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A DETERMINANT IN CONTINUOUS RINGS

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1. Introduction. In the theory, developed by Dieudonné [1], of determinants of nonsingular square matrices over a noncommutative field K the determinantal values are cosets modulo the commutator subgroup of $K^{\times}$, the multiplicative group of K. Since the matrix groups $M_n^{\times}(K)$ and their commutator subgroups C_n have the property that $M_n^{\times}(K)/C_n$ is independent of n, the latter cosets will serve just as well for determinantal values, at least for theorems involving only the multiplication of determinants.

The rings whose principal right ideal lattices form continuous geometries have many resemblances to matrix rings; in fact, the axioms of Continuous Geometry are satisfied by finite dimensional geometries over a field which are always equivalent to the right ideal lattice of some matrix ring. Irrespective of questions as to the existence or otherwise of fields in connection with a general continuous geometry playing a similar role to that of the field of coordinate values in the finite dimensional case we will show that multiplicative determinantal theorems can be obtained for the more general ring; the determinants will be cosets of the group of invertible ring elements modulo the closure of its commutator subgroup with respect to the rank-distance topology in the ring.

The definition of a complete rank ring is given by von Neumann [3, (iv)]. Essential properties of such a ring $\mathfrak{R}$ and the associated lattice of principal right ideals have been developed by von Neumann [3, 4] and Ehrlich [2]. We will assume throughout that $\mathfrak{R}$ is a complete rank ring, of characteristic not 2; and that if the discrete case (matrices over a field) applies, then the order of the matrices is at least 3.

2. Groups in a complete rank ring. Using a notation similar to that of [2], [3] we denote by $\mathfrak{G}$ the group of invertible ring elements; that is, $u \in \mathfrak{G} \subset \mathfrak{R}$ if and only if the rank $R(u)$ of u is 1.

DEFINITION 1. We denote by $\mathfrak{K}$ the closure of the commutator subgroup of $\mathfrak{G}$ in the rank-distance topology and by $\mathfrak{K}^{\dagger}$ the closure of the group generated by the elements of class 2 in $\mathfrak{G}$.

COROLLARY 1. *$\mathfrak{K}$ and $\mathfrak{K}^{\dagger}$ are groups.*

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Proof. Let $\{t_n;\ t_n \in \mathfrak{C},\ n=1, 2, \cdots\}$ be a converging sequence in $\mathfrak{R}$. Then $\lim_{n,m\to\infty} R(t_n-t_m)=0$ implies

$$\lim_{n,m\to\infty} R(t_n^{-1}-t_m^{-1})= \lim_{n,m\to\infty} R\{t_n^{-1}(t_m-t_n)t_m^{-1}\}=0$$

and hence $\lim_{n\to\infty} t_n^{-1}$ exists in $\mathfrak{R}$. By the continuity of multiplication $(\lim_{n\to\infty} t_n)(\lim_{n\to\infty} t_n^{-1})=1$ so that $\lim_{n\to\infty} t_n^{-1} \in \mathfrak{C}$. The result then follows routinely after the observation that the inverse of a commutator is a commutator and the inverse of the general class 2 element $1+r$ $(r^2=0)$ is $1-r$, also of class 2.

Lemma 1. *Let* $t \in C^2$ *(be of class* 2), $s \in \mathfrak{C}$. *Then* $sts^{-1} \in C^2$.

Corollary 2. *Let* $t \in C^2$, $s \in \mathfrak{C}$. *Then* $st=t_1s$ *for some* $t_1 \in C^2$.

Definition 2. We writh $u \cong s$ for nonsingular (invertible) u, $s \in \mathfrak{R}$ when $u=ts$ for some $t \in \mathfrak{R}^{\dagger}$.

Corollary 3. *The relation* $\cong$ *is an equivalence relation.*

Lemma 2. *Let* e *be any idempotent of rank* 1/2 *and* s *be nonsingular and otherwise arbitrary in* $\mathfrak{R}$. *Then for some* $t \in \mathfrak{R}$

$$s \cong e+(1-e)t(1-e)\,.$$

Proof. The existence of idempotents of rank 1/2 is assumed in continuous rings, that is, when the range of R is the unit interval. In the discrete case the result has no meaning if the order of the matrices is odd.

