ON THE INEQUALITY $\Delta u \geq f(u)$

ROBERT OSSERMAN
We are interested in solutions of the non-linear differential inequality

\[(1) \quad \Delta u \geq f(u)\]

where \(u(x_1, \ldots, x_n)\) is to be defined in some region of Euclidean \(n\)-space and \(\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}\) is the Laplacian of \(u\). Wittich [5] considered the corresponding equation

\[(1a) \quad \Delta u = f(u)\]

in two dimensions and found conditions on \(f(u)\) which guarantee that (1a) has no solution valid in the whole plane. Haviland [1] found a slightly weaker result in 3 dimensions, and Walter [4] generalized Wittich's theorem to \(n\)-dimensions. The method is essentially the same in all three papers, resulting on the one hand in the requirement that the function \(f(u)\) be convex, and on the other hand in a rather involved argument for the \(n\)-dimensional case. The proofs do extend immediately to the inequality (1).

In the present paper we deal directly with (1), and obtain in particular a simple proof of a stronger theorem (Theorem 1 below) where the convexity of \(f(u)\) is no longer required. Our method also yields much more precise information on the behavior of solutions.

Recently Redheffer [3] has obtained in the two-dimensional case an improvement of our Theorem 1, where the monotonicity of \(f(u)\) is not needed. Although Redheffer's theorem may very likely be extendable to \(n\) dimensions, it does not seem possible by his method to obtain the more precise results mentioned in the remarks following Theorem 1.

The present investigation resulted from an attempt to determine the type of a class of Riemann surfaces. One result, Theorem 2, is given here as an application of Theorem 1.

We should like to mention finally that the method presented here has been developed independently by Keller, who, in a paper to be published, derives further information on the behavior of solutions of (1a), and applies his results to an interesting physical problem described in [2].

Notation. Throughout this paper we shall reserve \(r\) for the polar
distance, \( r = \sqrt{x_1^2 + \cdots + x_n^2} \), in space of some fixed dimension \( n \geq 1 \). We note that if \( \varphi(r) \) is considered as a function in this space depending only on \( r \), then

\[
2 \quad \Delta \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{n-1}{r} \frac{\partial \varphi}{\partial r} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial \varphi}{\partial r} \right).
\]

**Lemma 1.** Let \( f(t) \) be a (weakly) monotone increasing continuous function defined for all \( t \). Suppose that there exists a function \( \varphi(x) \) satisfying

\[
3 \quad \varphi''(x) + \frac{n-1}{x} \varphi'(x) = f(\varphi)
\]

for \( 0 \leq x < R \), with \( \varphi'(0) = 0 \) and \( \varphi(x) \to +\infty \) as \( x \to R \). Then if \( u \) is any solution of \( (1) \) for \( r \leq R \), we have \( u(x_1, \ldots, x_n) \leq \varphi(r) \) at each point.

**Proof.** By \( (2) \) the function \( \varphi(r) \) satisfies \( \Delta \varphi = f(\varphi) \) for \( r < R \). We let \( v = u - \varphi \) and wish to show that \( v \leq 0 \) for \( r < R \). But suppose \( v > 0 \) at some point. Since \( v \to -\infty \) as \( r \to R \) it would follow that \( v \) would take on its maximum at some point \( P \) with \( r < R \). Then \( v > 0 \) in some neighborhood \( N \) of \( P \), that is \( u > \varphi \) throughout \( N \). This implies \( \Delta v = \Delta u - \Delta \varphi \geq f(u) - f(\varphi) \geq 0 \), so that \( \Delta v \) would be subharmonic in \( N \), contradicting that \( v \) had a maximum at \( P \).

**Lemma 2.** If \( f(t) > 0 \), \( f'(t) \) continuous, and \( f'(t) \geq 0 \) for all \( t \), then equation \( (1) \) has a solution \( u \) valid for all \( (x_1, \ldots, x_n) \) if and only if there is a solution of \( (3) \) valid for all \( x \geq 0 \), with \( \varphi'(0) = 0 \).

