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ON SEMI-NORMED *-ALGEBRAS

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1. Introduction. The notion of semi-normed algebras was introduced by Arens as a generalization of Banach algebras [2, 5]. They are called locally multiplicatively-convex algebras by Michael [16]. Various properties of Banach algebras have been generalized to semi-normed algebras [5, 16, 21, 22, 23].

We repeat here a few definitions. Let A be a linear algebra over the field K of complex or real numbers. A nonnegative real-valued function V defined on A is called a semi-norm if it satisfies the following conditions :

$V(x+y) \leq V(x) + V(y)$, $V(xy) \leq V(x)V(y)$, $V(\lambda x) = |\lambda|V(x)$. Suppose there is a family \mathscr{V} of semi-norms such that $V(x) = 0$ for all $V \in \mathscr{V}$ only if $x = 0$. A is a semi-normed algebra if all the translations of the sets on which $V(x) < e$, where e is real and $V \in \mathscr{V}$, are taken as a subbase of topology, and is complete if it is complete with respect to the uniform structure defined by the various relations $V(x-y) < e$. A is called an *-algebra if there is a semi-linear operation $*$ such that $(\lambda x - yz)^* = \overline{\lambda}x^* - z^*y^*$, $x^{**} = x$. A subset U of A is called idempotent if $UU \subset U$; it is called multiplicatively convex (m -convex) if it is convex and idempotent. A is locally m -convex if there exists a basis for the neighbourhoods of the origin consisting of sets which are m -convex and symmetric.

The present paper is devoted to generalizing the representation theorems for commutative and noncommutative Banach algebras to semi-normed algebras. An application of the Gelfand-Neumark-Arens representation theorem for commutative Banach algebras yields a simple proof of the spectral theorem for bounded self-adjoint operators in Hilbert space [14, p. 95]. Our generalized representation theorem for commutative semi-normed algebras gives rise to a similar proof of the spectral theorem for unbounded self-adjoint operators.

The characterization of the algebra $C(T, K)$ of all complex-valued continuous functions on a locally compact, paracompact Hausdorff space T has been treated by Arens [5, p. 469]. We have a characterization theorem for $C(T, K)$ where T is a locally compact completely regular space and also a uniqueness theorem for the space T [cf. the Banach-Stone theorem, 6, p. 170, 20, p. 469]: If $C(T_1, K)$, $C(T_2, K)$ are topo-

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logically isomorphic, then T_1 and T_2 are homeomorphic. If T_1, T_2 are Hewitt's Q -spaces [11, p. 85], the topological equivalence between the spaces follows from the algebraic isomorphism between $C(T_1, K)$ and $C(T_2, K)$, but not in general.

2. Functional representation.

2.1. THEOREM. *Let A be a complete commutative semi-normed *-algebra (with or without a unit) over the complex numbers K such that*

2.2. $V(xx^*) \geq k_V V(x^*)$, for all $V \in \mathcal{V} (k_V > 0)$. *Then A is topologically isomorphic to a complete self-adjoint subalgebra S of the algebra $C(T, K)$ of all continuous complex-valued functions (vanishing at infinity if A has no unit) on T with k -topology, where T is the union of the members of a family of pairwise disconnected and closed-open sets. (compact if A has a unit, otherwise locally compact).*

Proof. The elements x in A satisfying $V(x)=0$ form an ideal Z_V , a kernel ideal of A . The quotient algebra A/Z_V is a normed algebra when V is used to define a norm, and the completion B_V of A/Z_V is a commutative Banach *-algebra. By Gelfand-Neumark-Arens representation theorem [3, Theorem 1, p. 278], there exists a Hausdorff space (compact if A has a unit, otherwise locally compact) $Q_V=V$ -neighbourhood homomorphism, for which B_V is the class of all complex-valued continuous functions (vanishing at infinity if A has no unit) on Q_V such that

$$x_V^*(q) = \overline{x_V(q)} \quad (q \in Q_V, x \in B_V).$$

and

$$(2.3) \quad k_V V(x_V) \leq \sup_{q \in Q_V} |x_V(q)| \leq V(x_V).$$

Let

$$T = \bigcup_{V \in \mathcal{V}} Q_V.$$

Retaining the original weak* topology for Q_V and regarding all Q_V as pairwise disconnected and closed-open subsets, we have a locally compact completely regular space T . The complex-valued continuous functions on T are of the form $f(t) = \{f_V\}$, where $f_V(t) \in C(Q_V, K)$ and $f(t) = f_V(t)$ if $t \in Q_V$.

