

Pacific Journal of Mathematics

AN INVERSION OF THE STIELTJES TRANSFORM

RICHARD ROBINSON GOLDBERG

AN INVERSION OF THE STIELTJES TRANSFORM

RICHARD R. GOLDBERG

A generalized Lambert transform, or L -transform, is an integral of the form

$$H(x) = \int_0^{\infty} \sum_{k=1}^{\infty} a_k e^{-kxt} \psi(t) dt .$$

In this paper we shall invert the integral transform

$$(1) \quad G(x) = \int_0^{\infty} \frac{\psi(t)t dt}{x^2 + t^2} \quad 0 < x < \infty$$

by reducing it by means of a certain summation to an L -transform and then applying an inversion theorem for L -transforms.

From this we deduce an inversion formula for the Stieltjes transform. This is given in Theorem 3.

1. The inversion of the transform (1). We shall need the following theorem on L -transforms which is the case $r=1$ of Theorem 7.7 in [1].

THEOREM 1. *Let $\{a_k\}_{k=1}^{\infty}$ be a bounded sequence of non-negative numbers with $a_1 > 0$. Let $\{b_n\}_{n=1}^{\infty}$ be the (unique) sequence such that*

$$\sum_{d|m} a_d b_{m/d} = \begin{cases} 1, & m=1 \\ 0, & m=2, 3, \dots, \end{cases}$$

the summation running over all divisors d of m . If the b_n are also bounded and if

1. $K(t) = \sum_{k=1}^{\infty} a_k e^{-kt} \quad (0 < t < \infty)$
2. $H(x) = \int_0^{\infty} K(xt) \psi(t) dt$ converges for some $x > 0$
3. $\int_0^1 \frac{|\psi(t) \log t|}{t} dt < \infty$

then

$$\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p}{t} \right)^{p+1} \sum_{n=1}^{\infty} b_n n^p H^{(p)} \left(\frac{np}{t} \right) = \psi(t) \text{ almost everywhere } (0 < t < \infty) .$$

Received March, 21 1958.

Now let

$$G(x) = \int_0^{\infty} \frac{\phi(t)t \, dt}{x^2 + t^2}$$

where we assume

$$\int_0^{\infty} \frac{|\phi(t)|}{t} \, dt < \infty \quad \text{and} \quad \int_0^1 \frac{|\phi(t) \log t|}{t} \, dt < \infty .$$

To reduce $G(x)$ to an L -transform we define

$$H_N(x) = \frac{1}{x} \left[\frac{G(0)}{2} + \sum_{k=1}^N (-1)^k G\left(\frac{k\pi}{x}\right) \right] \quad N=1, 2, \dots$$

Then

$$(2) \quad H_N(x) = \int_0^{\infty} \phi(t) \left[\frac{1}{2xt} + \sum_{k=1}^N \frac{(-1)^k xt}{(xt)^2 + (k\pi)^2} \right] dt .$$

For $N=1, 2, \dots$ we have

$$\left| \sum_{k=1}^N \frac{(-1)^k xt}{(xt)^2 + (k\pi)^2} \right| < \frac{xt}{(xt)^2 + \pi^2} < \frac{1}{xt} \quad (0 < xt < \infty).$$

(This is because the terms of the sum alternate in sign and decrease in absolute value so that the modulus of the sum is less than that of its first term.) Hence for any $x > 0$

$$\int_0^{\infty} \left| \phi(t) \left[\frac{1}{2xt} + \sum_{k=1}^N \frac{(-1)^k xt}{(xt)^2 + (k\pi)^2} \right] \right| dt \ll \frac{3}{2x} \int_0^{\infty} \frac{|\phi(t)|}{t} \, dt < \infty .$$

This, by dominated convergence, allows us to let N become infinite under the integral sign in (2) and we obtain

$$H(x) \equiv \lim_{N \rightarrow \infty} H_N(x) = \int_0^{\infty} \phi(t) \left[\frac{1}{2xt} + \sum_{k=1}^{\infty} \frac{(-1)^k xt}{(xt)^2 + (k\pi)^2} \right] dt \quad (x > 0).$$

