SOME REMARKS ON A PAPER OF ARONSZAJN AND PANITCHPAKDI

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In the paper of the title [1], a number of problems are posed. Negative solutions of two of them (Problems 2 and 3) are derived in a straightforward way from a paper of L. Gillman and the present author [2].

Motivation will not be supplied since it is given amply in [1], but enough definitions are given to keep the presentation reasonably self-contained.

1. A Hausdorff space $X$ is said to satisfy $(Q_m)$, where $m$ is an infinite cardinal, if, whenever $U$ and $V$ are disjoint open subsets of $X$ such that each is a union of the closures of less than $m$ open subsets of $X$, then $U$ and $V$ have disjoint closures. In particular, a normal (Hausdorff) space $X$ satisfies $(Q_{\aleph_0})$ if and only if disjoint open $F_{\sigma}$-subsets of $X$ have disjoint closures. (For, an open set that is the union of less than $\aleph_0$ closed sets is a fortiori an $F_{\sigma}$. Conversely if $U$ is the union of countably many closed subsets $F_n$, then since $X$ is normal, for each $n$ there is an open set $U_n$ containing $F_n$ whose closure is contained in $U$. Thus $U$ is the union of the closures of the open sets $U_n$.) In Problem 3 of [1], it is asked if every compact (Hausdorff) space satisfying $(Q_m)$ for some $m>\aleph_0$ is necessarily totally disconnected, and it is remarked that this is the case if the first axiom of countability is also assumed.

If $X$ is a completely regular space, let $C(X)$ denote the ring of all continuous real-valued functions on $X$, and let $Z(f)=\{x\in X: f(x)=0\}$, $P(f)=\{x\in X: f(x)>0\}$, and let $N(f)=P(-f)$. As usual, let $\beta X$ denote the Stone-Čech compactification of $X$. If every finitely generated ideal of $C(X)$ is a principal ideal, then $X$ is called an $F$-space. The following are equivalent.

(i) $X$ is an $F$-space.

(ii) If $f\in C(X)$, then $P(f)$ and $N(f)$ are completely separated [2, Theorem 2.3].

(iii) If $f\in C(X)$, then every bounded $g\in C(X-Z(f))$ has an extension $\tilde{g}\in C(X)$ [2, Theorem 2.6].

A good supply of compact $F$-spaces is provided by the fact that if $X$ is locally compact and $\sigma$-compact, then $\beta X-X$ is an $F$-space [2, Theorem 2.7].

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We remark first that a normal (Hausdorff) space $X$ satisfies $(Q_{\aleph_1})$ if and only if it is an $F$-space.

For, suppose first that $X$ is an $F$-space, and let $U$, $V$ be disjoint open $F_\sigma$-subsets of $X$. Since $X-(U \cup V)$ is a closed $G_\delta$ in a normal space, there is a bounded $f \in C(X)$ such that $Z(f) = X-(U \cup V)$. Hence by (iii), there is a $g \in C(X)$ such that $g[U] = 0$ and $g[V] = 1$. In particular, $U$ and $V$ have disjoint closures, so $X$ satisfies $(Q_{\aleph_1})$. Conversely let $X$ satisfy $(Q_{\aleph_1})$, and let $f \in C(X)$. Then $P(f)$ and $N(f)$ are disjoint open $F_\sigma$-subsets of $X$, which by $(Q_{\aleph_1})$ have disjoint closures. So, by Urysohn's lemma, $P(f)$ and $N(f)$ are completely separated. Thus $X$ is an $F$-space by (ii).

Compact connected $F$-spaces exist. In particular it is known that if $R^+$ denotes the space of nonnegative real numbers, then $\beta R^+ - R^+$ is such a space [2, Example 2.8]. Hence Problem 3 of [1] has a negative solution.

We remark finally that if the first axiom of countability holds at a point of an $F$-space, then the point is isolated [2, Corollary 2.4]. In particular, every compact $F$-space satisfying the first axiom of countability is finite.

2. In Problem 2 of [1], it is asked (in different but equivalent language) if for every totally disconnected compact space $X$ satisfying $(Q_m)$ for some $m > \aleph_0$, the Boolean algebra $B(X)$ of open and closed subsets of $X$ has the property that every subset of less than $m$ elements has a least upper bound. A lattice is said to be (conditionally) $\sigma$-complete if every bounded countable subset has a least upper bound and a greatest lower bound. In view of the above (and since every subset of $B(X)$ is bounded), in case $m = \aleph_1$, the problem asks if for every compact totally disconnected $F$-space $X$, the Boolean algebra $B(X)$ is $\sigma$-complete.

In [3, Theorem 18], it is shown that if $X$ is compact and totally disconnected, then $B(X)$ is $\sigma$-complete if and only if $C(X)$ is $\sigma$-complete (as a lattice). It is noted in [2, Theorem 8.3, f.f.] that for a completely regular space $Y$, the lattice $C(Y)$ is $\sigma$-complete if and only if $f \in C(Y)$ implies $\overline{P}(f)$ and $\overline{N}(f)$ are disjoint open and closed subsets of $Y$ ($\overline{P}(f)$ denotes the closure of $P(f)$). It is easily seen that $Y$ has this latter property if and only if $\beta Y$ has [2, Lemma 1.6].

In [2, Example 8.10], an example is given of a completely regular space $X$ such that $\beta X$ is a totally disconnected $F$-space, and such that $C(X)$ is not $\sigma$-complete. By the above, it follows that $B(\beta X)$ yields a negative solution to Problem 2.

We remark also (as was pointed out by J. R. Isbell) that if $N$ denotes the countable discrete space, then $\beta N - N$ is also a totally disconnected compact $F$-space such that $B(\beta N - N)$ is not $\sigma$-complete. The
former assertion follows easily from the remarks in § 1, and the latter follows from the fact that $B(\beta N - N)$ is isomorphic to the Boolean algebra of all subsets of $N$ modulo the ideal of finite subsets of $N$ (under the correspondence induced by sending a subset of $N$ to the intersection of its closure in $\beta N$ with $\beta N - N$). It is easily verified that this latter Boolean algebra is not $\sigma$-complete.

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