The mapping

$$P : \quad x \in A \rightarrow x(t) = \{x_V(t)\} \in C(T, K)$$

maps A onto a subalgebra S of $C(T, K)$. P is isomorphic; for, if x maps to zero functional, then $V(x)=0$ for all $V \in \mathcal{V}$ and x is the zero

element of A .

In fact, P is a homeomorphism. Denote the open set in A consisting of all x such that $V(x) < e$ by $0(V, e)$ and the open set in $C(T, K)$ defined by $\sup_{q \in Q_V} |f(q)| < k_V e$ by $0'(Q_V, e)$. It follows from the inequalities 2.3 that P maps $0(V, e)$ onto a subset of $C(T, K)$ containing $0'(Q_V, e)$. This proves the continuity of the inverse mapping of P from S onto A .

Let W be a compact subset in T contained in the union of Q_{V_1}, \dots, Q_{V_n} . It is clear that P maps the intersection of $0(V_1, e), \dots, 0(V_n, e)$ onto a subset in $C(T, K)$ contained in the intersection of $0'(V_1, e/k_V), \dots, 0'(V_n, e/k_V)$, and S , that is, in the intersection of $0'(W, e/k_V)$ and S . P is therefore continuous.

The completeness of S is an immediate consequence of the completeness of A and inequalities 2.3.

2.4. COROLLARY. *Let M_V be a maximal ideal in B_V (the completion of the quotient ring $A_V = A/Z_V$) and let $f(t)$ be a complex-valued continuous function on the space T . Then $f(t)$ belongs to S if $f_V(M_V) = f_V(M_V)$ whenever $U \leq V$.*

Proof. M_V is actually a point in Q_V and $f_V(M_V)$ belongs to $C(Q_V, K)$. Let $\overline{\Pi}_{UV}$ be the natural mapping of B_V into B_U when $U \leq V$. Then $\overline{\Pi}_{UV}(f_V) = f_U$ whenever $U \leq V$ if $f_V(M_V) = f_U(M_U)$. Hence the corollary [16, Theorem 5.1].

This immediately yields the following result [cf. 5, p. 471].

2.5. THEOREM. *Let A be a commutative complete semi-normed *-algebra with a unit (without unit) satisfying 2.2. Then an element x in A has an inverse (reverse) if $x(M) \neq 0$ ($x(M) \neq -1$) for each closed maximal ideal M in A .*

3. Spectrum. An element h in a complete semi-normed *-algebra A satisfying 2.2 is called Hermitian, if $h^* = h$; and an Hermitian element h is called positive, if its spectrum consists of nonnegative numbers.

3.1. THEOREM. *The spectrum of every Hermitian element h is real.*

Proof. Suppose A has a unit. Let A_1 be the minimal complete *-subalgebra of A containing h . Then A_1 is commutative. By Theorem 2.1 A_1 is equivalent to a closed subalgebra S of $C(T, K)$. The corresponding function $h(M)$ of the element h in A is real-valued. For any nonreal number λ , the function $h(M) - \lambda$ is not equal to zero anywhere. The theorem follows from Theorem 2.5.

3.2. THEOREM. *Every closed self-adjoint subalgebra A_0 of a complete semi-normed *-algebra A with a unit (without unit) satisfying 2.2 contains inverses (reverses).*

Proof. Rickart has proved that $x_V \in A_{0V}$ (the completion of $A_{00} = A_0/Z_V$) has an inverse (reverse) iff both $x_V^*x_V$ and $x_Vx_V^*$ have inverses (reverses) and that the inverse (reverse) of x_V is contained in A_{0V} iff the inverses (reverses) of $x_V^*x_V$ and $x_Vx_V^*$ are contained in A_{0V} [18, pp. 531-532]. Since every closed maximal ideal in A contains a kernel ideal [5, p. 466], it follows from Theorem 2.5 that A_0 contains inverses (reverses) of its Hermitian elements, and hence of all its elements which have inverses (reverses) in A .

