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ON SEMI-NORMED *-ALGEBRAS

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1. Introduction. The notion of semi-normed algebras was introduced by Arens as a generalization of Banach algebras [**2, 5**]. They are called locally multiplically-convex algebras by Michael [**16**]. Various properties of Banach algebras have been generalized to semi-normed algebras [**5, 16, 21, 22, 23**].

We repeat here a few definitions. Let A be a linear algebra over the field K of complex or real numbers. A nonnegative real-valued function V defined on A is called a semi-norm if it satisfies the following conditions :

$V(x+y) \leqq V(x)+V(y)$, $V(xy) \leqq V(x)V(y)$, $V(\lambda x)=|\lambda| V(x)$. Suppose there is a family $\mathscr{V}$ of semi-norms such that $V(x)=0$ for all $V \in \mathscr{V}$ only if $x=0$. A is a semi-normed algebra if all the translations of the sets on which $V(x)<e$, where e is real and $V \in \mathscr{V}$, are taken as a subbase of topology, and is complete if it is complete with respect to the uniform structure defined by the various relations $V(x-y)<e$. A is called an *-algebra if there is a semi-linear operation * such that $(\lambda x-yz)^* = \bar{\lambda} x^* - z^* y^*$, $x^{**}=x$. A subset U of A is called idempotent if $UU \subset U$; it is called multiplicatively convex (m-convex) if it is convex and idempotent. A is locally m-convex if there exists a basis for the neighbourhoods of the origin consisting of sets which are m-convex and symmetric.

The present paper is devoted to generalizing the representation theorems for commutative and noncommutative Banach algebras to semi-normed algebras. An application of the Gelfand-Neumark-Arens representation theorem for commutative Banach algebras yields a simple proof of the spectral theorem for bounded self-adjoint operators in Hilbert space [**14**, p. 95]. Our generalized representation theorem for commutative semi-normed algebras gives rise to a similar proof of the spectral theorem for unbounded self-adjoint operators.

The characterization of the algebra $C(T, K)$ of all complex-valued continuous functions on a locally compact, paracompact Hausdorff space T has been treated by Arens [**5**, p. 469]. We have a characterization theorem for $C(T, K)$ where T is a locally compact completely regular space and also a uniqueness theorem for the space T [cf. the Banach-Stone theorem, **6**, p. 170, 20, p. 469]: If $C(T_1, K)$, $C(T_2, K)$ are topo-

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logically isomorphic, then T_1 and T_2 are homeomorphic. If T_1, T_2 are Hewitt's Q-spaces [**11**, p. 85], the topological equivalence between the spaces follows from the algebraic isomorphism between $C(T_1, K)$ and $C(T_2, K)$, but not in general.

2. Functional representation.

2.1. THEOREM. *Let A be a complete commutative semi-normed *-algebra* (with or without a unit) *over the complex numbers K such that*

2.2. $V(xx^*) \geqq k_V V(x^*)$, *for all* $V \in \mathscr{V} (k_V > 0)$. *Then A is topologically isomorphic to a complete self-adjoint subalgebra S of the algebra $C(T, K)$ of all continuous complex-valued functions (vanishing at infinity if A has no unit) on T with k-topology, where T is the union of the members of a family of pairwise disconnected and closed-open sets. (compact if A has a unit, otherwise locally compact).*

Proof. The elements x in A satisfying $V(x)=0$ form an ideal Z_V, a kernel ideal of A. The quotient algebra A/Z_V is a normed algebra when V is used to define a norm, and the completion B_V of A/Z_V is a commutative Banach *-algebra. By Gelfand-Neumark-Arens representation theorem [**3**, Theorem 1, p. 278], there exists a Hausdorff space (compact if A has a unit, otherwise locally compact) $Q_V =$ V-neighbourhood homomorphism, for which B_V is the class of all complex-valued continuous functions (vanishing at infinity if A has no unit) on Q_V such that

$$x_V^*(q) = \overline{x_V(q)} \qquad (q \in Q_V,\ x \in B_V).$$

and

$$k_V V(x_V) \leqq \sup_{q \in Q_V} |x_V(q)| \leqq V(x_V) \,. \tag{2.3}$$

Let

$$T = \bigcup_{V \in \mathscr{V}} Q_V \,.$$

Retaining the original weak* topology for Q_V and regarding all Q_V as pairwise disconnected and closed-open subsets, we have a locally compact completely regular space T. The complex-valued continuous functions on T are of the form $f(t) = \{f_V\}$, where $f_V(t) \in C(Q_V, K)$ and $f(t) = f_V(t)$ if $t \in Q_V$.