Now suppose the principal left ideal $((1-e)se)_l=(g_1)_l$ where $g_1=eg_1e$, $g_1^2=g_1$ [4, Chapter 15]. By the Pierce decomposition, s is the sum of the quantities in the blocks of

$$\begin{bmatrix} g_1sg_1 & g_1s(e-g_1) & \\ & & es(1-e) \\ (e-g_1)sg_1 & (e-g_1)s(e-g_1) & \\ (1-e)sg_1 & (1-e)s(e-g_1) & (1-e)s(1-e) \end{bmatrix}$$

where a matrix notation is used for clarity and to permit the comparison of later processes with standard matrix ones; we will simply equate such a partitioned array to the sum of its members. We have

$$g_1=y_1(1-e)se=y_1(1-e)seg_1=y_1(1-e)sg_1$$

for some $y_1 \in \mathfrak{R}$ so that

$$\{1+g_1(g_1-g_1sg_1)y_1(1-e)\}s$$
$$=\begin{bmatrix} g_1 & g_1s(e-g_1) & g_1s^*(1-e) \\ (e-g_1)sg_1 & (e-g_1)s(e-g_1) & (e-g_1)s(1-e) \\ (1-e)sg_1 & 0 & (1-e)s(1-e) \end{bmatrix}$$

for some $s^* \in \mathfrak{R}$ since

$$g_1sg_1+(g_1-g_1sg_1)y_1(1-e)sg_1=g_1sg_1+g_1-g_1sg_1=g_1$$

and

$$(1-e)s(e-g_1)=(1-e)se-(1-e)sg_1=(1-e)seg_1-(1-e)sg_1=0\,.$$

Multiplying on the left by $(1-(1-e)sg_1)(1-(e-g_1)sg_1)$ and on the right by $(1-g_1s(e-g_1))(1-g_1s^*(1-e))$ gives

$$t_1s=\begin{bmatrix} g_1 & 0 & 0 \\ 0 & (e-g_1)s_1(e-g_1) & (e-g_1)s_1(1-e) \\ 0 & (1-e)s_1(e-g_1) & (1-e)s_1(1-e) \end{bmatrix}=s_1$$

for some $s_1 \in \mathfrak{R}$ and some $t_1 \in \mathfrak{K}^\dagger$ by Corollary 2.

Define g_{n+1}, s_{n+1}, t_{n+1} for $n=1, 2, \cdots$ as follows.

Let $((1-e)s_n(e-g_1-\cdots-g_n))_l=(g_{n+1})_l$ where $g_{n+1}^2=g_{n+1}$ and $(e-g_1-\cdots-g_n)g_{n+1}(e-g_1-\cdots-g_n)=g_{n+1}$. We have, similarly to the above, the existence of a $t_{n+1} \in \mathfrak{K}^\dagger$ and an $s_{n+1} \in \mathfrak{R}$ such that

$$t_{n+1}s=\begin{bmatrix} g_1 & & & & 0 & 0 \\ & \ddots & & & & \\ & & g_n & & & \\ & & & g_{n+1} & (e-g_1-\cdots-g_{n+1})s_{n+1}(e-g_1-\cdots-g_{n+1}) & (e-g_1-\cdots-g_{n+1})s_{n+1}(1-e) \\ & 0 & & & (1-e)s_{n+1}(e-g_1-\cdots-g_{n+1}) & (1-e)s_{n+1}(1-e) \end{bmatrix}=s_{n+1}\,.$$

