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**UNIFORM CONTINUITY OF CONTINUOUS FUNCTIONS OF  
METRIC SPACES**

MASAHIKO ATSUJI

# UNIFORM CONTINUITY OF CONTINUOUS FUNCTIONS OF METRIC SPACES

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In this paper we intend to find equivalent conditions under which continuous functions of a metric space are always uniformly continuous. Isiwata has attempted to prove a theorem in a recently published paper [3] by a method that has a close relation with ours. Unfortunately he does not accomplish his purpose, so we shall give a correct theorem (Theorem 3) in the last part of this paper and, for this purpose, give a condition for the existence of a uniformly continuous unbounded function in a metric space (Theorem 2).

In this paper the space  $S$ , unless otherwise specified, is the metric space with a distance function  $d(x, y)$ , and, for a positive number  $\alpha$ , the  $\alpha$ -sphere about a subset  $A$   $\{x; d(A, x) < \alpha\}$  is denoted by  $S(A, \alpha)$ ; the function is the real valued continuous mapping.

DEFINITION 1. Let us consider a family of neighborhoods  $U_n$  of  $x_n$  such that  $\{x_n\}$  is a sequence of distinct points and  $U_m \cap U_n = \phi$  ( $=$ empty) for  $m \neq n$ . Let  $f_n(x)$  be a function such that  $f_n(x_n) = n$  and  $f_n(x) = 0$  for  $x \notin U_n$ . Then a mapping constructed from the family is a mapping  $f(x)$  defined by  $f(x) = f_n(x)$  for  $x$  belonging to some  $U_n$  and  $f(x) = 0$  for the other  $x$ .

LEMMA. Consider a family of neighborhoods  $U_n$  of  $x_n$  satisfying the following conditions:

- (1)  $\{x_n\}$ , which consists of distinct points, has no accumulation point,
- (2)  $\bar{U}_m \cap \bar{U}_n = \phi$ ,  $m \neq n$  ( $\bar{U}$  a closure of  $U$ ), and  $U_n \subset S(x_n, 1/n)$ ,
- (3) there is a sequence of points  $y_n$  such that distances of  $x_n$  and  $y_n$  converge to 0 and  $y_n$  does not belong to any  $U_m$ ; then the mapping constructed from the family is continuous and not uniformly continuous. When  $\{x_n\}$  is a sequence containing infinitely many distinct points and has no accumulation point, there is a family of neighborhoods of  $x_n$  satisfying (2); if  $\{x_n\}$  further contains infinitely many distinct accumulation points, then the family besides satisfies (3).

*Proof.* The continuity of the mapping constructed from the family follows from  $\overline{\cup U_{n_i}} = \cup \bar{U}_{n_i}$  for any subsequence  $\{n_i\}$  of indices; the mapping is not uniformly continuous by (3). Suppose  $\{x_n\}$  consists of distinct accumulation points and has no accumulation point, then, by an inductive process, we have neighborhood  $V_n$  of  $x_n$  such that  $V_n \subset S(x_n, 1/n)$  and  $\bar{V}_m \cap \bar{V}_n = \phi$ , and have  $y_n$  and a neighborhood  $U_n$  of  $x_n$

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such that  $U_n \not\supset y_n \in V_n$ ,  $U_n \subset V_n$ .

**DEFINITION 2.** Let  $x$  be isolated in a metric space, then we write  $I(x)$  for a supremum of positive numbers  $\alpha$  such that  $S(x, \alpha)$  consists of  $x$  alone.

**THEOREM 1.** *The following conditions on a metric space  $S$  are equivalent*

(1) *If  $\{x_n\}$  is a sequence of points without accumulation point, then all but finitely many members of  $x_n$  are isolated and  $\inf I(x_n)$  for the isolated points is positive.*

(2) *If a subset  $A$  of  $S$  has no accumulation point then all but finitely many points of  $A$  are isolated and  $\inf I(x)$  for all the isolated points of  $A$  is positive.*

(3) *The set  $A$  of all accumulation points in  $S$  is compact and  $\inf I(x_n)$  is positive for any sequence  $\{x_n\}$  in  $S-A$  which has no accumulation point (Iiwata [2], Theorem 2).*

