

# Pacific Journal of Mathematics

## **CHARACTERISTIC DIRECTION FOR EQUATIONS OF MOTION OF NON-NEWTONIAN FLUIDS**

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# CHARACTERISTIC DIRECTION FOR EQUATIONS OF MOTION OF NON-NEWTONIAN FLUIDS

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**1. Introduction.** According to the Reiner-Rivlin theory of non-Newtonian fluids,<sup>1</sup> the stress tensor  $t_j^i$  is given in terms of the rate of strain tensor  $d_j^i$  by relations of the form

$$(1) \quad t_j^i = -p\delta_j^i + \mathcal{F}_1 d_j^i + \mathcal{F}_2 d^k d_j^k,$$

where  $p$  is an arbitrary hydrostatic pressure, the  $\mathcal{F}$ 's are essentially arbitrary differentiable functions of

$$(2) \quad \text{II} = -\frac{1}{2} d_j^i d_i^j, \quad \text{III} = \det d_j^i,$$

and  $d_j^i$  satisfies the incompressibility condition

$$(3) \quad d_i^i = 0.$$

The tensors  $d_j^i$  and  $t_j^i$  are both symmetric.

It is known [2] that the characteristic directions of the corresponding equations of motion are the unit vectors  $\nu_i$  satisfying

$$(4) \quad F(\nu_i) \equiv 2U^2 + 2UU_i^i + (U_i^i)^2 - U_j U_j^i = 0,$$

where

$$\begin{aligned} U &= \mathcal{F}_1 + \mathcal{F}_2 \mu^i \nu_i, \\ U_j^i &= \mathcal{F}_2 (d_j^i - \nu^i \mu_j) + 2(\mu^i - \nu^i \mu_k \nu^k) \left( \mu^m d_{mj} \frac{\partial \mathcal{F}_1}{\partial \text{III}} - \mu_j \frac{\partial \mathcal{F}_1}{\partial \text{II}} \right) \\ &\quad + 2(d_{mj}^i \mu^m - \nu^i \mu_m \mu^m) \left( \mu^n d_{nj} \frac{\partial \mathcal{F}_2}{\partial \text{III}} - \mu_j \frac{\partial \mathcal{F}_2}{\partial \text{II}} \right), \\ \mu_i &= d_{ij} \nu^j. \end{aligned}$$

Since  $F(\nu_i)$  is a continuous function of  $\nu_i$  on the compact set  $\nu_i \nu^i = 1$ , a necessary and sufficient condition that no real characteristic directions exist is that  $F(\nu_i)$  be of one sign for all unit vectors. Using this fact, we obtain simpler necessary conditions which are shown to be sufficient when  $\mathcal{F}_2 \equiv 0$ .

**2. Necessary conditions.** Let  $d_1, d_2$  and  $d_3$  denote the eigenvalues of  $d_j^i$ . From (3),

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<sup>1</sup> This theory was proposed independently by Reiner [4] for compressible fluids, by Rivlin [5] for incompressible materials. We treat the latter case.

$$(5) \quad d_1 + d_2 + d_3 = 0 .$$

We restrict our attention to unit vectors  $\nu_i$  which are perpendicular to an eigenvector of  $d_j^i$  and note that  $F(\nu_i)$ , being a continuous function of  $\nu_i$ , must be of one sign for all unit vectors in order that no real characteristic directions exist. Given any unit vector  $\nu_i$  perpendicular to an eigenvector  $e_i$  corresponding to  $d_3$ , we may introduce a rectangular Cartesian coordinate system such that, at a point,  $\nu_i$  is parallel to the positive  $x^1$ -axis and  $e_i$  is parallel to the  $x^3$ -axis. Then

$$\begin{aligned} \nu_i &= \delta_{i1}, \quad d_{13} = d_{23} = d_{14}d_3^i = d_{21}d_3^i = 0, \\ 2d_{12} &= (d_1 - d_2) \sin 2\phi, \quad d_{33} = d_3, \end{aligned}$$