But for $z > 0$

$$\frac{1}{2z} + \sum_{k=1}^{\infty} \frac{(-1)^k z}{z^2 + (k\pi)^2} = \frac{\operatorname{cosech} z}{2} = \frac{1}{e^z - e^{-z}} = \sum_{k=1}^{\infty} e^{-(2k-1)z} ,$$

(see [3; 113]). Thus

$$(3) \quad H(x) = \int_0^{\infty} \sum_{k=1}^{\infty} e^{-(2k-1)xt} \phi(t) \, dt = \int_0^{\infty} K(xt) \phi(t) \, dt$$

where $K(t) = \sum_{k=1}^{\infty} a_k e^{-kt}$ and

$$(4) \quad a_{2k-1} = 1, \quad a_{2k} = 0, \quad k = 1, 2, \dots$$

It was shown in [2; 556] that the sequence $\{b_n\}_{n=1}^\infty$ defined in Theorem 1 corresponding to the a_k in (4) is

$$(5) \quad b_{2n-1} = \mu_{2n-1}, \quad b_{2n} = 0, \quad n = 1, 2, \dots$$

Here the μ_n are the Moebius numbers defined as $\mu_1 = 1$, $\mu_n = (-1)^s$ if n is the product of s distinct primes and $\mu_n = 0$ if n is divisible by a square factor. The b_n are bounded, so that we may apply Theorem 1 (with the a_k and b_n as in (4) and (5)) to invert the L -transform (3) and obtain $\phi(t)$ for almost all $t > 0$. These results are summarized in Theorem 2.

THEOREM 2. *Let*

$$G(x) = \int_0^\infty \frac{\phi(t)t \, dt}{x^2 + t^2}$$

where

$$\int_0^\infty \frac{|\phi(t)|}{t} \, dt < \infty$$

and

$$\int_0^1 \frac{|\phi(t) \log t|}{t} \, dt < \infty$$

Then

$$H(x) = \lim_{N \rightarrow \infty} \frac{1}{x} \left[\frac{G(0)}{2} + \sum_{k=1}^{\infty} (-1)^k G\left(\frac{k\pi}{x}\right) \right]$$

exists for all positive x and

$$H(x) = \int_0^\infty \sum_{k=1}^{\infty} e^{-(2k-1)x t} \phi(t) \, dt .$$

Moreover

$$\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p}{t}\right)^{p+1} \sum_{n=1}^{\infty} \mu_{2n-1} (2n-1)^p H^{(p)} \left[\frac{(2n-1)p}{t} \right] = \phi(t)$$

almost everywhere ($0 < t < \infty$).

2. The inversion of the Stieltjes transform. Let

$$(6) \quad F(x) = \int_0^\infty \frac{\varphi(t)}{x+t} \, dt$$

where

$$\int_0^{\infty} \frac{|\varphi(t)|}{t} dt < \infty \quad \text{and} \quad \int_0^1 \frac{|\varphi(t) \log t|}{t} dt < \infty .$$

Let $G(x) = \frac{1}{2}F(x^2)$, $\psi(t) = \varphi(t^2)$. Then

$$G(x) = \frac{1}{2} F(x^2) = \frac{1}{2} \int_0^{\infty} \frac{\varphi(t)}{x^2+t} dt = \int_0^{\infty} \frac{\varphi(t^2)t}{x^2+t^2} dt = \int_0^{\infty} \frac{\psi(t)t}{x^2+t^2} dt ;$$

also

$$\int_0^{\infty} \frac{|\psi(t)|}{t} dt = \int_0^{\infty} \frac{\varphi(t^2)}{t} dt = \frac{1}{2} \int_0^{\infty} \frac{\varphi(t)}{t} dt < \infty ;$$

similarly

$$\int_0^1 \frac{|\psi(t) \log t|}{t} dt < \infty .$$

The assumptions of Theorem 2 thus hold. We can therefore use Theorem 2 to obtain $\psi(t) = \varphi(t^2)$ for almost all $t > 0$. This gives us $\varphi(t)$ for almost all $t > 0$ and thus effects an inversion of the Stieltjes transform (6).

