ON THE PERIODICITY OF THE SOLUTION OF A CERTAIN NONLINEAR INTEGRAL EQUATION

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OLAVI HELLMAN

In the following paper we will study the nonlinear integral equation

\( E(t) = F(t) - \int_0^t G(t-\tau)N\{E(\tau)\} \, d\tau \)

where \( F(t) \) is a known periodic real function and \( G(t) \) and \( N(x) \) are known real functions. In particular we will investigate the behaviour of the solution \( E(t) \) of the equation (1) for large values of \( t \).

We assume that \( G \in L[0, \infty] \) and that \( N(x) \) is bounded almost everywhere and Borel-measurable in \([ -\infty, \infty ]\). Furthermore \( N(x) \) is assumed expressible in the form

\[ N(x) \sim N(0) + \int_{-\infty}^{+\infty} S(\lambda) e^{i\lambda x} \frac{1}{i\lambda} \, d\lambda \]

with \( \int_{-\infty}^{+\infty} |S(\lambda)| \, d\lambda < \infty \) and with finite \( N(0) \). This representation is to be valid almost everywhere in \([ -\infty, \infty ]\).

Because \( N(x) \) is Borel-measurable in \([ -\infty, \infty ]\) and \( |N(0)| < \infty \), the measurability of \( x \) implies the measurability of \( N(x) \). The following four classes of \( N(x) \)-functions are distinguished:

\[ \begin{align*}
N \in K_{1/2} & \quad \text{if} \quad x \in L[0, 1] \quad \text{implies} \quad N(x) \in L[0, 1] \\
N \in K_{1/1\infty} & \quad \text{if} \quad x \in L[0, 1] \quad \text{implies} \quad N(x) \in L[0, \infty] \\
N \in K_{1/\infty} & \quad \text{if} \quad x \in L[0, \infty] \quad \text{implies} \quad N(x) \in L[0, 1] \\
N \in K_{\infty/\infty} & \quad \text{if} \quad x \in L[0, \infty] \quad \text{implies} \quad N(x) \in L[0, \infty]
\end{align*} \]

The space of measurable and bounded functions defined on the finite interval \([0, A]\) will be denoted by \( M[0, A] \). The norm of \( x \in M[0, A] \) is defined, as usual, by

\[ \|x\| = \inf_{E \in \mathcal{E}} \left\{ \sup_{t \in [0, A] \setminus E} |x| \right\} \]

where \( E \) ranges over the sets of measure zero in \([0, A]\), and the distance of \( x \in M[0, A] \) and \( y \in M[0, A] \) by \( ||x-y|| \). The space \( M[0, 1] \) is complete.

The proofs in this paper will be based on the following theorem by Tihonov (see for instance [1]) which is valid in \( M[0, A] \): Let the operator \( B \) map \( M[0, A] \) into itself and let \( ||B(x) - B(y)|| \leq \beta ||x - y|| \) for all \( x \) and
y in $M[0, A]$, where $\beta < 1$. Then the equation $y = B(y)$ has a unique solution $\bar{y}$ in $M[0, A]$. The function $\bar{y}$ may be obtained by iteration:

$$\bar{y} = \lim_{n \to \infty} y_n$$

where $y_n = B(y_{n-1})$ and where $y_0$ may be taken arbitrarily from $M[0, A]$.

We will prove the following theorem.

**Theorem.** Suppose that $F(t)$ is a periodic function in $[0, \infty]$ with period $T$, and that $F \in M[0, T]$. Furthermore suppose that $G \in L[0, \infty]$, $N \in K_{10}$ and $E(t)$ is the solution of

$$\int_{-\infty}^{0} G(\tau)d\tau - \int_{-\infty}^{\infty} S(\lambda) e^{i\lambda(t-\tau)} - 1 d\lambda d\tau$$

If $E(t)$ is the solution of

$$(4) \quad E(t) = F(t) - N(0) \int_{t}^{0} G(u)du - \int_{0}^{t} G(t-\tau) \int_{-\infty}^{\infty} S(\lambda) e^{i\lambda B(\tau)} - 1 d\lambda d\tau$$

then $\lim_{n \to \infty} E(nt+u) = v(u)$ exists, as $n \to \infty$ through integer values. The convergence is uniform. Moreover, $v(u)$ has the period $T$, and satisfies

$$(5) \quad v(u) = F(u) - N(0) \int_{0}^{\infty} G(u)du - \int_{0}^{\infty} G(\tau) \int_{-\infty}^{\infty} S(\lambda) e^{i\lambda(u-\tau)} - 1 d\lambda d\tau$$

This equation can be solved by iteration stating with any element of $M[0, T]$. The solution of (5) is unique.