where  $\phi$  is the angle between  $\nu_i$  and an eigenvector corresponding to  $d_1$ . Making these substitutions in  $F(\nu_i)$ , given by (4), we obtain, by a routine calculation,

$$(6) \quad F(\nu_i) = 2[\mathcal{F}_1 - \mathcal{F}_2 d_2] \left\{ \mathcal{F}_1 - \mathcal{F}_2 d_3 - \frac{1}{2}(d_1 - d_2)^2 \sin^2 2\phi \left[ \frac{\partial \mathcal{F}_1}{\partial \text{II}} - d_3 \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_3 \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_3^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] \right\},$$

which must be of one sign for all real angles  $\phi$ . This is clearly true if and only if it is of the same sign for  $\phi=0$  and  $\phi=\pi/4$ . That is, either

$$(7) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_2][\mathcal{F}_1 - \mathcal{F}_2 d_3] > 0$$

and

$$(8) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_2] \left\{ \mathcal{F}_1 - \mathcal{F}_2 d_3 - \frac{1}{2}(d_1 - d_2)^2 \left[ \frac{\partial \mathcal{F}_1}{\partial \text{II}} - d_3 \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_3 \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_3^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] \right\} > 0,$$

or (7) and (8) hold simultaneously with the inequalities reversed. By similarly analyzing the cases where  $\nu_i$  is perpendicular to eigenvectors of  $d_j^i$  corresponding to  $d_1$  and  $d_2$ , we conclude that either

$$(9) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_i][\mathcal{F}_1 - \mathcal{F}_2 d_j] > 0 \quad (i \neq j),$$

and

$$(10) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_j] \left\{ \mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[ \frac{\partial \mathcal{F}_1}{\partial \text{II}} - d_k \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_k \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_k^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] \right\} > 0 \quad (i, j, k \neq),$$

or

$$(11) \quad [\mathcal{F}_1 - \mathcal{F}_2 d_i][\mathcal{F}_1 - \mathcal{F}_2 d_j] < 0 \quad (i \neq j),$$

and (10) holds with the inequality reversed. Now (11) cannot hold for all  $i$  and  $j$ , so this possibility is ruled out. We thus have

**THEOREM 1.** *A necessary and sufficient condition that no real characteristic directions exist is that  $F(\nu_i) > 0$ ; in order that there exist no real characteristic directions perpendicular to an eigenvector of  $d_i^j$ , it is necessary and sufficient that the inequalities (9) and (10) hold.*

For (9) and (10) to hold, it is necessary and sufficient that either

$$(12) \quad \mathcal{F}_1 - \mathcal{F}_2 d_i > 0$$

and

$$(13) \quad \mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[ \frac{\partial \mathcal{F}_1}{\partial \text{II}} - d_k \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_k \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_k^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] > 0 \quad (i, j, k \neq),$$

or

$$(14) \quad \mathcal{F}_1 - \mathcal{F}_2 d_i < 0$$

and

$$(15) \quad \mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2}(d_i - d_j)^2 \left[ \frac{\partial \mathcal{F}_1}{\partial \text{II}} - d_k \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_k \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_k^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right] < 0 \quad (i, j, k \neq).$$

**3. Equivalent conditions.** Let  $t_i$  denote the eigenvalues of the stress tensor corresponding to the eigenvalue  $d_i$  of  $d_{mn}$  so that from (1),

$$t_i = -p + \mathcal{F}_1 d_i + \mathcal{F}_2 d_i^2.$$

Using (5),

$$(16) \quad \begin{aligned} t_i - t_j &= [\mathcal{F}_1 + \mathcal{F}_2 (d_i + d_j)](d_i - d_j) \\ &= [\mathcal{F}_1 - \mathcal{F}_2 d_k](d_i - d_j) \end{aligned} \quad (i, j, k \neq).$$