3.3. COROLLARY. *Let A_0 be any closed self-adjoint subalgebra of A . Then the spectrum of $x \in A_0$ relative to A_0 is identical with the spectrum relative to A .*

3.4. THEOREM. *Let x be a normal element, that is, $xx^* = x^*x$, of A (with or without a unit) and let $f(\lambda)$ be a complex-valued continuous function (vanishing at infinity, if A has no unit) defined on the spectrum σ of x . Then $f(x)$ defines an element contained in every commutative closed self-adjoint subalgebra of A which contains x .*

Moreover if $s(\lambda) = f(\lambda) + g(\lambda)$, $p(\lambda) = f(\lambda)g(\lambda)$, $q(\lambda) = f(\lambda)$, $r(\lambda) = \lambda$, then $s(x) = f(x) + g(x)$, $p(x) = f(x)g(x)$, $q(x) = f(x)^*$, $r(x) = x$.

Proof. Let A_0 be a commutative closed self-adjoint subalgebra of A containing x and let M_V be a maximal ideal in A_{0V} . Then A_0 is equivalent to a closed self-adjoint subalgebra S of the algebra $C(T, K)$ of all complex-valued continuous functions on a locally compact completely regular space T and $f(x_V(M_U)) = f(x_V(M_V))$ whenever $U \leq V$. By Corollary 2.4, $f(x(M))$ determines a unique element, denoted by $f(x)$, contained in A_0 . The first part of the theorem is proved.

The second part of the theorem is obvious.

3.5. THEOREM. *The sum of two positive elements is positive.*

Proof. Suppose A has a unit. Let h and k be two positive elements in A and let A_0 be the minimal closed self-adjoint subalgebra of A containing $h+k$. Since the inverse of $h_V + k_V + \lambda e$ for any nonnegative number λ and each $V \in \mathcal{V}$. [13, p. 52] the function $h(M) + k(M) + \lambda$ does not vanish at any M . The theorem follows from Theorem 2.5.

3.6. THEOREM. *The Hermitian elements of a complete seminormed *-algebra satisfying the condition 2.2 constitute a lattice,*

Proof. To any Hermitian h , there is a positive element $|h|$ corresponding to the function $|\lambda|$ by Theorem 3.4. Let h and k be arbitrary Hermitian elements and define.

$h \vee k = \frac{1}{2}(h+k+|h-k|)$, $h \wedge k = \frac{1}{2}(h+k-|h-k|)$. Then the Hermitian elements constitutes a lattice.

4. Closed self-adjoint subalgebras.

4.1. THEOREM. A commutative complete semi-normed *-algebra A satisfying the condition 2.2 is equivalent to a closed, separating self-adjoint subalgebra S of the algebra $C(T_0, K)$ of all complex-valued continuous functions (vanishing at infinity, if A has no unit) on a completely regular space T_0 with a topology which has at most the open sets of the k -topology, that is, with a topology $\rho \leq k$.

Proof. By Theorem 2.1, A is equivalent to a closed self-adjoint subalgebra S of $C(T, K)$, where T is a union of pairwise disconnected and closed-open sets (compact if A has a unit, otherwise locally compact). Let $x(t)$ be the corresponding function in S of the element x in A . Denote by T_0 the class of all subsets of T :

$$L_a = \{t; x(t) = x(a) \text{ for each } x \in A\}.$$

Following Čech's notation, Let ρ denote the mapping:

$$a \in T \rightarrow L_a$$

and let $[f, I]$ denote those elements $\rho(t)$ of T_0 such that $f(t) \in I$, where $f(t)$ is a continuous real function belonging to S and I is an open interval. The topology generated by considering all these $[f, I]$ as a subbase is called ρ -topology.