In order to prove the theorem, we will first prove two lemmas. Put

$$H[\Delta(t+mT)] = \int_{0}^{t} G(\tau)d\tau \int_{-\infty}^{\infty} S(\lambda) e^{i\lambda E(t+mT+\tau)} - 1 d\lambda d\tau$$

where $\Delta(t+mT) = E(t+nT) - E(t+mT)$ and $0 \leq u \leq T$. Here $T$ is a finite positive real number, $t_0$ a positive real number which may be finite or infinite and $m$ and $n$ positive integers. $E(u+nT) \in M[0, T]$ and $E(u+mT) \in M[0, T]$ implies $\Delta(u+mT) \in M[0, T]$. The operator $H$ will play an important role in the following considerations. For this reason we will first establish some of its properties. We will write more briefly $H(\Delta(t+u)) = H(\Delta)$.

**Lemma 1.** Suppose that $G \in L[0, \infty]$, and suppose that the function $N(x)$ belongs to one of the classes $K_{11}$ and $K_{10}$. Then $\Delta \in M[0, T]$ implies $H(\Delta) \in M[0, T]$ and

$$\|H(\Delta_1) - H(\Delta_2)\| \leq \beta \|\Delta_1 - \Delta_2\|$$

where
\[ \beta = \left( \int_{0}^{\infty} |G(u)| du \right) \left( \int_{-\infty}^{+\infty} |S(\lambda)| d\lambda \right). \]

Now
\[
\int_{0}^{t} G(\tau) [N\{E_{1}(t-\tau)\} - N\{E_{2}(t-\tau)\}] d\tau = \int_{0}^{t} G(\tau) I(t_{0} - \tau) [N\{E_{3}(t-\tau)\} - N\{E_{4}(t-\tau)\}] d\tau
\]
where
\[
N(E) = \int_{-\infty}^{+\infty} S(\lambda) \frac{e^{i\lambda E} - 1}{i\lambda} d\lambda
\]
and
\[
I(t_{0} - t) = \begin{cases} 1 & \text{if } t \leq t_{0} \\ 0 & \text{if } t_{0} < t \end{cases}
\]
G \in L[0, \infty] implies \( G(\tau)I(t_{0} - \tau) \in L[0, \infty] \). Furthermore, from \( x \in M[0, T] \) and the properties of \( N(x) \) follows that \( N(x) \in M[0, T] \). Consequently \( N(x) \in L[0, T] \). From known properties of the convolution follows now that
\[
\int_{0}^{t} G(\tau) I(t_{0} - \tau) [N\{E_{1}(t-\tau)\} - N\{E_{2}(t-\tau)\}] d\tau \in L[0, T].
\]
Hence \( H(\lambda) \in L[0, T] \). Now, as is easily seen,
\[
||H(\lambda)|| \leq \left( \int_{-\infty}^{+\infty} |S(\lambda)| d\lambda \right) \int_{0}^{t} |G(u)||\lambda(t + mT - u)| du \leq \beta ||\lambda||
\]
which implies the boundedness of \( H(\lambda) \). The function \( H(\lambda) \) is thus measurable and bounded in \([0, T] \), \( H(\lambda) \in M[0, T] \). Furthermore
\[
||H(\lambda) - H(\lambda)|| = ||\int_{0}^{t} G(\tau) d\tau \int_{-\infty}^{+\infty} S(\lambda) e^{i\lambda E(u + mT - \tau)} e^{i\lambda \lambda_{2}(u + mT - \tau)} - e^{i\lambda \lambda_{1}(u + mT - \tau)} d\lambda\|
\]
\[
\leq \left( \int_{-\infty}^{+\infty} |S(\lambda)| d\lambda \right) \left( \int_{0}^{t} |G(\tau)||\lambda(t + mT - \tau) - \lambda_{1}(u + mT - \tau)| d\tau \right)
\]
\[
\leq \beta ||\lambda - \lambda_{1}||,
\]
which completes the proof.