The mapping

$$P: \qquad x \in A \rightarrow x(t) = \{x_V(t)\} \in C(T, K)$$

maps A onto a subalgebra S of $C(T, K)$. P is isomorphic; for, if x maps to zero functional, then $V(x)=0$ for all $V \in \mathscr{V}$ and x is the zero element of A.

In fact, P is a homeomorphism. Denote the open set in A consisting of all x such that $V(x) < e$ by $0(V, e)$ and the open set in $C(T, K)$ defined by $\sup_{q \in Q_V} |f(q)| < k_V e$ by $0'(Q_V, e)$. It follows from the inequalities 2.3 that P maps $0(V, e)$ onto a subset of $C(T, K)$ containing $0'(Q_V, e)$. This proves the continuity of the inverse mapping of P from S onto A.

Let W be a compact subset in T contained in the union of $Q_{V_1}, \cdots, Q_{V_n}$. It is clear that P maps the intersection of $0(V_1, e), \cdots, 0(V_n, e)$ onto a subset in $C(T, K)$ contained in the intersection of $0'(V_1, e/k_V), \cdots, 0(V_n, e/k_V)$, and S, that is, in the intersection of $0'(W, e/k_V)$ and S. P is therefore continuous.

The completeness of S is an immediate consequence of the completeness of A and inequalities 2.3.

2.4. COROLLARY. *Let M_V be a maximal ideal in B_V (the completion of the quotient ring $A_V = A/Z_V$) and let $f(t)$ be a complex-valued continuous function on the space T. Then $f(t)$ belongs to S if $f_V(M_U) = f_U(M_U)$ whenever $U \leqq V$.*

Proof. M_V is actually a point in Q_V and $f_V(M_U)$ belongs to $C(Q_V, K)$. Let $\overline{\Pi}_{UV}$ be the natural mapping of B_V into B_U when $U \leqq V$. Then $\overline{\Pi}_{UV}(f_V) = f_U$ whenever $U \leqq V$ if $f_V(M_U) = f_U(M_U)$. Hence the corollary [**16**, Theorem 5.1].

This immediately yields the following result [cf. 5, p. 471].

2.5. THEOREM. *Let A be a commutative complete semi-normed $*-$algebra with a unit* (without unit) *satisfying* 2.2. *Then an element x in A has an inverse* (reverse) *if $x(M) \neq 0$ ($x(M) \neq -1$) for each closed maximal ideal M in A.*

3. **Spectrum.** An element h in a complete semi-normed *-algebra A satisfying 2.2 is called Hermitian, if $h^* = h$; and an Hermitian element h is called positive, if its spectrum consists of nonnegative numbers.

3.1. THEOREM. *The spectrum of every Hermitian element h is real.*

Proof. Suppose A has a unit. Let A_1 be the minimal complete *-subalgebra of A containing h. Then A_1 is commutative. By Theorem 2.1 A_1 is equivalent to a closed subalgebra S of $C(T, K)$. The corresponding function $h(M)$ of the element h in A is real-valued. For any nonreal number λ, the function $h(M) - \lambda$ is not equal to zero anywhere. The theorem follows from Theorem 2.5.

3.2. THEOREM. *Every closed self-adjoint subalgebra* A_0 *of a complete semi-normed *-algebra* A *with a unit* (without unit) *satisfying* 2.2 *contains inverses* (reverses).

Proof. Rickart has proved that $x_V \in A_{0V}$ (the completion of $A_{0v}=A_0/Z_V$) has an inverse (reverse) iff both $x_V{}^*x_V$ and $x_Vx_V{}^*$ have inverses (reverses) and that the inverse (reverse) of x_V is contained in A_{0V} iff the inverses (reverses) of $x_V{}^*x_V$ and $x_Vx_V{}^*$ are contained in A_{0V} [**18**, pp. 531–532]. Since every closed maximal ideal in A contains a kernel ideal [**5**, p. 466], it follows from Theorem 2.5 that A_0 contains inverses (reverses) of its Hermitian elements, and hence of all its elements which have inverses (reverses) in A.

