

Pacific Journal of Mathematics

CERTAIN GENERALIZED HYPERGEOMETRIC IDENTITIES OF THE ROGERS-RAMANUJAN TYPE. II

V. N. SINGH

CERTAIN GENERALIZED HYPERGEOMETRIC IDENTITIES OF THE ROGERS-RAMANUJAN TYPE (II)

V. N. SINGH

1. Introduction. Nearly two years ago, Alder [1] established the following generalizations of the well-known Rogers-Ramanujan identities:

$$(1) \quad \prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-M})(1-x^{(2M+1)n-(M+1)})(1-x^{(2M+1)n})}{(1-x^n)} = \sum_{t=0}^{\infty} \frac{G_{M,t}(x)}{(x)_t},$$

$$(2) \quad \prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-1})(1-x^{(2M+1)n-2M})(1-x^{(2M+1)n})}{(1-x^n)} = \sum_{t=0}^{\infty} x^t \frac{G_{M,t}(x)}{(x)_t},$$

where $G_{M,t}(x)$ are polynomials which reduce to x^{t^2} for $M=2$ and

$$(x)_t = (1-x)(1-x^2)\cdots(1-x^t), \quad (x)_0 = 1.$$

In a recent paper [6] I gave a simple alternative proof of (1) and (2). We used the result

$$(3) \quad 1 + \sum_{s=1}^{\infty} (-1)^s k^{Ms} x^{\frac{1}{2}s(2M+1)s-1} (1-kx^{2s}) \frac{(kx)_{s-1}}{(x)_s} \\ = \prod_{n=1}^{\infty} (1-kx^n) \sum_{t=0}^{\infty} \frac{k^t G_{M,t}(x)}{(x)_t}, \quad M=2, 3, \dots$$

Alder in his paper states that identities involving the generating function for the number of partitions into parts not congruent to $0, \pm(M-r) \pmod{2M+1}$, where $0 \leq r \leq M-1$, can be obtained by his method and indicates the result for $r=1$.

In the present paper I give a simple method of obtaining the M identities for each modulus $(2M+1)$. In § 4 identities for which $r \geq \frac{1}{2}M$ have been deduced and in § 5 those for which $r \leq \frac{1}{2}M$ have been obtained for any r such that $0 \leq r \leq M-1$. The identities given in § 5 have not been mentioned by Alder. As a corollary, an interesting identity between two infinite series is given.

2. Notations. Assuming $|x| < 1$, let

$$(\alpha)_n \equiv (\alpha)_{x,n} = (1-\alpha)(1-\alpha x)\cdots(1-\alpha x^{n-1}), \quad (\alpha)_0 = 1,$$

$$(\alpha)_{-n} = (-1)^n x^{\frac{1}{2}n(n+1)} / \alpha^n (x/\alpha)_n,$$

$$x_n = 1 + x + x^2 + \cdots + x^{n-1}.$$

$$(4) \quad P_{m,t}(x) \equiv x^{\frac{1}{2}(\ell+1-m)\ell} \frac{(x)_m}{(x)_\ell(x)_{m-\ell+1}} (1-x^{m-2\ell+1}),$$

and let

$$(5) \quad \phi(M, x^r) = 1 + \sum_{s=1}^{\infty} (-1)^s x^{Mrs} x^{\frac{1}{2}s\{(2M+1)s-1\}} (1-x^{2s+r})(x^{s+1})_{r-1},$$

so that (3) can be written as

$$(6) \quad \phi(M, x^r) = \prod_{n=1}^{\infty} (1-x^n) \sum_{\ell=0}^{\infty} \frac{x^{r\ell} G_{M,\ell}(x)}{(x)_\ell}.$$

3. The polynomials $u_n(x)$. Before proceeding to deduce the generalized identities, we first give a few properties of a sequence of polynomials with the help of an operator. These we will need in later sections. Let us define a sequence $\{u_n(x)\}$ of polynomials by the relations