THEOREM 3. *Let*

$$F(x) = \int_0^{\infty} \frac{\varphi(t)}{x+t} dt$$

where

$$\int_0^{\infty} \frac{|\varphi(t)|}{t} dt < \infty$$

and

$$\int_0^1 \frac{|\varphi(t) \log t|}{t} dt < \infty .$$

Let

$$G(x) = \frac{1}{2}F(x^2)$$

and

$$H(x) = \frac{1}{x} \left[\frac{G(0)}{2} + \sum_{k=1}^{\infty} (-1)^k G\left(\frac{k\pi}{x}\right) \right]$$

(the sum converging by Theorem 2). *Then*

$$\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p}{\sqrt{t}} \right)^{p+1} \sum_{n=1}^{\infty} \mu_{2n-1} (2n-1)^p H^{(p)} \left[\frac{(2n-1)p}{\sqrt{t}} \right] = \varphi(t)$$

almost everywhere ($0 < t < \infty$).

Of course, the Stieltjes transform has been inverted under less restrictive conditions on $\varphi(t)$. We believe the interest of this note lies in the use of the μ_n as an inverting device.

REFERENCES

1. R. R. Goldberg, *Inversions of generalized Lambert transforms*, Duke Math J. To appear.
2. R. R. Goldberg and R. S. Varga, *Moebius inversion of Fourier transforms*, Duke Math J. **24** (1956), 553-560.
3. E. C. Titchmarsh, *Theory of functions*, Oxford 1932.

NORTHWESTERN UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. L. ROYDEN
Stanford University
Stanford, California

R. A. BEAUMONT
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

E. G. STRAUS
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
C. E. BURGESS
M. HALL
E. HEWITT

A. HORN
V. GANAPATHY IYER
R. D. JAMES
M. S. KNEBELMAN

L. NACHBIN
I. NIVEN
T. G. OSTROM
M. M. SCHIFFER

G. SZEKERES
F. WOLF
K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
OREGON STATE COLLEGE
UNIVERSITY OF OREGON
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF UTAH
WASHINGTON STATE COLLEGE
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
HUGHES AIRCRAFT COMPANY
THE RAMO-WOOLDRIDGE CORPORATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is \$12.00; single issues, \$3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues, \$1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 8, No. 2

April, 1958

John Herbert Barrett, <i>Second order complex differential equations with a real independent variable</i>	187
Avner Friedman, <i>Remarks on the maximum principle for parabolic equations and its applications</i>	201
Richard Robinson Goldberg, <i>An inversion of the Stieltjes transform</i>	213
Olavi Hellman, <i>On the periodicity of the solution of a certain nonlinear integral equation</i>	219
Gilbert Helmsberg, <i>A theorem on equidistribution on compact groups</i>	227
Lloyd Kenneth Jackson, <i>Subfunctions and the Dirichlet problem</i>	243
Naoki Kimura, <i>The structure of idempotent semigroups. I</i>	257
Stephen Kulik, <i>A method of approximating the complex roots of equations</i>	277
Ancel Clyde Mewborn, <i>A note on a paper of L. Guttman</i>	283
Zeev Nehari, <i>On the principal frequency of a membrane</i>	285
G. Pólya and I. J. Schoenberg, <i>Remarks on de la Vallée Poussin means and convex conformal maps of the circle</i>	295
B. M. Stewart, <i>Asymmetry of a plane convex set with respect to its centroid</i>	335
Hans F. Weinberger, <i>Lower bounds for higher eigenvalues by finite difference methods</i>	339
Edwin Weiss and Neal Zierler, <i>Locally compact division rings</i>	369
Bertram Yood, <i>Homomorphisms on normed algebras</i>	373