It is easy to see that ρ is a continuous mapping and that for any $a \in T$, there is an $[f, I]$ containing $\rho(a)$. Let $[f_1, I_1]$ and $[f_2, I_2]$ be any two open sets in T_0 containing $\rho(a)$. If both $f_1(a)$ and $f_2(a)$ are different from zero, we can assume without loss of generality that $f_1(a) = f_2(a)$ and that I_1 and I_2 are identical. We define $g_i(t) = f_i(t)$ if $f_i(t) \leq f_i(a)$, and $g_i(t) = 2f_i(a) - f_i(t)$ if $f_i(t) > f_i(a)$, $i = 1, 2$. Then $g_1(t)$ and $g_2(t)$ are continuous functions. Let $g(t) = g_1(t) \wedge g_2(t)$. It is clear that $[g, I] \subset [f_1, I] \cap [f_2, I]$. In case $f_1(a) = 0$ and $f_2(a) \neq 0$, we can assume that $f_1(t)$ and $f_2(t)$ are nonnegative. Let $g(t) = f_2(t) - f_1(t)$. An interval I can be so chosen that $[g, I] \subset [f_1, I] \cap [f_2, I]$. Hence T_0 is a topological space. Čech has proved that T_0 is Hausdorff and completely regular [8, p. 827].

Now the closed subalgebra S of $C(T, K)$ is a closed, separating subalgebra of $C(T_0, K)$.

4.2. **REMARK.** It is clear that the elements in the space T_0 are the closed maximal ideals in the algebra A and the ρ -topology is the weak* topology. Professor Arens has constructed examples to show that T_0 is not necessarily locally compact. He has also constructed a completely regular space T such that $C(T, K)$ with k -topology is not complete. [4, p. 234]. We have, however, the following.

4.3. **THEOREM.** *The necessary and sufficient condition that a commutative complete semi-normed *-algebra A satisfying the condition 2.2 be equivalent to $C(T, K)$, with k -topology, of all complex-valued continuous functions on a locally compact completely regular space T is:*

To any closed maximal ideal M_0 in A , there are an $x \in A$ and an $\epsilon > 0$ such that the intersection of the maximal ideals M satisfying the relation $|x(M_0) - x(M)| \leq \epsilon$ contains a kernel ideal.

Proof. The necessity is obvious. The sufficiency follows from Theorem 4.1 and Corollary 2.4.

4.4. **REMARK.** Theorem 4.3 generalizes the theorem of Arens characterizing the algebra $C(T, K)$, where T is a locally compact, paracompact Hausdorff space. [5, p. 469]. Let A be an algebra with a locally finite partition of unity. (For definition and notation, see 5, p. 463) To any maximal closed ideal M_0 , there exists an u_ν such that $u_\nu(M_0) = \delta \neq 0$, since M_0 contains a kernel ideal. There are only a finite number of W such that $W(u_\nu) \neq 0$, say, W_1, \dots, W_n . Let $W_0 = \max. (W_1, \dots, W_n)$. The intersection of the closed maximal ideals M satisfying $|u_\nu(M_0) - u_\nu(M)| \leq \delta/2$ evidently contains Z_{W_0} .

4.5. **THEOREM.** *For the algebra $C(T, K)$ of all complex-valued continuous functions (vanishing at infinity) on a locally compact completely regular space T with k -topology, there is one-to-one correspondence between closed ideals in $C(T, K)$ and the closed subsets of T .*

This is a generalization of a theorem due to Stone [20, Theorem 85] and the proof is straightforward.

4.6. **COROLLARY.** *For the algebra of all complex-valued continuous functions (vanishing at infinity) on a locally compact completely regular space with k -topology, there is one-to-one correspondence between the closed maximal (regular) ideals of the algebra and the points of the space (the point at infinity is not included).*

4.7. **THEOREM.** *The necessary and sufficient condition two locally compact completely regular spaces T and T' be homeomorphic is that the*

algebras $C(T, K)$ and $C(T', K)$ of all complex-valued continuous functions (vanishing at infinity) on the spaces with k -topology be topologically isomorphic.

Proof. Following Stone's idea, we define the closure of a family of closed maximal (regular) ideals in $C(T, K)$ as the hull of the kernel of the family [14, p. 56]. It is clear that a subset of the space T is closed iff it is equal to the hull of its kernel when it is considered as a set of the maximal (regular) ideals in $C(T, K)$.