We will now consider the norm
\[
(6) \quad ||E(u + nT) - E(u + mT) + \int_{0}^{f(m)} G(\tau)[N\{E(u + nT - \tau)\} - N\{E(u + mT - \tau)\}] d\tau|| = Q
\]
where \( m \) and \( n \) are positive integers, \( f(m) \) an arbitrary function of \( m \),
$T$ a finite positive number and $E \in M[0, T]$. Furthermore it will be assumed that $G \in L[0, \infty]$ and $N \in K_{1, \infty}$ and that they satisfy the condition

$$\left( \int_{-\infty}^{+\infty} |S(\lambda)| \, d\lambda \right) \left( \int_{0}^{\infty} |G(u)| \, du \right) < 1.$$  

The following lemma holds.

**Lemma 2.** For every $\varepsilon > 0$ there exists an integer $m_0$ such that $m \geq m_0$ and $n \geq m_1$ imply $Q < \varepsilon$, if and only if, with $v(u)$ from $M[0, T]$, $||E(u+pT) - v(u)|| \to 0$ as $p \to \infty$ through positive integral values.

Suppose first that $||E(u+pT) - v(u)|| \to 0$, as $p \to \infty$, where $E$ and $v$ are in $M[0, T]$. Now

$$\left| \int_{0}^{+\infty} G(\tau) \left[ N\{E(u+nT-\tau)\} - N\{E(u+mT-\tau)\} \right] d\tau \right|$$

$$= \left| \int_{0}^{+\infty} G(\tau) \left( \int_{-\infty}^{+\infty} S(\lambda) e^{i\lambda E(u+nT-\tau)} - e^{i\lambda E(u+mT-\tau)} \right) d\lambda d\tau \right|$$

$$\leq \left( \int_{0}^{+\infty} |G(u)| du \right) \left( \int_{-\infty}^{+\infty} |S(\lambda)| d\lambda \right) ||E(u+nT) - E(u+mT)||$$

and consequently

$$||E(u+nT) - E(u+mT)|| \leq \left[ 1 + \left( \int_{0}^{+\infty} |G(u)| du \right) \left( \int_{-\infty}^{+\infty} |S(\lambda)| d\lambda \right) \right] ||E(u+nT) - E(u+mT)||$$

where $\left( \int_{0}^{+\infty} |G(u)| du \right) \left( \int_{-\infty}^{+\infty} |S(\lambda)| d\lambda \right) < 1$. Because $||E(u+pT) - v(u)|| \to 0$, as $p \to \infty$, there exists for every $\varepsilon > 0$ an integer $m_1$ such that $m_1 \leq m < n$ implies

$$||E(u+nT) - E(u+mT)|| \leq \frac{\varepsilon}{1 + \beta}$$

from which the first part of the lemma follows.

Suppose now that (6) is valid for $m$ and $n$ greater than a given integer $m_2$. The inequality (6) may be written

$$(7) \quad ||\Delta(u+mT) + \int_{0}^{+\infty} G(\tau) \left( \int_{-\infty}^{+\infty} S(\lambda) e^{i\lambda E(u+mT-\tau)} - e^{i\lambda E(u+mT-\tau)} \right) d\lambda d\tau || \leq \varepsilon$$

where $\Delta(u+mT) = E(u+nT) - E(u+mT)$

Now let $\tilde{h}$ be a function in $M[0, T] \cap S(\varepsilon, 0)$ where $S(\varepsilon, 0)$ is the sphere with radius $\varepsilon$ and center at $\tilde{h} = 0$. Put
The functions $\Delta$ obtained by solving (8) for all $h \in M[0, T] \cap S(\varepsilon, 0)$ are those which satisfy (7). $E(u + mT)$ is a known function.

The equation

$$
\Delta(u + mT) = h(u) - \int_0^{f(m)} G(\tau) \left( S(\lambda)e^{i\lambda E(u + mT - \tau) - \frac{1}{i\lambda}} \right) d\lambda d\tau = h(u) - H[\Delta(u + mT)]
$$

where $H$ is the operator defined on page 3, may be solved by iteration. Indeed, by Lemma 1 the operator $H$ is defined in $M[0, T]$, $\Delta \in M[0, T]$ implies $H(\Delta) \in M[0, T]$ and

$$
|| h(u) - H(\Delta) - (h(u) - H(\Delta)) || = || H(\Delta) - H(\Delta) || = \beta || \Delta - \Delta ||
$$

where $\beta = \left( \int_0^{f(m)} |G(u)| du \right) \left( \int_{-\infty}^{+\infty} |S(\lambda)| d\lambda \right) < 1$.