- (i) $u_0(x) = 0$
- (ii) $u_n(x) = u_{n-1}(x) + x^{n-1}x_n, \quad n \geq 1.$

Let \mathcal{R} be an operator which replaces x_m by $u_m(x)$ in any $u_n(x)$, that is,

$$\mathcal{R}u_n(x) = \mathcal{R}u_{n-1}(x) + x^{n-1}u_n(x).$$

Also

$$\mathcal{R}^n u_m(x) = \mathcal{R}^{n-1} \{ \mathcal{R}u_m(x) \}.$$

Then we have

$$(7) \quad (\alpha)_n = 1 - \alpha x_n + \sum_{s=2}^n (-\alpha)^s x^{\frac{1}{2}s(s-1)} \mathcal{R}^{s-2} u_{n-s+1}(x).$$

As can be easily shown

$$(8) \quad \mathcal{R}^{s-2} u_{n-s+1}(x) = \frac{(x)_n}{(x)_s(x)_{n-s}}.$$

The above polynomials (8) have also recently occurred in a paper by Carlitz [3].

Comparing the coefficients of α^{s-1} in

$$(9) \quad (\alpha)_n = (-1)^n \alpha^n x^{\frac{1}{2}n(n-1)} (1/\alpha x^{n-1})_n,$$

we get the relation

$$(10) \quad \mathcal{R}^{s-3} u_{n-s+2}(x) = \mathcal{R}^{n-s-1} u_s(x), \quad s=1, \dots, (n+1).$$

We can thus write (7) as

$$(11) \quad (\alpha)_n = \sum_{s=0}^n (-\alpha)^s x^{\frac{1}{2}s(s-1)} \mathcal{P}^{s-2} u_{n-s+1}(x),$$

where negative indices of \mathcal{P} are defined by (10). Again comparing the coefficients of α^s in

$$(\alpha/x^{n-1})_{x,n} = (\alpha)_{1/x,n},$$

we get with the help of (11),

$$(12) \quad \mathcal{P}^{s-2} u_{n-s+1}(x) = x^{s(n-s)} \mathcal{P}^{s-2} u_{n-s+1}(x^{-1}).$$

In particular

$$(13) \quad u_n(x) = x^{2n-2} u_n(x^{-1}).$$

The following values of $\mathcal{P}^m u_n(x)$ will also be required:

$$\begin{aligned} \mathcal{P}^{n-3} u_2(x) &= x_n, && \text{from (10)} \\ u_1(x) &= 1 \\ u_3(x) &= 1 + x + 2x^2 + x^3 + x^4 \\ u_4(x) &= 1 + x + 2x^2 + 2x^3 + 2x^4 + x^5 + x^6 = \mathcal{P} u_3(x). \end{aligned}$$

4. Now we proceed to deduce identities involving the generating function for the number of partitions into parts not congruent to 0, $\pm(M-r) \pmod{2M+1}$. From (11), we have

$$\begin{aligned} &(x^{n-r+1})_{2r-1} (1-x^{2n}) \\ &= \left[1 + \sum_{s=1}^{2r-2} \{ (-x^{n-r+1})^s x^{\frac{1}{2}s(s-1)} \mathcal{P}^{s-2} u_{2r-s}(x) \} - x^{(2r-1)n} \right] (1-x^{2n}), \end{aligned}$$

whence

$$(14) \quad \begin{aligned} &1 + x^{(2r+1)n} \\ &= \left[x^{2n} + x^{(2r-1)n} - \left\{ \sum_{s=1}^{2r-2} x^{ns} (-1)^s x^{\frac{1}{2}s(s+1)-rs} \mathcal{P}^{s-2} u_{2r-s}(x) \right\} (1-x^{2n}) \right] \\ &\quad + (1-x^{2n})(x^{n-r+1})_{2r-1}. \end{aligned}$$

And since, because of (8) or (10), the terms equidistant from the two ends in the sum on the right of (14) have equal coefficients of powers of x^n , the expression in square brackets can be written as

$$(15) \quad \sum_{t=1}^r (-1)^{t-1} x^{tn} \{ 1 + x^{(2r-2t+1)n} \} U_{r,t}(x)$$

where

$$(16) \quad \begin{aligned} U_{r,t}(x) &= x^{\frac{1}{2}t(t+1)-tr} \mathcal{P}^{t-2} u_{2r-t}(x) - x^{\frac{1}{2}(t-2)(t-1)-(t-2)r} \mathcal{P}^{t-4} u_{2r-t+2}(x) \\ &= P_{2r,t}(x), && \text{using (4) and (8).} \end{aligned}$$

