_3 - d_1)$$

is negative, a possibility which many writers exclude on thermodynamic grounds.

4. **The case $F_2 = 0$.** When $F_2 = 0$, $F_1 \neq 0$, the characteristic equation (4) has been shown [2] to reduce to

$$G(\nu_i) \equiv F_1 + A^t B_i = 0 ,$$

where

$$A_i = 2(\mu^i - \nu^i \nu^k \nu^k) ,$$

$$B_i = \mu^i d^i_m \frac{\partial F_1}{\partial \Pi} - \mu^i \frac{\partial F_1}{\partial \Pi} .$$

In fact, $F(\nu_i) = 2F_1 G(\nu_i)$. When $F_2 = 0$, $F_1 = 0$, every direction is characteristic, a case which we exclude. Using the Hamilton-Cayley theorem,

$$d^i_j d^j_k d^k_m = \Pi \delta^i_m - \Pi d^i_m ,$$

we can reduce (19) to the form

$$G(\alpha, \beta) \equiv F_1 + 2(\Pi \alpha - \beta \alpha) \frac{\partial F_1}{\partial \Pi} + 2(\alpha^2 - \beta) \frac{\partial F_1}{\partial \Pi} = 0 ,$$

where

$$\alpha = \mu^i \nu^i = d^i_j \nu^i \nu^j , \quad \beta = \mu^i \mu^i = d^i_k d^k_{lm} \nu^l \nu^m .$$
Now (21) is a mapping of the unit sphere \( v_i \| = 1 \) onto a region \( R \) in the \( \alpha - \beta \) plane. The conditions
\[
\frac{\partial G}{\partial \alpha} = -2(\Pi + \beta) \frac{\partial \mathcal{F}_1}{\partial \Pi} + 4\alpha \frac{\partial \mathcal{F}_1}{\partial \Pi} = 0, \\
\frac{\partial G}{\partial \beta} = -2\alpha \frac{\partial \mathcal{F}_1}{\partial \Pi} - 2\frac{\partial \mathcal{F}_1}{\partial \Pi} = 0, \\
\pm d^2G = 4\left[ \frac{\partial \mathcal{F}_1}{\partial \Pi} d\alpha^2 - \frac{\partial \mathcal{F}_1}{\partial \Pi} d\alpha d\beta \right] \geq 0 \text{ for all } d\alpha, d\beta,
\]
must be satisfied at any interior point of \( R \) at which \( G \) is a maximum or minimum. These conditions cannot be satisfied unless \( \partial \mathcal{F}_1/\partial \Pi = \partial \mathcal{F}_1/\partial \Pi = 0 \), in which case \( G(v_i) \) is independent of \( v_i \), and \( \mathcal{F}_1 \neq 0 \) is then necessary and sufficient that there exist no real characteristics. From the implicit function theorem, values of \( v_i \) corresponding to boundary points of \( R \) are such that the equations
\[
d\alpha = 2d_i v^j d\nu^j, \quad d\beta = 2d_i \delta_{im} v^k d\nu^m, \quad 0 = v_i d\nu^i
\]
do not admit a unique solution for \( d\nu^i \) in terms of \( d\alpha \) and \( d\beta \). We thus have

**Theorem 3.** Maximum and minimum values of \( G(v_i) \), hence of \( F(v_i) \), hence of \( F(v_\alpha) \), occur only at values of \( v_i \) such that the vectors \( v_i, d_i v^j \) and \( d_i^k \delta_{km} v^m \) are linearly dependent or, equivalently, at values such that the determinant \( D \) of these three vectors vanishes.

Whatever be the unit vector \( v_i \), we can always choose rectangular Cartesian coordinates such that, at a point, \( v_i = \delta_{i1}, d_{i3} = 0 \). The condition \( D = 0 \) then reduces to
\[
0 = \begin{vmatrix} 1 & 0 & 0 \\ d_{i1} & d_{i2} & d_{i3} \\ d_{i1}^2 + d_{i2}^2 + d_{i3}^2 & d_{i2}(d_{i1} + d_{i3}) & d_{i3}(d_{i1} + d_{i3}) \end{vmatrix} = d_{i2}d_{i3}(d_{i3} - d_{i2}).
\]
If \( d_{i1} = 0(d_{i3} = 0) \), \( \delta_{i2}(\delta_{i3}) \) is an eigenvector of \( d_{ij} \). If \( d_{i2}d_{i3} \neq 0, d_{i3} = d_{i2} \), the vector with components \((0, d_{i1}, -d_{i2})\) is an eigenvector of \( d_{ij} \), whence follows

**Theorem 4.** The vectors \( v_i, d_i v^j, d_i^k \delta_{km} v^m \) can be linearly dependent only when \( v_i \) is perpendicular to an eigenvector of \( d_{ij} \).

Theorems 3 and 4 imply that, when \( \mathcal{F}_2 = 0 \), we will have \( F(v_i) > 0 \) for all unit vectors \( v_i \) if and only if \( F(v_i) > 0 \) for each unit vector \( v_i \) which is perpendicular to an eigenvector of \( d_{ij} \). From Theorem 1, we then deduce
Theorem 5. When $\mathcal{F}_2 \equiv 0$, a necessary and sufficient condition that there exist no real characteristic directions is that the inequalities (9) and (10) hold.

References


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