Now,

$$\frac{1}{2} \geqq R(g_1+\cdots+g_n)=R(g_1)+\cdots+R(g_n)=\sum_{i=1}^{n} R((1-e)s_i(e-g_1-\cdots-g_i))$$

so $\lim_{i\to\infty} R((1-e)s_i(e-g_1-\cdots-g_i))=0$ and in turn

$$\lim_{i\to\infty}(1-e)s_i(e-g_1-\cdots-g_i)=0 \tag{1}$$

More strongly,

$$\lim_{n,p\to\infty} R(g_{n+1}+\cdots+g_{n+p})=\lim_{n,p\to\infty}\{R(g_{n+1})+\cdots+R(g_{n+p})\}=0\,.$$

Hence, by [3, (iv), Section 3] $\lim_{n\to\infty}(g_1+\cdots+g_n)=g$, say, exists in $\mathfrak{R}$; also, by the continuity of multiplication, $g=ege$ and g is idempotent, being the limit of a sequence of idempotents.

In order to prove that $\lim_{n\to\infty} t_n$ exists in $\mathfrak{R}$ and so belongs to $\mathfrak{R}^\dagger$ we note that

$$\begin{aligned}(1-(1-e)s_ng_{n+1})&(1-(e-g_1-\cdots-g_{n+1})s_ng_{n+1})\\ &\cdot(1+g_{n+1}(g_{n+1}-g_{n+1}s_ng_{n+1})y_{n+1}(1-e))t_ns\\ &\cdot(1-g_{n+1}s_n(e-g_1-\cdots-g_{n+1}))(1-g_{n+1}s_n^*(1-e))=t_{n+1}s\end{aligned} \tag{2}$$

where $s_n^*\in\mathfrak{R}$ and y_{n+1} is defined by the condition $g_{n+1}=y_{n+1}(1-e)s_ne$ The last two factors on the left side of (2) may be transferred after a similarity transformation to the left of t_ns, by Corollary 2, giving

$$(1+\Phi(g_{n+1}))t_ns=t_{n+1}s$$

where $\Phi(g_{n+1})$ is an expression involving no more than $2^5-1=31$ terms, each containing g_{n+1} as a factor and so of rank $\leqq R(g_{n+1})$. Hence $t_{n+1}-t_n=\Phi(g_{n+1})t_n$ and

$$R(t_{n+1}-t_n)\leqq R\Phi(g_{n+1})\leqq 31R(g_{n+1})\,,$$

$$\begin{aligned}R(t_{n+p}-t_n)&\leqq\sum_{i=1}^{p}R(t_{n+i}-t_{n+i-1})\\ &\leqq 31\sum_{i=1}^{p}R(g_{n+i})\to 0 \text{ as } n,\,p\to\infty\,.\end{aligned}$$

[3, (iv), Equation 3, (iii)]

We conclude that

$$\lim_{n\to\infty}(1-g_1-\cdots-g_n)s_n(1-g_1-\cdots-g_n)=\lim_{n\to\infty}(t_ns-(g_1+\cdots+g_n))$$

exists in $\mathfrak{R}$. It equals $(1-g)t(1-g)$ for some $t\in\mathfrak{R}$. Moreover, $(1-e)\cdot t(e-g)=0$ by (1). Then

$$s\cong\begin{bmatrix} g & 0 & 0\\ 0 & (e-g)t(e-g) & (e-g)t(1-e)\\ 0 & 0 & (1-e)t(1-e)\end{bmatrix}$$

where $R((e-g)t(e-g)) \leqq 1/2$ and $(e-g)t(e-g)$ has an inverse in the subring $\mathfrak{R}(e-g)$.

By the proof of [4, Lemma 3.6], if $(1-e)h(1-e)=h$ is an idempotent of rank equal to $R(e-g)$, then $e-g$, h define quantities $x, y \in \mathfrak{R}$ such that

$$xh=(e-g)x=x\,, \quad hy=y(e-g)=y\,, \quad xy=e-g\,, \quad yx=h\,.$$

We have that $1+x$, $1+y \in C^2$ since $x^2=xh(e-g)x=0$, $y^2=y(e-g)hy=0$, and so $(1+x)(1-y)(1+x)=1-(e-g)-h+x-y \in \mathfrak{K}^\dagger$ whence