**Proof.** If such a function \( \varphi \) exists, then \( \varphi(r) \) is the desired solution of \( (1) \).

Conversely, suppose that no such function \( \varphi(x) \) exists. Given an arbitrary real number \( a \), there exists\(^1\) in any case a solution of \( (3) \) with initial values \( \varphi(0) = a \), \( \varphi'(0) = 0 \), valid in some interval \( 0 \leq x \leq x_0 \). Then there is a maximal interval \( 0 \leq x < R \) in which this solution exists.

Further, we have by \( (2) \) that \( \frac{d}{dx} (x^{n-1} \varphi') = x^{n-1} f(\varphi) > 0 \) for \( x > 0 \), so that \( x^{n-1} \varphi' \) is increasing, hence positive for \( x > 0 \) since \( \varphi'(0) = 0 \). Under these conditions we must have \( \varphi(x) \to +\infty \) as \( x \to R \). Then by Lemma 1 any solution \( u \) of \( (1) \) would satisfy \( u \leq \varphi \) for \( r < R \). In particular we would

\(^1\) The existence does not follow immediately from classical theorems, but may be proved by writing equation \( (3) \) in the integral form \( \varphi(x) = a + \int_0^x \frac{1}{s^{n-1}} \int_0^s t^{n-1} f(\varphi) \, dt \, ds \) and applying standard iteration procedure.
have \( u(0) \leq \varphi(0) = a \). But since \( a \) was arbitrary there could be no solution \( u \) valid in \( r < R \) for arbitrarily large \( R \).

**Lemma 3.** If \( f(t) > 0 \), \( f'(t) \) continuous, and \( f'(t) \geq 0 \) for all \( t \), then equation (3) has a solution \( \varphi \) with \( \varphi'(0) = 0 \) valid for all \( x \geq 0 \) if and only if

\[
\int_0^\infty \left( \int_0^t f(s) \, ds \right)^{-1/2} \, dt = \infty.
\]

**Proof.** Suppose first that there does not exist a solution of (3) valid for all \( x \geq 0 \). Then we have seen that if \( \varphi(x) \) satisfies (3) in some interval, with \( \varphi(0) = 0 \) and \( \varphi'(0) = 0 \), then for some \( R > 0 \) we will have \( \varphi(x) \to +\infty \) as \( x \to R \). Further we noted that for \( x > 0 \), \( \varphi'(x) > 0 \), and hence from equation (3), \( \varphi'' < f(\varphi) \). Thus \( \varphi'' < f(\varphi) \varphi' \) and integrating from \( x = 0 \) to \( x = t \) gives

\[
[\varphi'(t)]^2 < 2 \int_0^t f(\varphi) \varphi' \, d\varphi = 2 \int_0^{\varphi(t)} f(\varphi) \, d\varphi.
\]

Hence

\[
\left( \int_0^\varphi f(s) \, ds \right)^{-1/2} \, d\varphi < \sqrt{2} \, dt
\]

and integration from \( t = 0 \) to \( t = R \) gives

\[
\int_0^\infty \left( \int_0^t f(s) \, ds \right)^{-1/2} \, d\varphi < \sqrt{2} \, R.
\]

Suppose conversely that

\[
\int_0^\infty \left( \int_0^t f(s) \, ds \right)^{-1/2} \, dt < \infty.
\]

Then \( t \cdot \left( \int_0^t f(s) \, ds \right)^{-1/2} \to 0 \) as \( t \to \infty \) since \( \left( \int_0^t f(s) \, ds \right)^{-1/2} \) is monotone decreasing. Hence \( t^{-2} \cdot \int_0^t f(s) \, ds \to \infty \) and \( f(t)/t \to \infty \) since \( f(t) \) is monotone increasing. Thus for an arbitrary fixed \( a \), \( f(t) > t - a \), for \( t > t_0 \). Further, if \( \varphi \) is the solution of (3) with \( \varphi(0) = a \), \( \varphi'(0) = 0 \), then \( \varphi(x) \geq a \) for \( x \geq 0 \), and \( f(\varphi) \geq f(a) \). Hence \( (x^{n-1} \varphi')' \geq f(a) x^{n-1} \), and integrating twice we find \( x^{n-1} \varphi' \geq \frac{f(a)}{2n} x^n \), \( \varphi \geq \frac{f(a)}{2n} x^2 \). Thus \( \varphi(x) > t_0 \) for \( x > x_0 \). As above we note that

\[
\varphi'^2 < 2 \int_a^\varphi f(\varphi) \, d\varphi \leq 2(\varphi - a) f(\varphi) < 2[f(\varphi)]^2 \quad \text{for } \varphi > t_0.
\]