4.8. REMARK. The homeomorphism between the spaces T and T' does not follow from the algebraic isomorphism between $C(T, K)$ and $C(T', K)$. For example, the space $T_{\Omega+1} \otimes T_{\omega+1} - (\Omega, \omega)$ [11, p. 69] is pseudo-compact, completely regular, locally compact, and $C(T, K)$ and $C(\beta T, K)$ are algebraically isomorphic, while T and βT are not homeomorphic.

5. Spectral theorem for unbounded self-adjoint operators in Hilbert space.

5.1. Let L be the algebra of all real-valued continuous functions defined on a locally compact Hausdorff space T and vanishing off compact sets. It is well-known that every nonnegative linear functional on L is an integral [14, p. 44].

A family of real-valued functions on a space is called monotone if it is closed under the operations of taking monotone increasing and decreasing limits. The functions belonging to the smallest monotone family including L are called Baire functions.

A topological space T is called hemi-compact by Arens [1, p. 486] if there exists a sequence T_i of compact subsets of T such that $\bigcup_{i=1}^{\infty} T_i = T$ and every compact subset of T is contained in some T_i . Every topological space which is both σ -compact and locally compact is hemi-compact.

5.2. LEMMA. Let G be a *-representation of the algebra $C_0(T, K)$ of all complex-valued continuous functions vanishing outside compact sets on a hemi-compact Hausdorff space T , which is a union of pairwise disconnected, closed-open compact sets T_1, T_2, \dots , by a family \mathfrak{B} of operators in a Hilbert space H . Let H be spanned by a sequence of closed linear manifolds H_1, H_2, \dots , orthogonal in pairs, such that each operator of \mathfrak{B} is bounded on H_i and G is a bounded *-representation of the algebra $C(T_i, K)$ of all complex-valued continuous functions on T_i by a family of

operators on H_i . Then G can be extended to a $*$ -representation of the algebra $B(T, K)$ of all Baire functions bounded on compact subsets of T , and the extension is unique, subject to the condition that $J_{x,y}(f) = (Gx, y)$ is a complex-valued integral for every $x \in H, y \in H^*$.

Proof. The function $F(f_i, x, y) = (G_{f_i}x, y)$, defined for $f_i \in C(T_i), x \in H_i, y \in H_i^*$, is a bounded integral on $C(T_i)$ and thus is uniquely extensible to $B(T_i)$. [14, p. 93]. Hence the lemma [17, p. 312].

5.3. THEOREM. *To any self-adjoint operator R in a Hilbert space H , there exists a unique family of projections $\{E_\lambda\}$ depending on the parameter λ , satisfying*

- (a) $E_\lambda < E_\mu$ or $E_\lambda = E_\lambda E_\mu$ for $\lambda < \mu$,
 (b) $E_{\lambda+0} = E_\lambda$,
 (c) $\lim_{\lambda \rightarrow -\infty} E_\lambda = 0$ and $\lim_{\lambda \rightarrow \infty} E_\lambda = I$,

such that

$$R = \int_{-\infty}^{\infty} \lambda dE_\lambda.$$

Proof. Let b_i be a set of real numbers, $i=0, \pm 1, \pm 2, \dots$, such that

- (1) for all $i, b_i > b_{i-1}$;
 (2) $\lim_{t \rightarrow \infty} b_t = \infty$;
 (3) $\lim_{t \rightarrow -\infty} b_t = -\infty$.

Then there exists a set of closed linear manifolds $\{H_i\}, i=1, 2, \dots$, orthogonal in pairs, spanning H , and such that R is defined on H_i and satisfies the relation [15, 17]

$$b_i I \geq R \geq b_{i-1} I.$$

Let P_i be a projection on H such that $P_i x = x$ if $x \in H_i$, and $P_i x = 0$ otherwise. Now P_1, P_2, \dots , and R generate a commutative semi-normed $*$ -algebra A , the semi-norms of its elements being the norms of the operators in H_i . By Theorem 2.1, A is equivalent to a closed self-adjoint subalgebra S of the algebra $C(T, K)$ of all complex-valued continuous functions on a hemi-compact Hausdorff space T , which is a union of a sequence of pairwise disconnected, closed-open compact subsets T_1, T_2, \dots . S is, in fact, the algebra $C(T, K)$ itself.