$$s \cong \begin{bmatrix} g & 0 & 0 \\ 0 & 0 & (e-g)t^*(1-e) \\ 0 & -h(e-g)t(e-g) & (1-e)t^*(1-e) \end{bmatrix}$$

for some $t^* \in \mathfrak{R}$. Since

$$R(-h(e-g)t(e-g))=R(e-g)\,,$$

then

$$(-h(e-g)t(e-g))_\iota=(e-g)_\iota\,,$$

and by a similar argument to one above we have, for some $t' \in \mathfrak{R}$,

$$s \cong \begin{bmatrix} g & 0 & 0 \\ 0 & e-g & 0 \\ 0 & 0 & (1-e)t'(1-e) \end{bmatrix}.$$

This useful lemma permits us to obtain an analogue in continuous rings for a diagonalization theorem of Dieudonné [1, p. 30].

THEOREM 1. *In a continuous ring* $\mathfrak{R}$, *let* $e^2=e$, $R(e)<1$ *and* s *be nonsingular. Then, for some* $t \in \mathfrak{R}$,

$$s \cong e+(1-e)t(1-e)\,.$$

Proof. If $R(e)<1/2$, a similar proof to that of Lemma 2 yields the result.

We may suppose then, that

$$\sum_{i=1}^{p-1} 2^{-i} \leqq R(e) < \sum_{i=1}^{p} 2^{-i} \qquad \text{for } p>1$$

Let $e_1=ee_1e$ be an idempotent of rank $1/2$. Then, by Lemma 2, $t_1s=e_1+(1-e_1)s_1(1-e_1)$ for some $t_1 \in \mathfrak{K}^\dagger$ and $s_1 \in \mathfrak{R}$. If $p>2$, we let $e_2=(e-e_1)\cdot e_2(e-e_1)$ be an idempotent of rank $1/4$; then e_2 has normalized rank $1/2$ in the continuous ring $\mathfrak{R}(1-e_1)$ and $(1-e_1)s_1(1-e_1)$ is nonsingular in this ring. Hence, there exists t_2 in the group $\mathfrak{K}^\dagger$ of $\mathfrak{R}(1-e_1)$ such that

$$t_2(1-e_1)s_1(1-e_1)=e_2+(1-e_1-e_2)s_2(1-e_1-e_2)$$

where $s_2\in\mathfrak{R}(1-e_1)\subset\mathfrak{R}$. Then

$$(e_1+t_2)(e_1+(1-e_1)s_1(1-e_1))=e_1+e_2+(1-e_1-e_2)s_2(1-e_1-e_2);$$

moreover, $e_1+t_2\in\mathfrak{K}^\dagger$ as can be verified simply.

Proceeding in a similar fashion, we have eventually, for some s_{p-1} and independent idempotents $e_i=ee_ie$ $(i=1,\cdots,p-1)$ with $R(e_i)=2^{-i}$

$$s\cong e_1+\cdots+e_{p-1}+(1-e_1-\cdots-e_{p-1})s_{p-1}(1-e_1-\cdots-e_{p-1})\,.$$

Application of the first statement of the proof to the idempotent $e-e_1-\cdots-e_{p-1}$ in the subring $\mathfrak{R}(1-e_1-\cdots-e_{p-1})$ gives

$$\begin{aligned}t_p(1-e_1-\cdots-e_{p-1})&s_{p-1}(1-e_1-\cdots-e_{p-1})\\&=e-e_1-\cdots-e_{p-1}+(1-e)s_p(1-e)\end{aligned}$$

where

$$t_p\in\mathfrak{R}(1-e_1-\cdots-e_{p-1}),\ e_1+\cdots+e_{p-1}+t_p\in\mathfrak{K}^\dagger \text{ and } s_p\in\mathfrak{R}.$$

The result follows.

THEOREM 2. *In a continuous ring* $\mathfrak{K}=\mathfrak{K}^\dagger$.