3.3. COROLLARY. *Let* A_0 *be any closed self-adjoint subalgebra of* A*. Then the spectrum of* $x \in A_0$ *relative to* A_0 *is identical with the spectrum relative to* A.

3.4. THEOREM. *Let* x *be a normal element, that is,* $xx^*=x^*x$*, of* A (with or without a unit) *and let* $f(\lambda)$ *be a complex-valued continuous function* (vanishing at infinity, if A has no unit) *defined on the spectrum* σ *of* x*. Then* $f(x)$ *defines an element contained in every commutative closed self-adjoint subalgebra of* A *which contains* x.

Moreover if $s(\lambda)=f(\lambda)+g(\lambda)$, $p(\lambda)=f(\lambda)g(\lambda)$, $q(\lambda)=\bar{f}(\lambda)$, $r(\lambda)=\lambda$, *then* $s(x)=f(x)+g(x)$, $p(x)=f(x)g(x)$, $q(x)=f(x)^*$, $r(x)=x$.

Proof. Let A_0 be a commutative closed self-adjoint subalgebra of A containing x and let M_V be a maximal ideal in A_{0V}. Then A_0 is equivalent to a closed self-adjoint subalgebra S of the algebra $C(T, K)$ of all complex-valued continuous functions on a locally compact completely regular space T and $f(x_V(M_U))=f(x_U(M_U))$ whenever $U\leqq V$. By Corollary 2.4, $f(x(M))$ determines a unique element, denoted by $f(x)$, contained in A_0. The first part of the theorem is proved.

The second part of the theorem is obvious.

3.5. THEOREM. *The sum of two positive elements is positive.*

Proof. Suppose A has a unit. Let h and k be two positive elements in A and let A_0 be the minimal closed self-adjoint subalgebra of A containing $h+k$. Since the inverse of $h_V+k_V+\lambda e$ for any nonnegative number λ and each $V\in \mathscr{V}$. [**13**, p. 52] the function $h(M)+k(M)+\lambda$ does not vanish at any M. The theorem follows from Theorem 2.5.

3.6. THEOREM. *The Hermitian elements of a complete seminormed *-algebra satisfying the condition* 2.2 *constitute a lattice.*

Proof. To any Hermitian h, there is a positive element $|h|$ corresponding to the function $|\lambda|$ by Theorem 3.4. Let h and k be arbitrary Hermitian elements and define.

$h \vee k = \frac{1}{2}(h+k+|h-k|)$, $h \wedge k = \frac{1}{2}(h+k-|h-k|)$. Then the Hermitian elements constitutes a lattice.

4. Closed self-adjoint subalgebras.

4.1. THEOREM. *A commutative complete semi-normed *-algebra A satisfying the condition* 2.2 *is equivalent to a closed, separating self-adjoint subalgebra S of the algebra $C(T_0, K)$ of all complex-valued continuous functions* (vanishing at infinity, if A has no unit) *on a completely regular space T_0 with a topology which has at most the open sets of the k-topology, that is, with a topology $\rho \leqq k$.*

Proof. By Theorem 2.1, A is equivalent to a closed self-adjoint subalgebra S of $C(T, K)$, where T is a union of pairwise disconnected and closed-open sets (compact if A has a unit, otherwise locally compact). Let $x(t)$ be the corresponding function in S of the element x in A. Denote by T_0 the class of all subsets of T:

$$L_a = \{t\ ;\ x(t) = x(a) \text{ for each } x \in A\}\ .$$

Following Cěch's notation, Let ρ denote the mapping:

$$a \in T \rightarrow L_a$$

and let $[f, I]$ denote those elements $\rho(t)$ of T_0 such that $f(t) \in I$, where $f(t)$ is a continuous real function belonging to S end I is an open interval. The topology generated by considering all these $[f, I]$ as a subbase is called ρ-topology.