Hence \( \frac{n-1}{x} \varphi' < \frac{f(\varphi)}{2} \) for \( x > x_1 \), and consequently \( \varphi'' > \frac{1}{2} f(\varphi) \) for \( x > x_1 \).
Thus
\[ [\varphi'(x)]^2 - [\varphi'(x_1)]^2 \geq \int_{\varphi(x_1)}^{\varphi(x)} f(s) \, ds \]
or
\[ [\varphi'(x)]^2 \geq \int_{0}^{\varphi(x)} f(s) \, ds - C \]
whence
\[ \int_{0}^{\varphi(x)} \left( \int_{0}^{t} f(s) \, ds - C \right)^{-1/2} \, dt > x - x_1. \]

Since the constant C does not affect the convergence of the integral we have that x must be bounded, which completes the proof of the lemma.

We may note that the proof of Lemma 3 is essentially that of Haviland [1]. The assumption made by Haviland that \( f(t) \geq c > 0 \) is seen to be unnecessary, but it is interesting to note that the theorem is no longer true in \( n \geq 3 \) dimensions if we weaken the requirement to \( f(t) \geq 0 \). (If we allow \( f(t) = 0 \) we must speak of non-constant solutions of (3) for all x.) The reason for this is that a non-constant subharmonic function in one or two dimensions cannot be bounded above, while in three or more dimensions it can. Thus if we set \( f(t) = 0 \) for \( t \leq t \) and \( f(t) = t^2 \) for \( t > 0 \), we see that any negative subharmonic function \( \varphi \) (such as \( \varphi(r) = -1/(1+r^2) \) in 4 dimensions) satisfies \( \Delta \varphi \geq f(\varphi) \) throughout space, although the integral in (4) converges.

Combining these three lemmas we obtain the desired result:

**Theorem 1.** Let \( f(t) \) be positive, continuous, and monotone increasing for \( t \geq t_0 \), and suppose
\[ \int_{0}^{\infty} \left( \int_{0}^{t} f(s) \, ds \right)^{-1/2} \, dt < \infty. \]
Then a twice continuously differentiable function \( u \) cannot satisfy \( \Delta u > 0 \) throughout space and \( \Delta u \geq f(u) \) outside of some sphere \( S \).

**Proof.** Suppose such a function \( u \) exists. Then it has a maximum \( t_1 \) on \( S \), and \( \Delta u \) has a minimum \( m > 0 \) on \( S \). Define \( g(t) \) to be continuously differentiable for all \( t \), and such that
\begin{align*}
a) \quad & g'(t) \geq 0 \quad \text{for all } t \\
b) \quad & g(t) \leq m \quad \text{for } t \leq t_1 \\
c) \quad & g(t) \leq f(t) \quad \text{for all } t \\
d) \quad & g(t) \geq f(t) - 1 \quad \text{for } t \geq t_2.
\end{align*}
Then \( \Delta u \geq g(u) \) throughout space, so that by Lemma 2 there exists
a solution of (3) with $f$ replaced by $g$, and by Lemma 3 we would have

$$\int_{\infty}^{\sigma} \left( \int_{0}^{t} g(s) \, ds \right)^{-1/2} \, dt = \infty$$

which, in view of d), contradicts the hypothesis.

Remarks 1. That the integral condition on $f(t)$ is the best possible can be seen most easily, as was pointed out by Walter [4], by noting that for an arbitrary continuous positive function $f(t)$ we can define $u(x_\lambda)$ for $x_\lambda \ge 0$ as the inverse of $x_\lambda$ and for $x_\lambda < 0$ by $u(x_\lambda) = u(-x_\lambda)$. Then $\Delta u = \frac{\partial^2 u}{\partial x_\lambda^2} = f(u)$ in any number of dimensions, and if the integral diverges this will hold for all $x_\lambda$, and hence throughout space.