The conditions of the Tihonov's theorem are thus satisfied. We begin the iteration process with an $h$ from $M[0, T] \cap S(\varepsilon, 0)$:

$$
\Delta_1(u + nT) = h(u) - \int_0^{f(m)} G(\tau) \left( S(\lambda)e^{i\lambda E(u + nT - \tau) - \frac{1}{i\lambda}} \right) d\lambda d\tau
$$

and generally

$$
\Delta_{p+1}(u + nT) = h(u) - \int_0^{f(m)} G(\tau) \left( S(\lambda)e^{i\lambda E(u + mT - \tau) - \frac{1}{i\lambda}} \right) d\lambda d\tau
$$

The unique solution of (9) is then $\lim_{k \to \infty} \Delta_k(u + nT) = \Delta(u + mT)$ where $\Delta(u + mT)$ is in $M[0, T]$.

Now

$$
|| \Delta_{p+1} || \leq || h || + || \int_0^{f(m)} G(\tau) \left( S(\lambda)e^{i\lambda E(u + mT - \tau) - \frac{1}{i\lambda}} \right) d\lambda d\tau ||
$$

$$
\leq \varepsilon + \left( \int_0^{f(m)} |G(\lambda)| du \right) \left( \int_{-\infty}^{+\infty} |S(\lambda)| d\lambda \right) || \Delta_p || \leq \varepsilon + \beta || \Delta_p ||
$$

From this inequality one obtains now, remembering that $|| \Delta_0 || = || h || \leq \varepsilon$ and that $\beta < 1$,

$$
|| \Delta_{p+1} || \leq (1 + \beta + \beta^2 + \cdots + \beta^{p+1}) \varepsilon \leq \frac{\varepsilon}{1 - \beta}
$$

This inequality holds true for all $p$. Consequently
\[ ||A(u+nT)|| \leq \frac{\varepsilon}{1-\beta} \]

or, in view of the definition of \( A(u+nT) \),

\[ ||E(u+nT) - E(u+mT)|| \leq \frac{\varepsilon}{1-\beta} \]

for \( m \) and \( n \) greater than \( m_3 \). But such \( m_3 \) exists for every \( \varepsilon > 0 \). From this and from the completeness of the space \( M[0, T] \) follows that there exists a \( v_1 \in M[0, T] \) such that

\[ ||E(u+pT) - v_1(u)|| \to 0 \]
as \( p \to \infty \) through integral values.

We now proceed to prove the Theorem.

Because of the periodicity of \( F(t) \) one obtains from (1)

\[
E(u+nT) + \int_0^{u+nT} G(\tau)\{ E(u+nT-\tau) \} d\tau = E(u+mT) + \int_0^{u+mT} G(\tau)\{ E(u+mT-\tau) \} d\tau
\]

where \( 0 \leq u \leq T \) and where \( m \) and \( n \) are positive integers.

Suppose that \( m < n \) and \( t_d \leq mT \). Then

\[
E(u+nT) - E(u+mT) + \int_0^{u+nT} G(\tau)\{ E(u+nT-\tau) \} - E(u+mT-\tau) \} d\tau
\]

and

\[
||E(u+nT) - E(u+mT) + \int_0^{u+nT} G(\tau)\{ E(u+nT-\tau) \} - E(u+mT-\tau) \} d\tau||
\]

\[
\leq \left( \int_0^{u+nT} |G(\tau)||\{ E(u+nT-\tau) \} - E(u+mT-\tau) \} | d\tau \right) + \int_0^{u+nT} |G(\tau)||\{ E(u+mT-\tau) \} | d\tau||N||
\]

\[
\leq \left( \int_0^{u+nT} |G(\tau)||d\tau|| + \int_0^{u+mT} |G(\tau)||d\tau|| \right) ||N|| \leq 2||N|| \int_0^{\infty} |G(\tau)||d\tau||
\]

Because \( G \in L[0, \infty] \), there exists a positive integer \( m_3 \) for every \( \varepsilon > 0 \) such that for \( t_d = m_3T \)

\[
\int_{m_3T}^{\infty} |G(\lambda)||d\lambda| \leq \frac{\varepsilon}{2||N||}
\]

But \( m_3 \leq m < n \). Consequently, for every \( \varepsilon > 0 \) there exists a positive integer \( m_4 \) such that \( m_3 \leq m < n \) implies
\[ ||E(u+nT) - E(u+mT)|| \]
\[ + \int_{0}^{\frac{mT}{n}} G(\tau) [N\{E(u+nT-\tau)\} - N\{E(u+mT-\tau)\}] d\tau || \leq \varepsilon \]

By Lemma 2 it follows now that there exists a \( v \in M[0, T] \) such that \( ||E(u+pT) - v(u)|| \to 0 \) as \( p \to \infty \) through positive integral values. Consequently \( E(u+pT) \) converges uniformly to \( v(u) \) in \([0, T]\). That \( v(u) \) is periodic with period \( T \) is immediate.