It is easy to see that ρ is a continuous mapping and that for any $a \in T$, there is an $[f, I]$ containing $\rho(a)$. Let $[f_1, I_1]$ and $[f_2, I_2]$ be any two open sets in T_0 containing $\rho(a)$. If both $f_1(a)$ and $f_2(a)$ are different from zero, we can assume without loss of generality that $f_1(a) = f_2(a)$ and that I_1 and I_2 are identical. We define $g_i(t) = f_i(t)$ if $f_i(t) \leqq f_i(a)$, and $g_i(t) = 2f_i(a) - f_i(t)$ if $f_i(t) > f_i(a)$, $i = 1, 2$. Then $g_1(t)$ and $g_2(t)$ are continuous functions. Let $g(t) = g_1(t) \wedge g_2(t)$. It is clear that $[g, I] \subset [f_1, I] \cap [f_2, I]$. In case $f_1(a) = 0$ and $f_2(a) \neq 0$, we can assume that $f_1(t)$ and $f_2(t)$ are nonnegative. Let $g(t) = f_2(t) - f_1(t)$. An interval I can be so chosen that $[g, I] \subset [f_1, I] \cap [f_2, I]$. Hence T_0 is a topological space. Čech has proved that T_0 is Hausdorff and completely regular [8, p. 827].

Now the closed subalgebra S of $C(T, K)$ is a closed, separating subalgebra of $C(T_0, K)$.

4.2. REMARK. It is clear that the elements in the space T_0 are the closed maximal ideals in the algebra A and the ρ-topology is the weak* topology. Professor Arens has constructed examples to show that T_0 is not necessarily locally compact. He has also constructed a completely regular space T such that $C(T, K)$ with k-topology is not complete. [4, p. 234]. We have, however, the following.

4.3. THEOREM. *The necessary and sufficient condition that a commutative complete semi-normed *-algebra A satisfying the condition* 2.2 *be equivalent to $C(T, K)$, with k-topology, of all complex-valued continuous functions on a locally compact completely regular space T is:*

To any closed maximal ideal M_0 in A, there are an $x \in A$ and an $\varepsilon > 0$ such that the intersection of the maximal ideals M satisfying the relation $|x(M_0) - x(M)| \leqq \varepsilon$ contains a kernel ideal.

Proof. The necessity is obvious. The sufficiency follows from Theorem 4.1 and Corollary 2.4.

4.4. REMARK. Theorem 4.3 generalizes the theorem of Arens characterizing the algebra $C(T, K)$, where T is a locally compact, paracompact Hausdorff space. [5, p. 469]. Let A be an algebra with a locally finite partition of unity. (For definition and notation, see 5, p. 463) To any maximal closed ideal M_0, there exists an u_V such that $u_V(M_0) = \delta \neq 0$, since M_0 contains a kernel ideal. There are only a finite number of W such that $W(u_V) \neq 0$, say, $W_1, \cdots, W_n$. Let $W_0 = \max.(W_1, \cdots, W_n)$. The intersection of the closed maximal ideals M satisfying $|u_V(M_0) - u_V(M)| \leqq \delta/2$ evidently contains Z_{W_0}.

4.5. THEOREM. *For the algebra $C(T, K)$ of all complex-valued continuous functions* (vanishing at infinity) *on a locally compact completely regular space T with k-topology, there is one-to-one correspondence between closed ideals in $C(T, K)$ and the closed subsets of T.*

This is a generalization of a theorem due to Stone [20, Theorem 85] and the proof is straightforward.

4.6. COROLLARY. *For the glgebra of all complex-valued continuous functions* (vanishing at infinity) *on a locally compact completely regular space with k-topology, there is one-to-one correspondence between the closed maximal* (regular) *ideals of the algebra and the points of the space* (the point at infinity is not included).

4.7. THEOREM. *The necessary and sufficient condition two locally compact completely regular spaces T and T' be homeomorphic is that the* *algebras* $C(T, K)$ *and* $C(T', K)$ *of all complex-valued continuous functions* (vanishing at infinity) *on the spaces with k-topology be topologically isomorphic.*

Proof. Following Stone's idea, we define the closure of a family of closed maximal (regular) ideals in $C(T, K)$ as the hull of the kernel of the family [**14**, p. 56]. It is clear that a subset of the space T is closed iff it is equal to the hull of its kernel when it is considered as a set of the maximal (regular) ideals in $C(T, K)$.

