A NOTE ON A PAPER OF L. GUTTMAN

Ancel Clyde Mewborn
In a recent paper L. Guttman [2] obtained, using a result of von Neumann on the theory of games, lower bounds for the largest characteristic root of the matrix $AA'$ where $A$ is a real matrix of order $m \times n$. As Guttman points out his bounds are non-trivial only if some row or column of $A$ has only positive or only negative elements. I wish to show that Guttman’s results, and even a better result, are an immediate corollary of a well known theorem on Hermitian matrices: that each diagonal element lies between the smallest and largest characteristic roots (see e.g. [1]). Moreover, if $AA'$ be replaced by $AA^*$ then $A$ can be real or complex and a non-trivial result is always obtained.

**Theorem 1.** Let $A=(a_{ij})$ be an $m \times n$ matrix with real or complex elements. Let $\lambda$ be the largest characteristic root of the $m \times m$ non-negative definite Hermitian matrix $B=AA^*=(b_{ij})$. Then

\begin{align}
\lambda &\geq \max_i \max_j |a_{ij}|^2 \\
\lambda &\geq \max_j \sum_i |a_{ij}|^2
\end{align}

**Proof.** Let $b_{rr}$ be the largest diagonal element of $B$. Then

$$\lambda \geq b_{rr} = \sum_j |a_{ij}|^2 = \max_i \sum_j |a_{ij}|^2,$$

and (1) is proved. Now the non-zero characteristic roots of $AA^*$ are the same as those of $A^*A$. Then (2) follows as above if we consider $A^*A$ instead of $AA^*$.

The bounds in (1) and (2) can be replaced by the weaker bounds

\begin{align}
\lambda &\geq n \cdot \max_i \left( \min_j |a_{ij}|^2 \right) \\
\lambda &\geq m \cdot \max_j \left( \min_i |a_{ij}|^2 \right)
\end{align}

respectively, and even these bounds are obviously better than Guttman’s.

Theorem 1 can be improved further.

**Theorem 2.** Under the hypotheses of Theorem 1 we have

Received October 25, 1957. The author is a National Science Foundation fellow.
(5) \[ 2\lambda \geq \max_{i,j} \left[ \sum_{v=1}^{n} (|a_{iv}|^2 + |a_{iv}|^2) + \left\{ \sum_{v=1}^{n} (|a_{iv}|^2 - |a_{ij}|^2) \right\}^2 + 4 \left| \sum_{v=1}^{n} a_{iv} a_{iv} \right|^{1/2} \right] \]

(6) \[ 2\lambda \geq \max_{i,j} \left[ \sum_{v=1}^{m} (|a_{iv}|^2 + |a_{iv}|^2) + \left\{ \sum_{v=1}^{m} (|a_{iv}|^2 - |a_{ij}|^2) \right\}^2 + 4 \left| \sum_{v=1}^{m} a_{iv} a_{iv} \right|^{1/2} \right] \]

Proof. It was shown in [1] that the largest root of an Hermitian matrix is greater than or equal to the larger of the two roots of any principal minor of order two of the matrix. Suppose the principal minor or order two of \( B \) having the largest root lies in the \( r, s \) rows and columns of \( B \). Then

\[ 2\lambda \geq b_{rr} + b_{ss} + [(b_{rr} - b_{ss})^2 + 4 \left| b_{rs} \right|^2]^{1/2} \]

\[ = \sum_{v=1}^{n} (|a_{rv}|^2 + |a_{sv}|^2) + \left\{ \sum_{v=1}^{n} (|a_{rv}|^2 - |a_{sv}|^2) \right\}^2 + 4 \left| \sum_{v=1}^{n} a_{rv} a_{sv} \right|^{1/2} \]

and (3) follows. (4) is proved similarly by considering \( A^*A \) instead of \( B \).

References


University of North Carolina
Mathematical papers intended for publication in the Pacific Journal of Mathematics should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any of the editors. All other communications to the editors should be addressed to the managing editor, E. G. Straus at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published quarterly, in March, June, September, and December. The price per volume (4 numbers) is $12.00; single issues, $3.50. Back numbers are available. Special price to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues, $1.25.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 2120 Oxford Street, Berkeley 4, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 10, 1-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.
John Herbert Barrett, *Second order complex differential equations with a real independent variable* .................................................. 187
Avner Friedman, *Remarks on the maximum principle for parabolic equations and its applications* .................................................. 201
Richard Robinson Goldberg, *An inversion of the Stieltjes transform* ............ 213
Olavi Hellman, *On the periodicity of the solution of a certain nonlinear integral equation* ................................................................. 219
Gilbert Helmberg, *A theorem on equidistribution on compact groups* ............ 227
Lloyd Kenneth Jackson, *Subfunctions and the Dirichlet problem* ................. 243
Naoki Kimura, *The structure of idempotent semigroups. I* ......................... 257
Stephen Kulik, *A method of approximating the complex roots of equations* ................................................................. 277
Ancel Clyde Mewborn, *A note on a paper of L. Guttman* ............................ 283
Zeev Nehari, *On the principal frequency of a membrane* ........................... 285
B. M. Stewart, *Asymmetry of a plane convex set with respect to its centroid* ................................................................. 335
Hans F. Weinberger, *Lower bounds for higher eigenvalues by finite difference methods* .................................................. 339
Edwin Weiss and Neal Zierler, *Locally compact division rings* ..................... 369
Bertram Yood, *Homomorphisms on normed algebras* ................................. 373