The polynomials $U_{r,t}(x)$ may be called “reciprocal” since they are such that the terms equidistant from the two ends have equal coefficients. Taking $n=0$ in (14) we see that

$$(17) \quad \sum_{t=1}^r (-1)^{t-1} U_{r,t}(x) = 1 .$$

Also, with the help of (12), we have

$$(18) \quad U_{r,t}(x) = U_{r,t}(x^{-1}) .$$

Now from (15)

$$(19) \quad \begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n\{(2M+1)n+2r+1\}} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{\frac{1}{2}n\{(2M+1)n-1\}}}{x^{rn}} \{1 + x^{(2r+1)n}\} \\ &= 1 + \sum_{t=1}^r (-1)^{t-1} U_{r,t}(x) \sum_{n=1}^{\infty} \frac{(-1)^n x^{\frac{1}{2}n\{(2M+1)n-1\}}}{x^{(r-t)n}} \{1 + x^{(2r-2t+1)n}\} \\ & \quad + \sum_{n=1}^{\infty} \frac{(-1)^n x^{\frac{1}{2}n\{(2M+1)n-1\}}}{x^{rn}} (1-x^{2n})(x^{n-r+1})_{2r-1} . \end{aligned}$$

For $n=s+r$, the last series on the right-hand side of (19) becomes

$$(-1)^r x^{Mr^2 - \frac{1}{2}r(r+1)} \phi(M, x^{2r})$$

since the first $(r-1)$ terms of the series vanish because of the factor $(x^{n-r+1})_{2r-1}$. Then using (17) and writing

$$F(M, r) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n\{(2M+1)n+2r+1\}}$$

we obtain for (19) the form

$$(20) \quad F(M, r) = \sum_{t=1}^r (-1)^{t-1} U_{r,t}(x) F(M, r-t) + (-1)^r x^{Mr^2 - \frac{1}{2}r(r+1)} \phi(M, x^{2r}) .$$

Thus, using Jacobi’s classical identity

$$(21) \quad \sum_{n=-\infty}^{\infty} (-1)^n x^{n^2} z^n = \prod_{n=1}^{\infty} (1-x^{2n-1}z)(1-x^{2n-1}/z)(1-x^{2n})$$

to express the infinite series in (20) as infinite products, we could find for any given r , such that $0 \leq r \leq M-1$, an expression for the generating function for the number of partitions into parts not congruent to $0, \pm(M-r) \pmod{2M+1}$ in terms of generating functions for the number of partitions into parts not congruent to $0, \pm(M-s) \pmod{2M+1}$,

($s=0, 1, 2, \dots, r-1$). Since $F(M, 0)=\phi(M, 1)$, the F -series can be successively expressed in terms of ϕ -series and, with the help of (6), we get

THEOREM 1.

$$(22) \quad \prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-(M-r)})(1-x^{(2M+1)n-(M+r+1)})(1-x^{(2M+1)n})}{(1-x^n)} = \sum_{t=0}^{\infty} \frac{A_r(x, t)G_{M,t}(x)}{(x)_t}$$

where

$$A_r(x, t) = \sum_{s=0}^r (-1)^s x^{Ms^2 - \frac{1}{2}s(s+1) + 2st} U'_{r, s+1}(x).$$

The polynomials $U'(x)$ are of the "reciprocal" kind, with

$$\begin{aligned} U'_{r, r+1}(x) &= 1 \\ U'_{r, s+1}(x) &= \sum_{m=1}^{r-s} (-1)^{m-1} U_{r, m}(x) U'_{r-m, s+1}(x), \quad s \neq r. \end{aligned}$$

so that

$$U'_{r, 1}(x) = 1, \quad \text{because of (17)}$$

and

$$U'_{r, m}(x) = U'_{r, m}(x^{-1}), \quad \text{because of (18).}$$

As an example of Theorem 1, taking the case $r=1$, we have

$$1 + x^{2n} = x^n(1 + x^n) + (1 - x^n)(1 - x^{2n}).$$

Therefore

$$(23) \quad \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n\{(2M+1)n+3\}} = \phi(M, 1) - x^{M-1}\phi(M, x^2)$$

which is equivalent to equation (23) of Alder [1].