From (2) and (5),

$$(17) \quad \begin{aligned} \text{II} &= -\frac{1}{2}(d_1^2 + d_2^2 + d_3^2) = -\frac{1}{4}(d_i - d_j)^2 - \frac{3}{4}d_k^2, \\ \text{III} &= d_1 d_2 d_3 = \frac{1}{4}d_k [d_k^2 - (d_i - d_j)^2] \end{aligned} \quad (i, j, k \neq).$$

Using (16) and (17) to express  $t_i - t_j$  as a function of  $d_i - d_j$  and  $d_k (i, j, k \neq)$ , we calculate

$$(18) \quad \frac{\partial(t_i - t_j)}{\partial(d_i - d_j)} \Big|_{d_k = \text{const.}}$$

$$= \mathcal{F}_1 - \mathcal{F}_2 d_k - \frac{1}{2} (d_i - d_j)^2 \left[ \frac{\partial \mathcal{F}_1}{\partial \text{II}} - d_k \frac{\partial \mathcal{F}_2}{\partial \text{II}} + d_k \frac{\partial \mathcal{F}_1}{\partial \text{III}} - d_k^2 \frac{\partial \mathcal{F}_2}{\partial \text{III}} \right].$$

From (12), (13), (14), (15), (16), (18) and Theorem 1, we have

**THEOREM 2.** *When the eigenvalues of  $d_j^i$  are all unequal, a necessary and sufficient condition that there exist no real characteristic direction perpendicular to an eigenvector of  $d_j^i$  is that either*

$$(t_i - t_j)/(d_i - d_j) > 0 \quad \text{and} \quad \partial(t_i - t_j)/\partial(d_i - d_j) \Big|_{d_k = \text{const.}} > 0,$$

or

$$(t_i - t_j)/(d_i - d_j) < 0 \quad \text{and} \quad \partial(t_i - t_j)/\partial(d_i - d_j) \Big|_{d_k = \text{const.}} < 0 \quad (i, j, k \neq).$$

When (12) holds, the stress power  $\Phi$ , given by

$$3\Phi = 3t_j^i d_j^i = (t_1 - t_2)(d_1 - d_2) + (t_2 - t_3)(d_2 - d_3) + (t_3 - t_1)(d_3 - d_1)$$

is negative, a possibility which many writers exclude on thermodynamic grounds.

**4. The case  $\mathcal{F}_2 \equiv 0$ .** When  $\mathcal{F}_2 \equiv 0$ ,  $\mathcal{F}_1 \neq 0$ , the characteristic equation (4) has been shown [2] to reduce to

$$(19) \quad G(\nu_i) \equiv \mathcal{F}_1 + A^i B_i = 0,$$

where

$$A^i = 2(\mu^i - \nu^i \mu_k \nu^k),$$

$$B_i = \mu^m d_{mi} \frac{\partial \mathcal{F}_1}{\partial \text{III}} - \mu_i \frac{\partial \mathcal{F}_1}{\partial \text{II}}.$$

In fact,  $F(\nu_i) = 2\mathcal{F}_1 G(\nu_i)$ . When  $\mathcal{F}_2 = 0$ ,  $\mathcal{F}_1 = 0$ , every direction is characteristic, a case which we exclude. Using the Hamilton-Cayley theorem,

$$d_j^i d_k^j d_m^k = \text{III} \delta_m^i - \text{II} d_m^i,$$

we can reduce (19) to the form

$$(20) \quad G(\alpha, \beta) \equiv \mathcal{F}_1 + 2(\text{III} - \text{II}\alpha - \beta\alpha) \frac{\partial \mathcal{F}_1}{\partial \text{III}} + 2(\alpha^2 - \beta) \frac{\partial \mathcal{F}_1}{\partial \text{II}} = 0,$$

where

$$(21) \quad \alpha = \mu_i \nu^i = d_{ij} \nu^i \nu^j, \quad \beta = \mu^i \mu_i = d_k^i d_{im} \nu^k \nu^m.$$