(4)  *$\bar{A} \cap \bar{B} = \phi$  implies  $S(A, \alpha) \cap S(B, \alpha) = \phi$  for some  $\alpha$  (Nagata [4], Lemma 1).*

(5)  *$\bigcap_{n=1}^{\infty} \bar{A}_n = \phi$  implies  $\bigcap_{n=1}^{\infty} S(A_n, \alpha) = \phi$  for some  $\alpha$ .*

(6) *For any function  $f(x)$ , there is a positive integer  $n$  such that every point of  $A = \{x; |f(x)| \geq n\}$  is isolated and  $\inf_{x \in A} I(x)$  is positive.*

(7) *All functions of  $S$  are uniformly continuous.*

(8) *All continuous mappings of  $S$  into an arbitrary uniform space  $S'$  are uniformly continuous.*

*Proof.* Since the equivalence of (1) and (3) is simple, we shall show (1)  $\rightarrow$  (8)  $\rightarrow$  (7)  $\rightarrow$  (6)  $\rightarrow$  (5)  $\rightarrow$  (4)  $\rightarrow$  (2)  $\rightarrow$  (1).

(1)  $\rightarrow$  (8): If a continuous mapping  $f(x)$  of  $S$  is not uniformly continuous, there is an "entourage"  $V$  (in the sense of Bourbaki) of  $S'$  such that  $d(x_n, y_n) < 1/n$  and  $(f(x_n), f(y_n)) \notin V$  for any positive integer  $n$  and for some  $x_n$  and  $y_n$ .  $\{x_n\}$  contains infinitely many distinct points. If  $\{x_n\}$  has an accumulation point  $x$ , there are subsequences  $\{x_{n_i}\}$  and  $\{y_{n_i}\}$  of  $\{x_n\}$  and  $\{y_n\}$  converging to  $x$ , and, since  $f(x)$  is continuous,  $(f(x), f(x_{n_i})) \in W$  and  $(f(x), f(y_{n_i})) \in W$  for  $W$  satisfying  $W \cdot W \subset V$  (we may assume  $W^{-1} = W$ ) and for all sufficiently large  $i$ . Hence we have  $(f(x_{n_i}), f(y_{n_i})) \in V$ , which is excluded. Consequently  $\{x_n\}$  has no accumulation point and  $\inf I(x_n) = r > 0$  for all sufficiently large  $n$ , which contradicts the first inequality of  $f$  for  $n$  satisfying  $r > 1/n$ .

(8)  $\rightarrow$  (7) is obvious.

(7)  $\rightarrow$  (6): If, for some function  $f(x)$  and every  $n$ , there is an accumulation point  $x_n$  such that  $|f(x_n)| \geq n$ ,  $\{x_n\}$  contains infinitely many distinct elements and has no accumulation point, then, by the Lemma, we have

a function which is not uniformly continuous. Suppose that every point of  $A = \{x; |f(x)| \geq n\}$  is isolated and  $\inf I(x) = 0$ . Then there is a sequence  $\{x_n\}$  in  $A$  such that  $\inf I_n = 0$ ,  $I_n = I(x_n)$ .  $\{x_n\}$  has no accumulation point, and we may assume  $I_n < 1/n$ . If distances of distinct points of  $\{x_n\}$  are greater than a positive number  $e$ , then, for all  $n$  satisfying  $e > 4I_n$ ,  $x_n$  and  $y_n$  ( $\neq x_n, \in S(x_n, 2I_n)$ ) satisfy the conditions of the Lemma. In the other case, there are arbitrarily large  $m$  and  $n$  satisfying  $d(x_m, x_n) < e$  for any positive number  $e$ , and we have, by an inductive process, a subsequence  $\{y_i\}$  of  $\{x_n\}$  satisfying  $d(y_{2i-1}, y_{2i}) < 1/i$ . Then  $y_{2i-1}$  and  $y_{2i}$  satisfy the conditions of the Lemma.