From (23), using (21) and (6), we get the identity

$$(24) \quad \prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-(M-1)})(1-x^{(2M+1)n-(M+2)})(1-x^{(2M+1)n})}{(1-x^n)} = \sum_{t=0}^{\infty} \frac{(1-x^{M+2t-1})}{(x)_t} G_{M,t}(x).$$

For $r=2$,

$$(25) \quad \prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-(M-2)})(1-x^{(2M+1)n-(M+3)})(1-x^{(2M+1)n})}{(1-x^n)} = \sum_{t=1}^{\infty} \frac{\{1 - U'_{2,2}(x)x^{M+2t-1} + x^{4M+4t-3}\}}{(x)_t} G_{M,t}(x),$$

where

$$U'_{2,2}(x) = x^{-1} + 1 + x .$$

Similarly for $r=3$ we get

$$\begin{aligned} U'_{3,2}(x) &= x^{-2} + x^{-1} + 2 + x + x^2 \\ U'_{3,3}(x) &= x^{-2} + x^{-1} + 1 + x + x^2 , \end{aligned}$$

and so on for any r such that $0 \leq r \leq M-1$.

5. In this section identities involving the generating function for the number of partitions into parts not congruent to $0, \pm r \pmod{2M+1}$ are obtained.

From (11) we have

$$\begin{aligned} &(x^{n-r+2})_{2r-2} (1-x^{2n+1}) \\ &= \left[1 + \left\{ \sum_{s=1}^{2r-3} (-x^{n-r+2})^s x^{\frac{1}{2}s(s-1)} \mathcal{R}^{s-2} u_{2r-s-1}(x) \right\} + x^{(2n+1)(r-1)} \right] (1-x^{2n+1}) , \end{aligned}$$

whence

$$\begin{aligned} &1 - x^{(2n+1)r} \\ (26) \quad &= \left[x^{2n+1} - x^{(2n+1)(r-1)} - \left\{ \sum_{s=1}^{2r-3} x^{ns} (-1)^s x^{\frac{1}{2}s(s+3)-rs} \mathcal{R}^{s-2} u_{2r-s-1}(x) \right\} (1-x^{2n+1}) \right] \\ &\quad + (1-x^{2n+1})(x^{n-r+2})_{2r-2} . \end{aligned}$$

In the expression in square brackets in (26), the terms containing x^{nr} cancel and the other terms can again be grouped in pairs to give

$$(27) \quad 1 - x^{(2n+1)r} = \sum_{t=1}^{r-1} (-1)^{t-1} V_{r,t}(x) x^{tn} \{ 1 - x^{(2n+1)(r-t)} \} + (1-x^{2n+1})(x^{n-r+2})_{2r-2} ,$$

where

$$\begin{aligned} (28) \quad V_{r,t}(x) &= x^{\frac{1}{2}t(t+3)-rt} \mathcal{R}^{t-2} u_{2r-t-1}(x) - x^{\frac{1}{2}t(t-1)-r(t-2)} \mathcal{R}^{t-4} u_{2r-t+1}(x) \\ &= x^{\frac{1}{2}t} P_{2r-1,t}(x) . \end{aligned}$$

The polynomials $V(x)$ are less symmetric than $U(x)$. In particular, corresponding to (17) and (18), they satisfy the relations

$$(29) \quad \sum_{t=1}^{r-1} (-1)^{t-1} V_{r,t}(x) x_{r-t} = x_r ,$$

and

$$(30) \quad V_{r,t}(x) = x^t V_{r,t}(x^{-1}) .$$

Now

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n\{(2M+1)(n+1)\}-rn} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{1}{2}n^2(2M+1)+(M-\frac{1}{2})n}}{x^{(r-1)n}} \{1-x^{(2n+1)r}\}.$$

Denoting the left-hand side of the last equation by $\phi(M, r)$ and using (27) and (5), we get, after slight simplification,

$$(31) \quad \phi(M, r) = \sum_{t=1}^{r-1} (-1)^{t-1} \phi(M, r-t) V_{r,t}(x) + (-1)^{r-1} x^{\frac{1}{2}(2M-1)r(r-1)} \phi(M, x^{2r-1}).$$