4.8. REMARK. The homeomorphism between the spaces T and T' does not follow from the algebraic isomorphism between $C(T, K)$ and $C(T', K)$. For example, the space $T_{\Omega+1} \otimes T_{\omega+1} - (\Omega, \omega)$ [**11**, p. 69] is pseudo-compact, completely regular, locally compact, and $C(T, K)$ and $C(\beta T, K)$ are algebraically isormorphic, while T and βT are not homeomorphic.

5. Spectral theorem for unbounded self-adjoint operators in Hilbert space.

5.1. Let L be the algebra of all real-valued continuous functions defined on a locally compact Hausdorff space T and vanishing off compact sets. It is well-known that every nonnegative linear functional on L is an integral [**14**, p. 44].

A family of real-valued functions on a space is called monotone if it is closed under the operations of taking monotone increasing and decreasing limits. The functions belonging to the smallest monotone family including L are called Baire functions.

A topological space T is called hemi-compact by Arens [**1**, p. 486] if there exists a sequence T_i of compact subsets of T such that $\bigcup_{i=1}^{\infty} T_i = T$ and every compact subset of T is contained in some T_i. Every topological space which is both σ-compact and locally compact is hemi-compact.

5.2. LEMMA. *Let* G *be a* $*$*-representation of the algebra* $C_0(T, K)$ *of all complex-valued continuous functions vanishing outside compact sets on a hemi-compact Hausdorff space* T*, which is a union of pairwise disconnected, closed-open compact sets* $T_1, T_2, \cdots$ *, by a family* $\mathfrak{B}$ *of operators in a Hilbert space* H*. Let* H *be spanned by a sequence of closed linear manifolds* $H_1, H_2, \cdots$ *, orthogonal in pairs, such that each operator of* $\mathfrak{B}$ *is bunded on* H_i *and* G *is a bounded* $*$*-representation of the algebra* $C(T_i, K)$ *of all complex-valued continuous functions on* T_i *by a family of* *operators on H_i. Then G can be extented to a *-representation of the algebra $B(T, K)$ of all Baire functions bounded on compact subsets of T, and the extension is unique, subject to the condition that $J_{x,y}(f)=(G_f x, y)$ is a complex-valued integral for every $x \in H, y \in H^*$.*

Proof. The function $F(f_i, x, y)=(G_{f_i}x, y)$, defined for $f_i \in C(T_i)$, $x \in H_i$, $y \in H_i^*$, is a bounded integral on $C(T_i)$ and thus is uniquely extensible to $B(T_i)$. [**14**, p. 93]. Hence the lemma [**17**, p. 312].

5.3. THEOREM. *To any self-adjoint operator R in a Hilbert space H, there exists a unique family of projections $\{E_\lambda\}$ depending on the parameter λ, satisfying*

(a) $$E_\lambda < E_\mu \quad \text{or} \quad E_\lambda = E_\lambda E_\mu \quad \text{for} \quad \lambda < \mu ,$$

(b) $$E_{\lambda+0} = E_\lambda ,$$

(c) $$\lim_{\lambda\to-\infty} E_\lambda = 0 \quad \textit{and} \quad \lim_{\lambda\to\infty} E_\lambda = I ,$$

such that

$$R = \int_{-\infty}^{\infty} \lambda dE_\lambda .$$

Proof. Let b_i be a set of real numbers, $i=0, \pm 1, \pm 2, \cdots$, such that

(1) $$\text{for all } i, b_i > b_{i-1} ;$$

(2) $$\lim_{i\to\infty} b_i = \infty ;$$

(3) $$\lim_{i\to-\infty} b_i = -\infty .$$

Then there exists a set of closed linear manifolds $\{H_i\}, i=1, 2, \cdots$, orthogonal in pairs, spanning H, and such that R is defined on H_i and satisfies the relation [**15**, 17]

$$b_i I \geqq R \geqq b_{i-1} I .$$

Let P_i be a projection on H such that $P_i x = x$ if $x \in H_i$, and $P_i x = 0$ otherwise. Now $P_1, P_2, \cdots$, and R generate a commutative semi-normed *-algebra A, the semi-norms of its elements being the norms of the operators in H_i. By Theorem 2.1, A is equivalent to a closed self-adjoint subalgebra S of the algebra $C(T, K)$ of all complex-valued continuous functions on a hemi-compact Hausdorff space T, which is a union of a sequence of pairwise disconnected, closed-open compact subsets $T_1, T_2, \cdots$. S is, in fact, the algebra $C(T, K)$ itself.