(6)→(5): Let  $\bigcap_n S(A_n, 1/m) \neq \phi$  for every  $m$  in spite of  $\bigcap_n \bar{A}_n = \phi$ . We have a point  $x_1$  contained in  $\bigcap_n S(A_n, 1)$  and a point  $y_1$  distinct from  $x_1$  satisfying  $d(x_1, y_1) < 1$ . Suppose  $B_i = \{x_1, \dots, x_i\}$  consists of distinct points such that  $x_j \in \bigcap_n S(A_n, 1/j)$ ,  $x_j$  and  $y_j$  are distinct and  $d(x_j, y_j) < 1/j$ ,  $j = 1, \dots, i$ . Since, for any point  $x$ ,  $\bigcap_n S(A_n, 1/m)$  does not contain  $x$  for a sufficiently large  $m$ ,  $\bigcap_n S(A_n, 1/(i+1))$  contains a point  $x_{i+1}$  being not contained in  $B_i$ , and some  $A_n$  contains  $y_{i+1}$  distinct from  $x_{i+1}$  satisfying  $d(x_{i+1}, y_{i+1}) < 1/(i+1)$ . Thus we have a sequence  $\{x_n\}$  of distinct points and  $\{y_n\}$  such that  $x_m \in \bigcap_n S(A_n, 1/m)$ ,  $x_n$  and  $y_n$  are distinct, and  $d(x_n, y_n) < 1/n$ .  $\{x_n\}$  has no accumulation point because of  $\bigcap_n \bar{A}_n = \phi$ . The function obtained from the Lemma does not satisfy the condition (6) whether all but finitely many members of  $x_n$  are isolated or not.

(5)→(4) is obvious.

(4)→(2): Suppose  $A$  has infinitely many accumulation points  $x_n$ ,  $n = 1, 2, \dots$ . Since  $B = \{x_n\}$  has no accumulation point, there is a sequence  $C = \{y_n\}$  having no accumulation point such that  $d(x_n, y_n) < 1/n$ ,  $B \cap C = \phi$ .  $\bar{B} \cap \bar{C} = B \cap C = \phi$ , and  $S(B, \alpha) \cap S(C, \alpha) = \phi$  for no  $\alpha$ . If every point of  $A$  is isolated and  $\inf I(x) = 0$ , we have a sequence  $\{x_n\}$  such that  $\lim I(x_n) = 0$ , and have a sequence  $\{y_n\}$  with the same properties as the above.

(2)→(1) is obvious.

Recently Isiwata has stated a theorem ([3], Theorem 4) which is related to our Theorem 1. However the first step in his proof is wrong. We shall give a correct form of the theorem in Theorem 3. Let us first give a counterexample for the statement "In a connected metric space which is not totally bounded, there exists a sequence  $\{x_n\}$  and a uniformly continuous function  $f$  such that  $f(x_n) = n$ ".

EXAMPLE. Denoting the points of the plane by polar-coordinate,

we consider the following subsets of the plane :

$$A_m = \{(r, \theta) ; 0 \leq r \leq 1, \theta = \pi/m\},$$

$$S = \bigcup_{m=1}^{\infty} A_m .$$

We define the distance of the points of  $S$  by

$$\begin{aligned} d((r, \theta), (r', \theta')) &= |r - r'| && \text{as } \theta = \theta' \text{ or } rr' = 0, \\ &= r + r' && \text{as } \theta \neq \theta', \end{aligned}$$

then  $S$  is obviously a connected metric space which is not totally bounded. When  $f(x)$ ,  $x \in S$ , is a uniformly continuous function of  $S$ , there is a positive integer  $n$  such that  $d(x, y) < 1/n$  implies  $|f(x) - f(y)| < 1$ . If  $x$  is contained in  $A_m$ , there are points  $y_0 = 0 = \text{pole}$ ,  $y_1, \dots, y_r = x$ ,  $r \leq n+1$ , of  $A_m$  such that  $d(y_{i-1}, y_i) < 1/n$ ,  $i = 1, \dots, r$ .

$$|f(0) - f(x)| \leq |f(0) - f(y_1)| + \dots + |f(y_{r-1}) - f(x)| \leq n+1 ;$$

namely  $f(x)$  is bounded.

**DEFINITION 3.** Let  $e$  be a positive number, then the finite sequence of points  $x_0, x_1, \dots, x_m$  satisfying  $d(x_{i-1}, x_i) < e$ ,  $i = 1, \dots, m$ , is said to be an  $e$ -chain with length  $m$ . If, for any positive number  $e$ , there are finitely many points  $p_1, \dots, p_i$  and a positive integer  $m$  such that any point of the space can be bound with some  $p_j$ ,  $1 \leq j \leq i$ , by an  $e$ -chain with length  $m$ , then the space is said to be *finitely chainable*.