2. We may note that in the proof of Lemma 3 we have obtained somewhat more than the non-existence of a solution for all $x$. Namely, we have an upper bound on the values of $x$ for which (3) can hold. However, the expression obtained is not a very convenient one, and in any case does not give the best possible bound. The advantage of Lemma 1 is that it allows us to give the best bound whenever we can find the function $\varphi$ explicitly. For example, if we have the inequality $\Delta u \ge \varepsilon e^{\alpha u}$, $\varepsilon > 0$, in two dimensions, then we can easily verify that

$$\varphi = \log \frac{2R}{\sqrt{\varepsilon} (R^2 - r^2)}$$

satisfies the hypotheses of Lemma 1, so that $u(0) \le \varphi(0) = \log \frac{2}{R\sqrt{\varepsilon}}$.

We may therefore state the following result:

*If $u$ satisfies $\Delta u \ge \varepsilon e^{\alpha u}$ for $r \le R$ and $u(0) = a$, then $R \le \frac{2}{e^{\alpha \sqrt{\varepsilon}}}$.***

3. We note that in the proof of Lemma 1 we need only assume that $\varphi$ satisfies the inequality $\varphi'' + \frac{n-1}{x} \varphi' \le f(\varphi)$. In many cases it may be possible to find an explicit solution of this inequality, but not of equation (3). For example, if $f(\varphi) = \varepsilon |\varphi|^\alpha$, $\alpha > 1$, then the function

$$\varphi = \frac{cR^{2m}}{(R^2 - r^2)^m}, \quad c > 0$$
satisfies in \( n \) dimensions

\[
\Delta \varphi = 2mR^{-4}c^{-2/m}(nR^2 + (2m + 2 - n)r^2)^{\varphi^{1+2/m}} 
\leq 4m(m+1)R^{-2}c^{-2/m}\varphi^{1+2/m}
\text{ if } 2m + 2 \geq n, \ r < R.
\]

Hence \( \Delta \varphi \leq \varepsilon u^{1+2/m} \) if \( R \geq 2(m+1)e^{-1/2}c^{-1/m} \). We can therefore state the following:

If \( u \) satisfies \( \Delta u \geq \varepsilon |u|^\alpha \) for \( r \leq R \) in \( n \)-dimensions, where \( \varepsilon > 0 \) and \( \alpha > 1 \), and if \( u(0) = a > 0 \), then \( R \leq (m+1)e^{-1/2}a^{-1/m} \), where

\[
m = \max \left\{ \frac{n}{2} - 1, \frac{2}{\alpha - 1} \right\}.
\]

4. The above remarks may also be viewed from the other direction. That is, if a function \( u \) is known to satisfy (1) for \( r \leq R \), then we get a pointwise upper bound on \( u \) in terms of the solution of (3). Furthermore, if we know that \( u \leq M \) for \( r = R \), then we can improve these bounds. Namely, we have \( u \leq \varphi \), where \( \varphi \) is the solution of (3) with \( \varphi'(0) = 0 \) and \( \varphi(R) = M \). Finally, these bounds are again the best possible since \( \varphi(r) \) itself satisfies (1).

We turn next to an application of Theorem 1.

**Theorem 2.** If a simply-connected surface \( S \) has a Riemannian metric whose Gauss curvature \( K \) satisfies \( K \leq -\varepsilon < 0 \) everywhere, then \( S \) is conformally equivalent to the interior of the unit circle.

**Proof.** Considering \( S \) as a Riemann surface, we know that it can be mapped conformally onto either the interior of the unit circle or else the whole plane. We proceed by contradiction. Suppose we could map \( S \) conformally onto the \( x, y \)-plane. The Riemannian metric on \( S \) could then be expressed as \( ds^2 = \lambda^2(dx^2 + dy^2) \), and we have for the Gauss curvature:

\[
K = -\frac{\Delta \log \lambda}{\lambda^2}.
\]

\( K \leq -\varepsilon \) means that the function \( u = \log \lambda \) would have to satisfy \( \Delta u \geq \varepsilon e^{2u} \) throughout the plane, contradicting Theorem 1.