Any real continuous function $f(t)$ on the space T is a Baire function. Define a continuous function f_n such that $f_n(t)=f(t)$ if $t \in T_1 \cup \dots \cup T_n$ and $f_n(t)=0$ otherwise. Let $g_n^m \in L$ so that $g_n^m \uparrow m f_n$, where g_n^m vanish outside the sets T_1, \dots, T_n , and let $g_n = g_n^1 \vee \dots \vee g_n^n$. Then $g_n \uparrow f$ and f is a Baire function. Also the characteristic functions of closed subsets in T are Baire functions.

Let \hat{R} be the image of the operator R . Given $\epsilon > 0$, we can choose $\lambda_i, i=0, \pm 1, \pm 2, \dots$ such that $\lambda_i \rightarrow \infty, \lambda_{-i} \rightarrow -\infty$ as $i \rightarrow \infty$ and, for all $i, \lambda_i > \lambda_{i-1}, \lambda_i - \lambda_{i-1} < \epsilon$. Let \hat{E}_λ be the characteristic function of the closed set where $\hat{R} \leq \lambda$, and choose λ_i' from the interval $[\lambda_{i-1}, \lambda_i]$. Then

$$\left\| \hat{R} - \sum_{-\infty}^{\infty} \lambda_i' (\hat{E}_{\lambda_i} - \hat{E}_{\lambda_{i-1}}) \right\|_{\infty} < \epsilon$$

and hence

$$\left\| R - \sum_{-\infty}^{\infty} \lambda_i' (E_{\lambda_i} - E_{\lambda_{i-1}}) \right\|_V < \epsilon \text{ for each } V \in \mathcal{V}.$$

The theorem is proved.

6. Imbedding algebras into rings of operators in Hilbert space.

6.1. THEOREM. *Every complete semi-normed *-algebra A with or without a unit, satisfying the condition $V(xx^*) = V(x)V(x^*)$ for each $V \in \mathcal{V}$, can be isomorphically mapped onto a closed self-adjoint subalgebra A_1 of the algebra of all linear operators in a Hilbert space $H = \sum_{V \in \mathcal{V}} H_V$ such that if $x \in A$ maps to $X \in A_1$, then X is bounded in each H_V and $V(x) = \|x\|_V$ for each $V \in \mathcal{V}$, where $\|x\|_V$ denotes the norm of X in H_V .*

Proof. By Gelfand-Neumark representation theorem [10, Theorem 1; 12, p. 409], the completed quotient algebra A_V can be isometrically mapped onto a closed self-adjoint subalgebra of the algebra of all bounded operators in Hilbert space H_V .

Let

$$H = \sum_{V \in \mathcal{V}} H_V$$

be the set of all complexes $h = \{h_V\}, h_V \in H_V$, with

$$\sum_{V \in \mathcal{V}} \|h\|_V^2 < \infty .$$

The algebraic operations and inner products are defined as follows:

$$\lambda h = \{\lambda h_V\}, h_1 + h_2 = \{h_{1V} + h_{2V}\}, (h_1, h_2) = \sum_{V \in \mathcal{V}} (h_{1V} - h_{2V}) .$$

Let $h_i = \{h_{iV}\}$. Then $\|h_i - h_j\|^2 = \sum_{V \in \mathcal{V}} \|h_{iV} - h_{jV}\|^2$. $\|h_i - h_j\| \rightarrow 0$ implies $\|h_{iV} - h_{jV}\| \rightarrow 0$ for each V . For any fixed V , h_{iV} approaches to an element h_{0V} in H_V as a limit when i approaches infinity. Then $h_i \rightarrow h_0 = \{h_{0V}\}$ which belongs to H , and H is complete.

The corresponding operator X in H of an element $x \in A$ is defined as $X = \{X_V\}$, where X_V is the operator in H_V corresponding to $x_V \in \overline{A_V}$. Now $Xh = \{X_V h_V\}$ with

$$\sum_{V \in \mathcal{V}} \|X_V h_V\|^2 < \infty.$$

The domain of X is dense in H , for it contains all those elements $\{h_V\}$ where h_V are 0 except for a finite number of them. It is clear that $X(H) \subset H$ and $X(H_V) \subset H_V$.

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