**THEOREM 2.** A metric space  $S$  admits a uniformly continuous unbounded function if and only if  $S$  is not finitely chainable.

*Proof.* Verification of "only if" part is analogous to that stated in the above example, hence is passed over. Let  $S$  be not finitely chainable, then there is a positive number  $e$  such that, for any finitely many points and a positive integer  $n$ , there is a point which cannot be bound with any one of points selected above by an  $e$ -chain with length  $n$ . We denote by  $A_0^n$  the set of all points which can be bound with a fixed  $x_0$  by an  $e$ -chain with length  $n$ .

(1) When  $A_0^n \neq A_0^{n+1}$  for every  $n$ , we put

$$f(x) = (n-1)e + d(x, A_0^{n-1})$$

for  $x$  belonging to  $A_0^n$  and not to  $A_0^{n-1}$ , and  $f(x) = 0$  for  $x \in A_0 = \bigcup_n A_0^n$  ( $f(x) = d(x_0, x)$  for  $x \in A_0$ ). Since  $S(A_0, e) = A_0$ ,  $f(x)$  is uniformly continuous on  $S$  if it is so on  $A_0$ . Let  $A_0^n \ni x \notin A_0^{n-1}$  and  $d(x, y) < e' < e$ , then  $A_0^{n+1} \ni y \notin A_0^{n-2}$ . (i) When  $y$  is in  $A_0^{n-1}$ , then

$$f(y) = (n-2)e + d(y, A_0^{n-2})$$

and  $d(x, A_0^{n-1}) < e'$ ,  $d(y, A_0^{n-2}) < e$ , hence  $f(y) \leq f(x)$ . If  $d(y, A_0^{n-2}) < e - e'$ , then  $d(y, y') < e - e'$  for some  $y'$  of  $A_0^{n-2}$  and  $d(x, y') \leq d(x, y) + d(y, y') < e$ , so that  $x$  is in  $A_0^{n-1}$ , which is excluded. Therefore  $d(y, A_0^{n-2}) \geq e - e'$  and

$$\begin{aligned} |f(x) - f(y)| &= f(x) - f(y) = e + d(x, A_0^{n-1}) - d(y, A_0^{n-2}) \\ &< e + e' - (e - e') = 2e'. \end{aligned}$$

(ii) When  $y$  is in  $A_0^n$  and not in  $A_0^{n-1}$ , then

$$f(y) = (n-1)e + d(y, A_0^{n-1}),$$

and we have

$$|f(x) - f(y)| = |d(x, A_0^{n-1}) - d(y, A_0^{n-1})| \leq d(x, y) < e'$$

(cf. the proof of Prop. 3 of §2, [1]). (iii) The remaining case for  $y$  is similar to (i). Consequently  $f(x)$  is uniformly continuous on  $A_0$ .

(2) When  $A_0^n = A_0^{n+1}$  for some  $n$ , then  $A_0^m = A_0^n$  for every  $m \geq n$ , and, in the similar way to (1),  $A_1 = \cup A_1^n$  is obtained from a point of  $S - A_0$ . If we can make an unbounded function which is uniformly continuous on  $A_1$ , our proof will be complete.

(3) When we cannot, for every  $m (0 \leq m \leq n)$ , construct a desired function on  $A_m$  obtained in the same way as (2),  $A_0, \dots, A_n$  cannot cover the space, because the space is not finitely chainable; namely we have a sequence of infinitely many subsets  $A_0, A_1, \dots$  when our proof is not complete in the similar way to (2). Then we put  $f(x) = n$  for  $x$  of  $A_n$  and  $f(x) = 0$  for  $x$  which is not in any  $A_n$ . Then, since  $S(A_m, e) \cap A_n = \phi$  for any  $m \neq n$  and  $S(\cup A_n, e) = \cup A_n$ ,  $f(x)$  is uniformly continuous.