Using (21), the generating function for the number of partitions into parts not congruent to $0, \pm r \pmod{2M+1}$ can now be expressed in terms of the generating function for the number of partitions into parts not congruent to $0, \pm s \pmod{2M+1}$, ($s=1, 2, \dots, r-1$). Thus, we finally have

THEOREM 2.

$$(32) \quad \prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-r})(1-x^{(2M+1)n-(2M+1-r)})(1-x^{(2M+1)n})}{(1-x^n)} = \sum_{t=0}^{\infty} \frac{B_r(x, t) G_{M,t}(x)}{(x)_t},$$

where

$$B_r(x, t) = \sum_{s=1}^r (-1)^{s-1} x^{\frac{1}{2}(2M-1)s(s-1)+(2s-1)t} V'_{r,s}(x),$$

and $V'_{r,s}(x)$ are polynomials with

$$\begin{aligned} V'_{r,r}(x) &= 1 \\ V'_{r,s}(x) &= \sum_{m=1}^{r-s} (-1)^{m-1} V_{r,m}(x) V'_{r-m,s}(x), \quad s \neq r, \end{aligned}$$

so that

$$V'_{r,1}(x) = x_r$$

and

$$V'_{r,t}(x) = x^{r-t} V'_{r,t}(x^{-1}).$$

As an illustration, for $r=2$ in Theorem 2, we have

$$1-x^{4n+2} = x^n(1+x)(1-x^{2n+1}) + (1-x^{2n+1})(1-x^n)(1-x^{n+1}).$$

Therefore

$$\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n^2(2M+1)+(M-\frac{3}{2})n} = (1+x)\phi(M, x) - x^{2M-1}\phi(M, x^3),$$

which gives us the identity

$$(33) \quad \prod_{n=1}^{\infty} \frac{(1-x^{(2M+1)n-2})(1-x^{(2M+1)n-(2M-1)})(1-x^{(2M+1)n})}{(1-x^n)} = \sum_{t=0}^{\infty} \frac{\{(1+x)x^t-x^{2M+3t-1}\}}{(x)_t} G_{M,t}(x).$$

COROLLARY. If r is replaced by $M-r$ in Theorem 2 then the left-hand sides of (22) and (32) become the same and we have

$$(34) \quad \sum_{t=0}^{\infty} \frac{A_r(x,t)G_{M,t}(x)}{(x)_t} = \sum_{t=0}^{\infty} \frac{B_{M-r}(x,t)G_{M,t}(x)}{(x)_t},$$

$r=0, 1, \dots, M-1, M=2, 3, \dots$

For $M=2$ and $r=0$ and 1 we get respectively the relations

$$\sum_{t=0}^{\infty} \frac{x^t}{(x)_t} = \sum_{t=0}^{\infty} \frac{\{(1+x)x^t-x^{3(t+1)}\}}{(x)_t} x^{t^2}$$

$$\sum_{t=0}^{\infty} \frac{(1-x^{2t+1})}{(x)_t} x^{t^2} = \sum_{t=0}^{\infty} \frac{x^{t+t^2}}{(x)_t}$$

the truth of which can easily be verified.

Some time ago, Slater ([4] and [5]) gave a very large number of identities of the Rogers-Ramanujan type using Bailey’s summation theorem [2] for a well-poised ${}_6P_6$. It is interesting to note that, as special cases of our identities, we get some of those given by Slater, differing only in form as can be easily verified. To mention an example, let us take equation (90) of Slater [5]:

$$(35) \quad \prod_{n=1}^{\infty} \frac{(1-x^{27n-3})(1-x^{27n-24})(1-x^{27n})}{(1-x^n)} = \sum_{t=0}^{\infty} \frac{(x^3)_{x^3,t} x^{t(t+3)}}{(x)_t(x)_{2t+2}},$$

If we put $M=13, r=3$ in Theorem 2, we obtain another series for the product on the left of (35). I propose to study the equivalence of identities (22) and (32) above and those of Slater in a subsequent paper, as also identities involving products in which the powers increase by $2M$.

I would like to express my gratitude to Dr. R. P. Agarwal for suggesting the present work and for his kind help in the preparation of this paper.