Proof. The equation $utu^{-1}=t^2$ is satisfied by any $t\in C^2$, for some $u\in\mathfrak{G}$ depending on t [2, Theorem 2.12]. Hence the arbitrary $t\in C^2$ satisfies

$$(3)\qquad t=utu^{-1}t^{-1}$$

and $\mathfrak{K}^\dagger\subseteqq\mathfrak{K}$.

By Lemma 2, if $a_1,\ a_2\in\mathfrak{G}$ and e is an idempotent such that $R(e)=1/2$, then $a_1=b_1d_1$, $a_2=b_2d_2$ where $b_1,\ b_2\in\mathfrak{K}^\dagger$ and

$$d_1=e+(1-e)d_1(1-e)\,,\quad d_2=e+(1-e)d_2(1-e)\,.$$

The commutator $a_1a_2a_1^{-1}a_2^{-1}$ has the form $bd_1d_2d_1^{-1}d_2^{-1}$ with $b\in\mathfrak{K}^\dagger$ by Corollary 2. It is sufficient to show that $d_1d_2d_1^{-1}d_2^{-1}\in\mathfrak{K}^\dagger$ and we need only show that $d_1d_2=b^{(1)}d_2d_1b^{(2)}$ where $b^{(1)},\ b^{(2)}\in\mathfrak{K}^\dagger$. Write $(1-e)d_1(1-e)=\lambda$, $(1-e)d_2(1-e)=\mu$.

Now e, $1-e$ define a matrix basis s_{ij} with $s_{11}=e$, $s_{22}=1-e$, $s_{12}=es_{12}=s_{12}(1-e)$, $s_{21}=(1-e)s_{21}=s_{21}e$ [4, Chapter 3]. Then

$$(1+s_{12})(1-s_{21})(1+s_{12})=-s_{21}+s_{12}$$

and

$$(-s_{21}+s_{12})^2=-s_{11}-s_{22}=-1$$

belong to $\mathfrak{K}^\dagger$.

Noticing that λ has an inverse in $\mathfrak{R}(1-e)$ we obtain without difficulty

$$(4)\qquad d_1d_2=\begin{bmatrix} e & 0 \\ 0 & \lambda\mu \end{bmatrix}\cong\begin{bmatrix} e & s_{12}\mu \\ 0 & \lambda\mu \end{bmatrix}\cong\begin{bmatrix} e & s_{12}\mu \\ -\lambda s_{21} & 0 \end{bmatrix}\cong\begin{bmatrix} 0 & s_{12}\mu \\ -\lambda s_{21} & 0 \end{bmatrix}$$

and on left multiplying the last member of (4) by $-(-s_{21}+s_{12})$

$$d_1d_2\cong\begin{bmatrix} s_{12}\lambda s_{21} & 0 \\ 0 & \mu \end{bmatrix}\cong\begin{bmatrix} 0 & s_{12}\lambda \\ -\mu s_{21} & 0 \end{bmatrix}.$$

Retracting the steps of (4) we obtain the result.

REMARK 1. When $\mathfrak{R}$ is a matrix ring over a field (discrete ring), $\mathfrak{K}$, $\mathfrak{K}^\dagger$ are respectively the commutator group and the group generated by the elements of class 2. Provided the order of the matrices exceeds two, as we assume, (3) holds and again $\mathfrak{K}^\dagger\subseteqq\mathfrak{K}$; also $\mathfrak{K}^\dagger$ contains the group generated by the transvections which is shown by Dieudonné [1, p. 31] to itself contain $\mathfrak{K}$. Hence Theorem 2 holds for rings of matrices of order greater than two.

3. Determinants in a complete rank ring.

DEFINITION 3. Let $\mathfrak{R}$ be a continuous or discrete ring. We define the determinant $\Delta(a)$ $(a\in\mathfrak{G})$ as the coset $\mathfrak{K}_a$.

We now proceed to obtain generalizations of some well-known results in determinants; the restrictions on characteristic and order apply and the determinants, we note, are defined only for nonsingular ring elements. Theorem 2, Remark 1 and the commutativity of the cosets are used freely without additional reference.