**THEOREM 3.** *If  $S$  is a connected metric space which is not finitely chainable, then the set of all uniformly continuous functions of  $S$  does not form a ring.*

*Proof.* The following verification is essentially due to Isiwata [3]. There is, by Theorem 2, a uniformly continuous unbounded function  $f(x)$  of the space, and we have a sequence  $A = \{x_n; n = 1, 2, \dots\}$  such that  $f(x_n) = a_n$ ,  $a_{n+1} - a_n \geq 1$ ,  $a_1 \geq 1$ ;  $A$  has no accumulation point. For some positive number  $\alpha$ ,  $d(x, y) < \alpha$  implies  $|f(x) - f(y)| < 1/3$ , and so  $S(x_m, \alpha) \cap S(x_n, \alpha) = \phi$  for  $m \neq n$ . We put

$$h(x) = 1 - d(A, x)/\alpha \quad \text{and} \quad G = \bigcup_n S(x_n, \alpha)$$

and

$$f'(x) = \begin{cases} h(x) & \text{for } x \in G, \\ 0 & \text{for } x \notin G. \end{cases}$$

$h(x)$  is uniformly continuous on the space, because  $d(A, x)$  is so (cf. Prop. 3 of §2, [1]).  $h(x) > 0$  and  $h(y) \leq 0$  for  $x$  of  $G$  and  $y$  of  $S-G$  respectively, so we have

$$|h(x) - h(y)| = h(x) - h(y) \geq h(x) = |f'(x) - f'(y)|.$$

Hence  $f'(x)$  is uniformly continuous on the space.  $g(x) = f(x)f'(x)$  is not uniformly continuous. In fact, if it is uniformly continuous,  $d(x, y) < \beta$  implies

$$(*) \quad |g(x) - g(y)| < 1 \quad \text{and} \quad |f(x) - f(y)| < 1$$

for some  $\beta (\leq \alpha)$ . We select a positive integer  $n$  such that  $a_n$  is greater than  $1 + 4\alpha/\beta$ , and take a point  $y$  such that  $\beta/2 \leq d(x_n, y) < \beta$  (it is possible to take such a point because of the connectedness of the space). Then, by (\*), we have  $|a_n - f(y)| < 1$ ,  $f(y) > a_n - 1 \geq 0$ , and

$$\begin{aligned} |g(x_n) - g(y)| &= |a_n - (1 - d(A, y)/\alpha)f(y)| = |a_n - f(y) + d(x_n, y)f(y)/\alpha| \\ &\geq |d(x_n, y)f(y)/\alpha| - |a_n - f(y)| > d(x_n, y)f(y)/\alpha - 1 \\ &> \beta(a_n - 1)/2\alpha - 1 > \beta(1 + 4\alpha/\beta - 1)/2\alpha - 1 = 1, \end{aligned}$$

which contradicts (\*).

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Shimshon A. Amitsur, <i>Commutative linear differential operators</i> .....	1
Masahiko Atsuji, <i>Uniform continuity of continuous functions of metric spaces</i> .....	11
S. P. Avann, <i>A numerical condition for modularity of a lattice</i> .....	17
Raymond G. D. Ayoub, <i>A mean value theorem for quadratic fields</i> .....	23
Errett Albert Bishop, <i>Subalgebras of functions on a Riemann surface</i> .....	29
Shaul Foguel, <i>The relations between a spectral operator and its scalar part</i> .....	51
John Rolfe Isbell, <i>Euclidean and weak uniformities</i> .....	67
Samuel Karlin and James L. McGregor, <i>Many server queueing processes with Poisson input and exponential service times</i> .....	87
Paul Joseph Kelly and Ernst Gabor Straus, <i>Curvature in Hilbert geometries</i> .....	119
John W. Lamperti, <i>Stationary measures for certain stochastic processes</i> ....	127
Richard Scott Pierce, <i>Distributivity and the normal completion of Boolean algebras</i> .....	133
F. M. Ragab, <i>Transcendental addition theorems for the hypergeometric function of Gauss</i> .....	141
William T. Reid, <i>Principal solutions of non-oscillatory self-adjoint linear differential systems</i> .....	147
Maurice Sion, <i>On general minimax theorems</i> .....	171
Chien Wenjen, <i>On semi-normed *-algebras</i> .....	177