LOCALLY COMPACT DIVISION RINGS

Edwin Weiss and Neal Zierler
Let $K$ be a division ring with a non-discrete topology $T$ with respect to which both the additive group $K^+$ and the multiplicative group $K^*$ of $K$ are locally compact topological groups.\(^1\) If $m$ is Haar measure for $K^+$ and $a \in K$, the function $m'(E) = m(aE)$ is clearly an invariant Borel measure for $K^*$. Hence there exists a real number $\phi(a)$ such that $m'(E) = \phi(a)m(E)$ for all Borel subsets $E$ of $K^*$. The real-valued function $\phi$ on $K$ (which is essentially the Radon-Nikodym derivative of $m$ with respect to left-invariant Haar measure on $K^*$) evidently has the first two of the following three properties.

1. $\phi(a) \geq 0$; $\phi(a) = 0$ if and only if $a = 0$.
2. $\phi(ab) = \phi(a)\phi(b)$.
3. There exists $M > 0$ such that $\phi(a) \leq 1$ implies $\phi(1+a) \leq M$.

We shall show that $\phi$ satisfies (3) also, i.e., is a valuation for $K$, and that the topology $T_\phi$ for $K$ defined by $\phi$ coincides with $T$.\(^2\) The classification of $K$ then follows from known results.

**Lemma 1.** $\phi$ is continuous.

**Proof.** Let $\varepsilon$ be a positive number and let $E$ be a compact set of positive measure. By the regularity of Haar measure we may choose an open set $U$ containing $E$ such that $m(U) - m(E) < \varepsilon m(E)$. Choose a neighborhood $V$ of 1 with $V = V^{-1}$ and $V \cdot E \subset U$. Then for $x$ in $V$, $\phi(x) = m(xE)/m(E) \leq m(U)/m(E) < 1 + \varepsilon$; since $x^{-1} \in V$, $\phi(x) = (\phi(x^{-1}))^{-1} > (1 + \varepsilon)^{-1}$. Hence $1 - \varepsilon < \phi(x) < 1 + \varepsilon$ and the continuity of $\phi$ on $K^*$ follows.\(^2\)

Now choose an open set $U$ with $m(U) < \varepsilon m(E)$ and a neighborhood $V$ of 0 with $V \cdot E \subset U$. Then for $a$ in $V$, $\phi(a) = m(aE)/m(E) \leq m(U)/m(E) < \varepsilon$ and $\phi$ is continuous at 0.

**Lemma 2.** $S = \{a \in K : \phi(a) \leq 1\}$ is compact.
Proof. Let $C$ be a compact neighborhood of 0 and choose a neighborhood $V$ of 0 such that $V \cdot C \subset C$. Let $a \in V \cap C$ such that $0 < \phi(a) < 1$. If $a^n S \subset C$ holds for no $n = 1, 2, \ldots$, we select for each $n$ an $s_n \in S$ such that $a^n s_n \notin C$. Since $\phi(a^n) \to 0$ and all the $a^n$ lie in the compact set $C$, $a^n \to 0$ and hence $a^n s_n \in C$ for sufficiently large $k$. We may therefore choose $k_n \geq n$ such that $a^n s_n \notin C$ but $a^{k_n + 1} s_n \in C$. Then the sequence $\{a^n s_n\}$ of elements of the compact set $a^{-1} C$ has a cluster point $c$ in $a^{-1} C$. Hence $\phi(a^n s_n) = \phi(a)^n \phi(s_n) \leq \phi(a)^n$ has $\phi(c)$ as a cluster point by the continuity of $\phi$; thus $\phi(c) = 0$ and $c = 0$, which contradicts $a^n s_n \notin C$. It follows that $S$ is a subset of the compact set $a^{-n} C$ for some $n$ and so, being closed by virtue of the continuity of $\phi$, is compact.

COROLLARY. $\phi$ is a valuation.

Proof. $\phi(1+S)$, the continuous image of the compact set $1+S$, is bounded.

LEMMA 3. $T_\phi = T$.

Proof. Let $V \in T \setminus \{\phi\}$, $a \in V$ and $B_n = \{b \in K : \phi(b-a) < 2^{-n}\}$. Suppose we can choose $b_n \in B_n$ with $b_n \notin V$ for each $n = 1, 2, \ldots$. But then the points $b_n - a$, all of which lie in the compact set $S$, have a cluster point $c$ in $S$ which must be 0 since $\phi(c) = 0$. Hence $b_n \to a$ contrary to our assumption and it follows that $T \subset T_\phi$. Since the opposite inclusion is an immediate consequence of the continuity of $\phi$, the proof is complete.

If $K$ is connected, it is the real, complex or quaternion field (Pontryagin [10]); in particular, $\phi$ is archimedean. Conversely, if $\phi$ is archimedean, the theorem of Ostrowski [8, p. 278] asserts that the center of $K$ is either the real or complex field and so $K$, not being totally disconnected, is connected.\footnote{\textit{Note:} $K$ is either connected or totally disconnected: if the component $C$ of 0 contains $a \notin 0$ then $ba^{-1} C$ is a connected set containing 0 and $b \in K$.}

If $K$ is totally disconnected, $\phi$ is non-archimedean (and conversely, according to the above) and results due to van Dantzig [2], Hasse [4], Hasse and Schmidt [5], Jacobson and Taussky [6] and Jacobson [7] assert that $K$ is of one of the following three types:\footnote{Otobe [9] shows that $a \to a^{-1}$ need not be assumed to be continuous; cf. our final remark in this connection.}

(i) the completion of an algebraic number field at a finite prime,
(ii) the completion of an algebraic function field in one variable.

Alternatively, if $K$ is connected, it is not difficult to show that $\phi$ is archimedean; then $K$ is a vector space over the reals (Ostrowski) with $\phi$ as a norm, hence is the real, complex or quaternion field (Arens [1] Tornheim [13]), proving Pontryagin’s theorem.
over a finite field $H$,

(iii) a division ring $D$ obtained from a field $F$ of type (ii) by redefining $x$. $a = a^\sigma$, $x, a \in H$, $\sigma$ a fixed non-trivial automorphism of $H$, the elements of $D$ and $F$ being regarded as power series $\sum_{n=0}^{\infty} a_n x^n$ in an indeterminate $x$ over $H$ with coefficients in $H$.

**Remark.** Continuity of $a \rightarrow a^{-1}$ need not be assumed, for it appears in the connected case only in the proof that $K$ is not compact in the proof of the Pontrjagin theorem [11, p. 173, Theorem 45]. If $K$ were compact, $\phi(a) = m(aK)/m(K) \leq 1$ for all $a \in K$. But, as in the proof of the continuity of $\phi$ at 0 in Lemma 1, we can find $a \in K$ such that $0 < \phi(a) < 1$; then $\phi(a^{-1}) > 1$ and it follows that $K$ is not compact. If $K$ is totally disconnected we have only to apply to $T, K^*$ the following unpublished theorem of A. M. Gleason: Let $G$ be a group with a totally disconnected topology $T$ under which the group operation is continuous from $G \times G$ to $G$. Then $T, G$ is a topological group.

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