(i) *A theorem on minors of the inverse.*

THEOREM 3. *Let c be nonsingular and e any idempotent in $\mathfrak{R}$. Then*

$$\Delta(1-e+ec^{-1}e)\Delta(c)=\Delta(e+(1-e)c(1-e)).$$

Proof. $\Delta(1-e+ec^{-1}e)\Delta(c)=\Delta\{(1+ec^{-1}(1-e))(1-e+ec^{-1}e)\}\Delta(c)$

$$=\Delta((1-e)c+e)$$

$$=\Delta\{(1-(1-e)ce)((1-e)ce+(1-e)c(1-e)+e)\}$$
$$=\Delta(e+(1-e)c(1-e))\,.$$

(ii) *The Laplace development.* (Compare [1, p. 37].)

THEOREM 4. *Let* $e^2=e$, $x\in\mathfrak{R}$. *If* $R(exe)=R(e)$, *then*

$$\Delta(x)=\Delta(exe+(1-e))\Delta(e+(1-e)x(1-e)-(1-e)xe\cdot eye\cdot ex(1-e))$$

where eye is the inverse of exe in $\mathfrak{R}(e)$.

Proof. $\Delta(x)=\Delta\{(1-(1-e)xe\cdot eye)x\}$

$$=\Delta(exe+ex(1-e)+(1-e)x(1-e)-(1-e)xe\cdot eye\cdot ex(1-e))$$
$$=\Delta\{(exe+ex(1-e)+(1-e)x(1-e)-(1-e)xe\cdot eye\cdot ex(1-e))$$
$$\cdot(1-eye\cdot ex(1-e))\}$$
$$=\Delta(exe+(1-e)x(1-e)-(1-e)xe\cdot eye\cdot ex(1-e))$$
$$=\Delta(exe+(1-e))\cdot\Delta(e+(1-e)x(1-e)$$
$$-(1-e)xe\cdot eye\cdot ex(1-e))\,.$$

(iii) *Cramer's rule.*

THEOREM 5. *Let* $ax=b$ *be satisfied by* a, b, $x\in\mathfrak{R}$. *Then*

$$\Delta(be+a(1-e))=\Delta(a)\Delta(exe+(1-e))$$

for any idempotent e.

Proof. $ax=b$ implies $axe=be$ and so

$$\Delta(be+a(1-e))=\Delta(axe+a(1-e))$$
$$=\Delta(a)\Delta(xe+(1-e))$$
$$=\Delta(a)\Delta\{(exe+(1-e)xe+(1-e))(1-(1-e)xe)\}$$
$$=\Delta(a)\Delta(exe+(1-e)).$$

REMARK 2. The fact that Theorem 5 includes Cramer's rule can be seen as follows.

The matrix equation $Ax=b$ with $A=(a_{ij})$ an $n\times n$ matrix and $x=\{x_1,\cdots,x_n\}$, $b=\{b_1,\cdots,b_n\}$, the components being in a field K, can be expressed

$$(a_{ij})\begin{pmatrix} x_1 & & x_1 \\ \vdots & \cdots & \vdots \\ x_n & & x_n \end{pmatrix} = \begin{pmatrix} b_1 & & b_1 \\ \vdots & \cdots & \vdots \\ b_n & & b_n \end{pmatrix}$$

where each vector is replaced by a ring element with identical columns.

Taking $e=e_i=\operatorname{diag}(0, 0, \cdots, 1, \cdots)$ with 1 in the ith place, Theorem 5 gives

$$\Delta\begin{pmatrix} a_{11} & & b_1 & a_{i+1,1} & \\ \vdots & \cdots & \vdots & \vdots & \cdots \\ a_{1n} & & b_n & a_{i+1,n} & \end{pmatrix} = \Delta(A)\Delta\{\operatorname{diag}(1, \cdots, x_i, 1, \cdots)\} .$$

If C is the commutator subgroup of $K^\times$, the isomorphism of $M_n^\times(K)/C_n$ and $M^\times/C$ implies the preceding equation holds when we interpret Δ as the Dieudonné determinant (K noncommutative) or as the ordinary determinant (K commutative).

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