We remark finally that the condition \( K \leq -\varepsilon \) can be weakened slightly to \( K < 0 \), and

\[
\int_R \int R K d\omega \leq -\varepsilon < 0
\]

\[
\int_R d\omega
\]
for every region $R$ on the surface including some fixed compact set $D$, where $d\omega$ is the area element on the surface. The proof of this again involves assuming that the surface may be mapped conformally onto the whole $x, y$-plane and defining $z(r)$ as the mean value of $K$ over the disk $x^2 + y^2 \leq r^2$. Simple inequalities yield

$$z'' + \frac{3z'}{r} \geq e^{rz}$$ for $r$ sufficiently large,

and

$$z'' + \frac{3z'}{r} > 0 \quad \text{everywhere.}$$

Since $z'' + \frac{3z'}{r}$ is just the Laplacian of $z$ in four dimensions, we again have a contradiction to Theorem 1.

REFERENCES

5. H. Wittich, Ganze Lösungen der Differentialgleichung $\Delta u = e^u$, Math. Z. 49 (1944), 579–582.

STANFORD UNIVERSITY
PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. L. ROYDEN
Stanford University
Stanford, California

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

R. A. BEAUMONT
University of Washington
Seattle 5, Washington

E. G. STRAUS
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
C. E. BURGESS
M. HALL
E. HEWITT

A. HORN
V. GANAPATHY IYER
R. D. JAMES
M. S. KNEBELMAN

L. NACHBIN
I. NIVEN
T. G. OSTROM
M. M. SCHIFFER

G. SZEKERES
F. WOLF
K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
THE RAMO-WOOLDRIDGE CORPORATION

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.
Robert George Buschman, *A substitution theorem for the Laplace transformation and its generalization to transformations with symmetric kernel* ................................................................. 1529

S. D. Conte, *Numerical solution of vibration problems in two space variables* ................................................................. 1535

Paul Dedecker, *A property of differential forms in the calculus of variations* ........................................................................ 1545

H. Delange and Heini Halberstam, *A note on additive functions* .......... 1551

Jerald L. Ericksen, *Characteristic direction for equations of motion of non-Newtonian fluids* ................................................................. 1557

Avner Friedman, *On two theorems of Phragmén-Lindelöf for linear elliptic and parabolic differential equations of the second order* .......... 1563

Ronald Kay Getoor, *Additive functionals of a Markov process* ................................................................. 1577

U. C. Guha, *(γ, k)-summability of series* ................................................................. 1593

Alvin Hausner, *The tauberian theorem for group algebras of vector-valued functions* ................................................................. 1603

Lester J. Heider, *T-sets and abstract (L)-spaces* ................................................................. 1611

Melvin Henriksen, *Some remarks on a paper of Aronszajn and Panitchpakdi* ................................................................. 1619


Robert Osserman, *On the inequality Δu ≥ f (u)* ................................................................. 1641

Calvin R. Putnam, *On semi-normal operators* ................................................................. 1649

Binyamin Schwarz, *Bounds for the principal frequency of the non-homogeneous membrane and for the generalized Dirichlet integral* ................................................................. 1653

Edward Silverman, *Morrey’s representation theorem for surfaces in metric spaces* ................................................................. 1677

V. N. Singh, *Certain generalized hypergeometric identities of the Rogers-Ramanujan type. II* ................................................................. 1691

R. J. Smith, *A determinant in continuous rings* ................................................................. 1701

Drury William Wall, *Sub-quasigroups of finite quasigroups* ................................................................. 1711

Sadayuki Yamamuro, *Monotone completeness of normed semi-ordered linear spaces* ................................................................. 1715

C. T. Rajagopal, *Simplified proofs of “Some Tauberian theorems” of Jakimovski: Addendum and corrigendum* ................................................................. 1727

N. Aronszajn and Prom Panitchpakdi, *Correction to: “Extension of uniformly continuous transformations in hyperconvex metric spaces”* ................................................................. 1729

Alfred Huber, *Correction to: “The reflection principle for polyharmonic functions”* ................................................................. 1731