Now (21) is a mapping of the unit sphere  $\nu_i\nu^i=1$  onto a region  $R$  in the  $\alpha-\beta$  plane. The conditions

$$\begin{aligned} \frac{\partial G}{\partial \alpha} &= -2(\text{II}+\beta)\frac{\partial \mathcal{F}_1}{\partial \text{III}} + 4\alpha\frac{\partial \mathcal{F}_1}{\partial \text{II}} = 0, \\ \frac{\partial G}{\partial \beta} &= -2\alpha\frac{\partial \mathcal{F}_1}{\partial \text{III}} - 2\frac{\partial \mathcal{F}_1}{\partial \text{II}} = 0, \\ \pm d^2G &= \pm 4\left[\frac{\partial \mathcal{F}_1}{\partial \text{II}}d\alpha^2 - \frac{\partial \mathcal{F}_1}{\partial \text{III}}d\alpha d\beta\right] \geq 0 \text{ for all } d\alpha, d\beta, \end{aligned}$$

must be satisfied at any interior point of  $R$  at which  $G$  is a maximum or minimum. These conditions cannot be satisfied unless  $\partial \mathcal{F}_1/\partial \text{II} = \partial \mathcal{F}_1/\partial \text{III} = 0$ , in which case  $G(\nu_i)$  is independent of  $\nu_i$ , and  $\mathcal{F}_1 \neq 0$  is then necessary and sufficient that there exist no real characteristics. From the implicit function theorem, values of  $\nu_i$  corresponding to boundary points of  $R$  are such that the equations

$$d\alpha = 2d_{i_j}\nu^j d\nu^j, \quad d\beta = 2d_{i_k}^k d_{i_m}\nu^m d\nu^m, \quad 0 = \nu_i d\nu^i$$

do not admit a unique solution for  $d\nu^i$  in terms of  $d\alpha$  and  $d\beta$ . We thus have

**THEOREM 3.** *Maximum and minimum values of  $G(\nu_i)$ , hence of  $F(\nu_i)$ , hence of  $F(\nu_i)$ , occur only at values of  $\nu_i$  such that the vectors  $\nu_i$ ,  $d_{i_j}\nu^j$  and  $d_i^k d_{k_m}\nu^m$  are linearly dependent or, equivalently, at values such that the determinant  $D$  of these three vectors vanishes.*

Whatever be the unit vector  $\nu_i$ , we can always choose rectangular Cartesian coordinates such that, at a point,  $\nu_i = \delta_{i1}$ ,  $d_{23} = 0$ . The condition  $D=0$  then reduces to

$$\begin{aligned} 0 &= \begin{vmatrix} 1 & 0 & 0 \\ d_{11} & d_{21} & d_{31} \\ d_{11}^2 + d_{12}^2 + d_{13}^2 & d_{21}(d_{11} + d_{22}) & d_{31}(d_{11} + d_{33}) \end{vmatrix} \\ &= d_{21}d_{31}(d_{33} - d_{22}). \end{aligned}$$

If  $d_{21} = 0$  ( $d_{31} = 0$ ),  $\delta_{i2}$  ( $\delta_{i3}$ ) is an eigenvector of  $d_{ij}$ . If  $d_{21}d_{31} \neq 0$ ,  $d_{33} = d_{22}$ , the vector with components  $(0, d_{31}, -d_{21})$  is an eigenvector of  $d_{ij}$ , whence follows

**THEOREM 4.** *The vectors  $\nu_i$ ,  $d_{i_j}\nu^j$ ,  $d_i^k d_{k_m}\nu^m$  can be linearly dependent only when  $\nu_i$  is perpendicular to an eigenvector of  $d_{ij}$ .*

Theorems 3 and 4 imply that, when  $\mathcal{F}_2 \equiv 0$ , we will have  $F(\nu_i) > 0$  for all unit vectors  $\nu_i$  if and only if  $F(\nu_i) > 0$  for each unit vector  $\nu_i$  which is perpendicular to an eigenvector of  $d_{ij}$ . From Theorem 1, we then deduce

**THEOREM 5.** *When  $\mathcal{F}_2 \equiv 0$ , a necessary and sufficient condition that there exist no real characteristic directions is that the inequalities (9) and (10) hold.*

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