REFERENCES

1. H. L. Alder, *Generalizations of the Rogers-Ramanujan identities*, Pacific J. Math., **4** (1954), 161-168.
2. W. N. Bailey, *Series of the hypergeometric type which are infinite in both directions*, Quart. J. Math., **7** (1936), 105-115.
3. L. Carlitz, *The expansion of certain products*, Proc. Amer. Math. Soc., **7** (1956), 558-564.

4. L. J. Slater, *A new proof of Rogers's transformation of infinite series*, Proc. London Math. Soc. (2), **53** (1951), 460-475.
5. ———, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2), **54** (1952), 147-167.
6. V. N. Singh, *Certain generalized hypergeometric identities of the Rogers-Ramanujan type* Pacific J. Math. **7** (1957), 1011-1014.

Corrigenda. In [6] the following corrections may be noted:

p. 1011. The series for $T_{n,m}$ runs up to

$$t_n = \left[\frac{M-n-1}{M-n} t_{n-1} \right].$$

p. 1012. In the line immediately preceding (3.3), a_{2Mn-1} should be a_{2M+1} .

In the right hand side of (3.4) a factor $(cn;t)$ should be inserted in the denominator of the outer series.

p. 1014. In the right hand side of the last identity of the paper, we should have Π instead of π .

UNIVERSITY OF LUCKNOW
LUCKNOW, INDIA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. L. ROYDEN
Stanford University
Stanford, California

R. A. BEAUMONT
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

E. G. STRAUS
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
C. E. BURGESS
M. HALL
E. HEWITT

A. HORN
V. GANAPATHY IYER
R. D. JAMES
M. S. KNEBELMAN

L. NACHBIN
I. NIVEN
T. G. OSTROM
M. M. SCHIFFER

G. SZEKERES
F. WOLF
K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
THE RAMO-WOOLDRIDGE CORPORATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Robert Geroge Buschman, <i>A substitution theorem for the Laplace transformation and its generalization to transformations with symmetric kernel</i>	1529
S. D. Conte, <i>Numerical solution of vibration problems in two space variables</i>	1535
Paul Dedecker, <i>A property of differential forms in the calculus of variations</i>	1545
H. Delange and Heini Halberstam, <i>A note on additive functions</i>	1551
Jerald L. Ericksen, <i>Characteristic direction for equations of motion of non-Newtonian fluids</i>	1557
Avner Friedman, <i>On two theorems of Phragmén-Lindelöf for linear elliptic and parabolic differential equations of the second order</i>	1563
Ronald Kay Gettoor, <i>Additive functionals of a Markov process</i>	1577
U. C. Guha, <i>(γ, k)-summability of series</i>	1593
Alvin Hausner, <i>The tauberian theorem for group algebras of vector-valued functions</i>	1603
Lester J. Heider, <i>T-sets and abstract (L)-spaces</i>	1611
Melvin Henriksen, <i>Some remarks on a paper of Aronszajn and Panitchpakdi</i>	1619
H. M. Lieberstein, <i>On the generalized radiation problem of A. Weinstein</i>	1623
Robert Osserman, <i>On the inequality $\Delta u \geq f(u)$</i>	1641
Calvin R. Putnam, <i>On semi-normal operators</i>	1649
Binyamin Schwarz, <i>Bounds for the principal frequency of the non-homogeneous membrane and for the generalized Dirichlet integral</i>	1653
Edward Silverman, <i>Morrey's representation theorem for surfaces in metric spaces</i>	1677
V. N. Singh, <i>Certain generalized hypergeometric identities of the Rogers-Ramanujan type. II</i>	1691
R. J. Smith, <i>A determinant in continuous rings</i>	1701
Drury William Wall, <i>Sub-quasigroups of finite quasigroups</i>	1711
Sadayuki Yamamuro, <i>Monotone completeness of normed semi-ordered linear spaces</i>	1715
C. T. Rajagopal, <i>Simplified proofs of "Some Tauberian theorems" of Jakimovski: Addendum and corrigendum</i>	1727
N. Aronszajn and Prom Panitchpakdi, <i>Correction to: "Extension of uniformly continuous transformations in hyperconvex metric spaces"</i>	1729
Alfred Huber, <i>Correction to: "The reflection principle for polyharmonic functions"</i>	1731