Any real continuous function $f(t)$ on the space T is a Baire function. Define a continuous function f_n such that $f_n(t)=f(t)$ if $t \in T_1 \cup \cdots \cup T_n$ and $f_n(t)=0$ otherwise. Let $g_n^m \in L$ so that $g_n^m \uparrow m f_n$, where g_n^m vanish outside the sets $T_1, \cdots, T_n$, and let $g_n = g_1^n \vee \cdots \vee g_n^n$. Then $g_n \uparrow f$ and f is a Baire function. Also the characteristic functions of closed subsets in T are Baire functions.

Let $\hat{R}$ be the image of the operator R. Given $\varepsilon > 0$, we can choose λ_i, $i=0, \pm 1, \pm 2, \cdots$ such that $\lambda_i \to \infty$, $\lambda_{-i} \to -\infty$ as $i \to \infty$ and, for all i, $\lambda_i > \lambda_{i-1}$, $\lambda_i - \lambda_{i-1} < \varepsilon$. Let $\hat{E}_\lambda$ be the characteristic function of the closed set where $\hat{R} \leqq \lambda$, and choose λ_i' from the interval $[\lambda_{i-1}, \lambda_i]$.
Then

$$\left\| \hat{R} - \sum_{-\infty}^{\infty} \lambda_i' (\hat{E}_{\lambda_i} - \hat{E}_{\lambda_{i-1}}) \right\|_\infty < \varepsilon$$

and hence

$$\left\| R - \sum_{-\infty}^{\infty} \lambda_i' (E_{\lambda_i} - E_{\lambda_{i-1}}) \right\|_V < \varepsilon \text{ for each } V \in \mathscr{V}.$$

The theorem is proved.

6. Imbedding algebras into rings of operators in Hilbert space.

6.1. THEOREM. *Every complete semi-normed *-algebra A with or without a unit, satisfying the condition $V(xx^*) = V(x)V(x^*)$ for each $V \in \mathscr{V}$, can be isomorphically mapped onto a closed self-adjoint subalgebra A_1 of the algebra of all linear operators in a Hilbert space $H = \sum_{V \in \mathscr{V}} H_V$ such that if $x \in A$ maps to $X \in A_1$, then X is bounded in each H_V and $V(x) = \|x\|_V$ for each $V \in \mathscr{V}$, where $\|x\|_V$ denotes the norm of X in H_V.*

Proof. By Gelfand-Neumark representation theorem [10, Theorem 1; 12, p. 409], the completed quotient algebra A_V can be isometrically mapped onto a closed self-adjoint subalgebra of the algebra of all bounded operators in Hilbert space H_V.

Let

$$H = \sum_{V \in \mathscr{V}} H_V$$

be the set of all complexes $h = \{h_V\}$, $h_V \in H_V$, with

$$\sum_{V \in \mathscr{V}} \|h\|_V^2 < \infty .$$

The algebraic operations and inner products are defined as follows:

$$\lambda h = \{\lambda h_V\}, \quad h_1 + h_2 = \{h_{1V} + h_{2V}\} , \quad (h_1, h_2) = \sum_{V \in \mathscr{V}} (h_{1V} - h_{2V}) .$$

Let $h_i=\{h_{iV}\}$. Then $\|h_i-h_j\|^2=\sum_{V\in\mathcal{V}}\|h_{iV}-h_{jV}\|^2$. $\|h_i-h_j\|\to 0$ implies $\|h_{iV}-h_{jV}\|\to 0$ for each V. For any fixed V, h_{iV} approaches to an element h_{0V} in H_V as a limit when i approaches infinity. Then $h_i\to h_0=\{h_{0V}\}$ which belongs to H, and H is complete.

The corresponding operator X in H of an element $x\in A$ is defined as $X=\{X_V\}$, where X_V is the operator in H_V corresponding to $x_V\in \overline{A_V}$. Now $Xh=\{X_Vh_V\}$ with

$$\sum_{V\in\mathcal{V}}\|X_Vh_V\|^2<\infty\ .$$

The domain of X is dense in H, for it contains all those elements $\{h_V\}$ where h_V are 0 except for a finite number of them. It is clear that $X(H)\subset H$ and $X(H_V)\subset H_V$.

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