We substitute now

\[ E(u+nT) = v(u) + H_n(u) \]

where \( H_n \in M[0, T] \) and \( ||H_n|| \to 0 \), as \( n \to \infty \) and where \( 0 \leq u \leq T \), into (1) and obtain

\[ v(u) + H_n(u) = F(u) - N(0) \int_{0}^{\frac{u+nT}{n}} G(\tau) d\tau \]

As is seen at once, this may be rewritten as follows:

\[ v(u) - F(u) + \int_{0}^{\infty} G(\tau) \int_{-\infty}^{\infty} S(\lambda) \frac{e^{i\lambda v(u-\tau)} - 1}{i\lambda} d\lambda d\tau + \int_{0}^{\infty} G(\tau) d\tau \]

\[ + \int_{0}^{\infty} G(\tau) \int_{-\infty}^{\infty} S(\lambda) \frac{e^{i\lambda v(u-\tau)} - 1}{i\lambda} d\lambda d\tau + H_n(u) + \int_{-\infty}^{\infty} G(\tau) \int_{-\infty}^{\infty} S(\lambda) \frac{e^{i\lambda v(u-\tau)} - 1}{i\lambda} d\lambda d\tau - N(0) \int_{\frac{nT+u}{n}}^{\infty} G(\tau) d\tau = 0 \]

which yields the inequality

\[ ||v(u) - F(u)|| + \int_{0}^{\infty} G(\tau) \int_{-\infty}^{\infty} S(\lambda) \frac{e^{i\lambda v(u-\tau)} - 1}{i\lambda} d\lambda d\tau \]

\[ \leq ||H_n(u)|| + \left( \int_{0}^{\infty} |G(u)|du \right) \left( \int_{-\infty}^{\infty} |S(\lambda)|d\lambda \right) ||H_n(u)|| \]

\[ + \left( \int_{\frac{nT+u}{n}}^{\infty} |G(u)|du \right) \left( \int_{-\infty}^{\infty} |S(\lambda)|d\lambda \right) ||v(u)|| + H_n(u) + \int_{\frac{nT+u}{n}}^{\infty} |G(u)|du \]

\[ = (1 + \beta)||H_n(u)|| + \left( \int_{\frac{nT+u}{n}}^{\infty} |G(u)|du \right) \left( \int_{-\infty}^{\infty} |S(\lambda)|d\lambda \right) ||v(u)|| + ||H_n(u)|| \]

\[ + N(0) \int_{\frac{nT+u}{n}}^{\infty} |G(u)|du + \lambda \]

But \( \beta, \int_{-\infty}^{\infty} |S(\lambda)|d\lambda, ||v(u)|| \) and \( N(0) \) are finite, \( ||H_n|| \to 0 \) as \( n \to \infty \) and \( \int_{\frac{nT+u}{n}}^{\infty} |G(u)|du \to 0 \) as \( n \to \infty \). Consequently
\[ ||v(u) - (F(u) - N(0)) \int_0^\infty G(u) du - \int_0^\infty G(\tau) S(\omega) \frac{e^{i\lambda(u-\tau)} - 1}{i\lambda} d\lambda d\tau|| \to 0, \]

as \( n \to \infty \) through integral values, from which the equation (5) follows for \( v(u) \).

The right side of (5) satisfies the conditions of Tihonov's theorem. This follows by Lemma 1 where we substitute \( t_0 = \infty, \) \( E(mT+u-\tau) = 0 \) and \( \Delta(mT+u-\tau) = v(u-\tau) \). If the right side of (5) is denoted by \( c(v) \), then, by Lemma 1, \( v \in M[0, T] \) implies \( c(v) \in M[0, T] \) and \( ||c(v_1) - c(v_2)|| \leq \beta ||v_1 - v_2|| \) for \( v_1 \) and \( v_2 \) from \( M[0, T] \). By Tihonov's theorem it follows then that the equation (5) has a unique solution \( v \in M[0, T] \) which may be obtained by iteration, beginning with an arbitrary function from \( M[0, T] \).

Reference

1. Г. Эдсго, Качественная теория дифференциальных уравнений. Гостехиздат 1955, Москга

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